REDUCING REDUCIBLE LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH FUNCTION COEFFICIENTS TO LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract
In this article, we propose a generalized method for obtaining a substitution for reducing a reducible linear ordinary differential equation with function coefficients (RLDEF) to a linear ordinary differential equation with constant coefficient (LDE). This proposed method was also used to obtain the already known substitutions for the Euler's and Legendre's homogeneous second order linear differential equation. The derived method is able to reduce quite a large number of RLDEF to LDE including the Euler's and Legendre's homogeneous second order linear differential equation. However, these RLDEF (homogeneous and inhomogeneous) must satisfy the condition for reducibility, which is also proposed before the substitution is derived. the condition for reducibility is based on the order of the differential equation. In this article, the condition for reducibility is presented for a second and third order LDEF.

Keywords: *reducibility, generalized, differential equations.*

1.0 Introduction
The solutions to RLDEF have been one of the major problems in solving linear ordinary differential equations (ODEs). These RLDEF can be solved either by substitution which transforms it into a LDE or by knowing one of the solutions of the RLDEF.
However, reducing these RLDEF by substitution to a LDE requires getting the right substitution. These substitutions are majorly gotten by trial and error method especially when it is not the Euler's or Legendre's homogeneous differential equation form which has standard substitutions [1, 2].
[3] discussed the reducibility of second order differential operators with rational coefficients of the form $D^2+\alpha(x)D+\beta(x)$ where $D^n=\frac{d^n}{dx^n}$, $\alpha(x)=\frac{p(x)}{m(x)}$, and $\beta(x)=\frac{q(x)}{n(x)}$ are rational and P(x), q(x), M(x) and n(x) are polynomial with complex coefficient. [4] dealt with the factorization of self-adjoint differential operators

$L_{(2n)}=\frac{1}{\rho}\frac{d^n}{dx^n}\left(\rho\beta^n\frac{d^n}{dx^n}\right)$ and their spectral type differential equations and the work provided sufficient conditions of factorizing self adjoint differential equations

$\frac{1}{\rho}\frac{d^n}{dx^n}\left(\rho\beta^n\frac{d^n y}{dx^n}\right)-\mu y=0,$ where $\rho(x), \beta(x)$ are scalar functions and μ is a positive constant.

This paper proposes a standard method of determining if a linear ordinary differential equation with function coefficients is reducible and subsequently a method on how to determine the substitution that will reduce it if reducible to LDE, which hitherto does not exist in literature to the best of our knowledge.

2.0 Proposition 1

The homogeneous second order ordinary differential equation with function coefficient $f(x)\frac{d^2y}{dx^2}+g(x)\frac{dy}{dx}+ay=0$, where a is a constant, is reducible if $g(x)=\frac{1}{2}f'(x)+b\sqrt{f(x)}$ holds and it is reducible to a differential equation with constant coefficient by the substitution $z(x)=\int\frac{1}{\sqrt{f(x)}}dx$

Proof.

Given a second order differential equation of the form

$$f(x)\frac{d^2y}{dx^2}+g(x)\frac{dy}{dx}+ay=0 \quad 2.0$$

Where a is a constant

Consider the substitution

$$z=r(x) \quad 2.1$$

$$\frac{dz}{dx}=r'(x)$$

$$\frac{dy}{dx}=r'(x)\frac{dy}{dz}$$

$$g(x)\frac{dy}{dx} = g(x)r'(x)\frac{dy}{dz} \quad 2.2$$

$$\frac{d^2y}{dx^2}=r''(x)\frac{dy}{dz}+(r'(x))^2\frac{d^2y}{dz^2}$$

$$f(x)\frac{d^2y}{dx^2}= f(x)r''(x)\frac{dy}{dz}+ f(x)(r'(x))^2\frac{d^2y}{dz^2} \quad 2.3$$

Substitute (2.2) and (2.3) into (2.0) and simplify gives

$$f(x)(r'(x))^2\frac{d^2y}{dz^2}+\left(f(x)r''(x)+g(x)r'(x)\right)\frac{dy}{dz}+ay=0 \quad 2.4$$

$$f(x)r''(x)+g(x)r'(x)=b \quad 2.5$$

Where b is a constant

$$f(x)(r'(x))^2 =1 \quad 2.6$$

So that we have

$$\frac{d^2y}{dz^2}+\frac{bdy}{dz}+ay=0 \quad 2.7$$

From (2.6)

$$(r'(x))^2=\frac{1}{f(x)}$$

$$r'(x)=\sqrt{\frac{1}{f(x)}} \qquad 2.8$$

$$r(x)=\int\frac{1}{\sqrt{f(x)}}dx \qquad 2.9$$

$$r''(x)=-\frac{1}{2}\sqrt{\frac{1}{(f(x))^3}}\,f'(x) \qquad 2.10$$

Substitute (2.8) and (2.10) into (2.5) and after simplification we have

$$g(x)=\frac{1}{2}f'(x)+b\sqrt{f(x)} \qquad 2.11$$

Also from (2.8) and (2.1), we have

$$z=r(x)=\int\frac{1}{\sqrt{f(x)}}dx \qquad 2.12$$

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2.0) is reducible if (2.11) holds and (2.12) is a substitution required to reduce (2.0) to a LDE

3.0 Proposition 2

The homogeneous third order ordinary differential equation with function coefficient $f(x)\frac{d^3y}{dx^3}+g(x)\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+ay=0$ where a is a constant, is reducible if $g(x)=f'(x)+b\sqrt[3]{(f(x))^2}$ and $p(x)=\frac{1}{3}f''(x)-\frac{(f'(x))^2}{9f(x)}+b\frac{f'(x)}{\sqrt[3]{f(x)}}+c\sqrt[3]{f(x)}$ hold and it is reducible to a differential equation with constant coefficient by the substitution $z=\int\frac{1}{\sqrt[3]{f(x)}}dx.$

Proof

Consider a third order ordinary differential equation of the form:

$$f(x)\frac{d^3y}{dx^3}+g(x)\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+ay=0 \qquad 3.0$$

Where a is a constant

Let $z=r(x)$, 3.1

So,

$$\frac{dz}{dx}=r'(x)$$

$$\frac{dy}{dx}=r'(x)\frac{dy}{dz}$$

$$p(x)\frac{dy}{dx}=p(x)r'(x)\frac{dy}{dz} \qquad 3.2$$

$$\frac{d^2y}{dx^2}=r''(x)\frac{dy}{dz}+(r'(x))^2\frac{d^2y}{dz^2}$$

$$g(x)\frac{d^2y}{dx^2} = g(x)r''(x)\frac{dy}{dz} + g(x)(r'(x))^2\frac{d^2y}{dz^2} \quad 3.3$$

$$\frac{d^3y}{dx^3} = r'''(x)\frac{dy}{dz} + 3r''(x)r'(x)\frac{d^2y}{dz^2} + (r'(x))^3\frac{d^3y}{dz^3}$$

$$f(x)\frac{d^3y}{dx^3} = f(x)r'''(x)\frac{dy}{dz} + 3f(x)r''(x)r'(x)\frac{d^2y}{dz^2} + f(x)(r'(x))^3\frac{d^3y}{dz^3} \quad 3.4$$

Substitute (3.2), (3.3) and (3.4) into (3.0) and after simplification gives

$$f(x)(r'(x))^3\frac{d^3y}{dz^3} + \left(g(x)(r'(x))^2 + 3f(x)r''(x)r'(x)\right)\frac{d^2y}{dz^2} + \left(p(x)r'(x) + g(x)r''(x) + f(x)r'''(x)\right)\frac{dy}{dz} + ay = 0$$

$$g(x)(r'(x))^2 + 3f(x)r''(x)r'(x) = b \quad 3.5$$

$$p(x)r'(x) + g(x)r''(x) + f(x)r'''(x) = c \quad 3.6$$

Where b and c are constants

$$f(x)(r'(x))^3 = 1 \quad 3.7$$

So that we have

$$\frac{d^3y}{dz^3} + b\frac{d^2y}{dz^2} + c\frac{dy}{dz} + ay = 0 \quad 3.8$$

From (3.7)

$$(r'(x))^3 = \frac{1}{f(x)}$$

$$r'(x) = \frac{1}{\sqrt[3]{f(x)}} \quad 3.9$$

$$r(x) = \int \frac{1}{\sqrt[3]{f(x)}}dx \quad 3.10$$

$$r''(x) = -\frac{1}{3}\frac{1}{\sqrt[3]{(f(x))^4}}f'(x) \quad 3.11$$

$$r'''(x) = -\frac{1}{3}\frac{1}{\sqrt[3]{(f(x))^4}}f''(x) + \frac{4}{9}\frac{1}{\sqrt[3]{(f(x))^7}}(f'(x))^2 \quad 3.12$$

Substitute (3.9) and (3.11) into (3.5) and simplify

$$g(x) = f'(x) + b\sqrt[3]{(f(x))^2} \quad 3.13$$

Substitute (3.9), (3.11) and (3.12) into (3.6) and simplify

$$p(x) = \frac{1}{3}f''(x) - \frac{(f'(x))^2}{9f(x)} + b\frac{f'(x)}{\sqrt[3]{f(x)}} + c\sqrt[3]{f(x)} \quad 3.14$$

Also from (3.1) and (3.10), we have

$$z = r(x) = \int \frac{1}{\sqrt[3]{f(x)}}dx \quad 3.15$$

4.0 Proposition 3

The homogeneous third order ordinary differential equation with function coefficient $f(x)\frac{d^4y}{dx^4}+g(x)\frac{d^3y}{dx^3}+p(x)\frac{d^2y}{dx^2}+h(x)\frac{dy}{dx}+ay=0$ where a is a constant, is reducible if $g(x)=\frac{3}{2}f'(x)+b\sqrt[4]{(f(x))^3}$, $p(x)=f''(x)-\frac{5(f'(x))^2}{16f(x)}+b\frac{f'(x)}{\sqrt[4]{f(x)}}+c\sqrt{f(x)}$, and $h(x)=\frac{1}{4}f'''-\frac{35(f'(x))^3}{64(f(x))^2}+\frac{45}{64}\left(\frac{f'(x)}{f(x)}\right)^2-\frac{5}{16}f'(x)f''(x)+b\left(4\frac{f''(x)}{\sqrt[4]{f(x)}}-\frac{(f'(x))^2}{\sqrt[4]{(f(x))^5}}\right)+\frac{cf'(x)}{\sqrt[4]{(f(x))^2}}+d\sqrt[4]{f(x)}$ hold and it is reducible to a differential equation with constant coefficient by the substitution $z=\int\frac{1}{\sqrt[4]{f(x)}}dx.$

Proof

Consider a fourth order ordinary differential equation of the form:

$$f(x)\frac{d^4y}{dx^4}+g(x)\frac{d^3y}{dx^3}+p(x)\frac{d^2y}{dx^2}+h(x)\frac{dy}{dx}+ay=0 \qquad 4.0$$

Where a is a constant

Let $z=r(x)$, 4.1

So,

$$\frac{dz}{dx}=r'(x)$$

$$\frac{dy}{dx}=r'(x)\frac{dy}{dz}$$

$$h(x)\frac{dy}{dx}=h(x)r'(x)\frac{dy}{dz} \qquad 4.2$$

$$\frac{d^2y}{dx^2}=r''(x)\frac{dy}{dz}+(r'(x))^2\frac{d^2y}{dz^2}$$

$$p(x)\frac{d^2y}{dx^2}=p(x)r''(x)\frac{dy}{dz}+p(x)(r'(x))^2\frac{d^2y}{dz^2} \qquad 4.3$$

$$\frac{d^3y}{dx^3}=r'''(x)\frac{dy}{dz}+3r''(x)r'(x)\frac{d^2y}{dz^2}+(r'(x))^3\frac{d^3y}{dz^3}$$

$$g(x)\frac{d^3y}{dx^3}=g(x)r'''(x)\frac{dy}{dz}+3g(x)r''(x)r'(x)\frac{d^2y}{dz^2}+g(x)(r'(x))^3\frac{d^3y}{dz^3} \qquad 4.4$$

$$\frac{d^4y}{dx^4}=r''''(x)\frac{dy}{dz}+4r'''(x)r'(x)\frac{d^2y}{dz^2}+3(r''(x))^2\frac{d^2y}{dz^2}+6r''(x)(r'(x))^2\frac{d^3y}{dz^3}+(r'(x))^4\frac{d^4y}{dz^4}$$

$$f(x)\frac{d^4y}{dx^4}=f(x)r''''(x)\frac{dy}{dz}+f(x)\left(4r'''(x)r'(x)+3(r''(x))^2\right)\frac{d^2y}{dz^2}+$$

$$6f(x)r''(x)(r'(x))^2\frac{d^3y}{dz^3}+f(x)(r'(x))^4\frac{d^4y}{dz^4} \qquad 4.5$$

Substitute (4.2), (4.3), (4.4) and (4.5) into (4.0) and after simplification gives

$$f(x)(r'(x))^4 \frac{d^4y}{dz^4} + \left(g(x)(r'(x))^3 + 6f(x)r''(x)(r'(x))^2\right)\frac{d^3y}{dz^3} +$$

$$\left(p(x)(r'(x))^2 + 3g(x)r''(x)r'(x) + f(x)\left(4r'''(x)r'(x) + 3(r''(x))^2\right)\right)\frac{d^2y}{dz^2} +$$

$$(h(x)r'(x) + p(x)r''(x) + g(x)r'''(x) + f(x)r''''(x))\frac{dy}{dz} + ay = 0 \qquad 4.6$$

$$g(x)(r'(x))^3 + 6f(x)r''(x)(r'(x))^2 = b \qquad 4.7$$

$$p(x)(r'(x))^2 + 3g(x)r''(x)r'(x) + f(x)\left(4r'''(x)r'(x) + 3(r''(x))^2\right) = c \qquad 4.8$$

$$h(x)r'(x) + p(x)r''(x) + g(x)r'''(x) + f(x)r''''(x) = d \qquad 4.9$$

Where b, c and d are constants

$$f(x)(r'(x))^4 = 1 \qquad 4.10$$

So that we have

$$\frac{d^4y}{dz^4} + b\frac{d^3y}{dz^3} + c\frac{d^2y}{dz^2} + d\frac{dy}{dz} + ay = 0 \qquad 4.11$$

From (4.10) $(r'(x))^4 = \frac{1}{f(x)}$

$$r'(x) = \frac{1}{\sqrt[4]{f(x)}} \qquad 4.12$$

$$r(x) = \int \frac{1}{\sqrt[4]{f(x)}}dx \qquad 4.13$$

$$r''(x) = -\frac{1}{4}\frac{1}{\sqrt[4]{(f(x))^5}}f'(x) \qquad 4.14$$

$$r'''(x) = -\frac{1}{4}\frac{1}{\sqrt[4]{(f(x))^5}}f''(x) + \frac{5}{16}\frac{1}{\sqrt[4]{(f(x))^9}}(f'(x))^2 \qquad 4.15$$

$$r''''(x) = -\frac{1}{4}\frac{1}{\sqrt[4]{(f(x))^5}}f'''(x) + \frac{5}{16}\frac{1}{\sqrt[4]{(f(x))^5}}f'(x)f''(x) + \frac{5}{8}\frac{1}{\sqrt[4]{(f(x))^9}}f'(x)f''(x) -$$

$$\frac{45}{64}\frac{1}{\sqrt[4]{(f(x))^{13}}}(f'(x))^2 \qquad 4.16$$

Substitute (4.12) and (4.14) into (4.7) and simplify

$$g(x) = \frac{3}{2}f'(x) + b\sqrt[4]{(f(x))^3} \qquad 4.17$$

Substitute (5.12), (5.14) and (5.15) into (4.8) and simplify

$$p(x) = f''(x) - \frac{5(f'(x))^2}{16f(x)} + b\frac{f'(x)}{\sqrt[4]{f(x)}} + c\sqrt{f(x)} \qquad 4.18$$

Substitute (5.12), (5.14), (5.15) and (5.16) into (4.9) and simplify

$$h(x) = \frac{1}{4}f''' - \frac{35(f'(x))^3}{64(f(x))^2} + \frac{45}{64}\left(\frac{f'(x)}{f(x)}\right)^2 - \frac{5}{16}f'(x)f''(x) + b\left(4\frac{f''(x)}{\sqrt[4]{f(x)}} - \frac{(f'(x))^2}{\sqrt[4]{(f(x))^5}}\right) +$$

$$\frac{cf'(x)}{\sqrt[4]{(f(x))^2}} + d\sqrt[4]{f(x)} \qquad 4.19$$

Also from (4.1) and (4.13), we have

$$z = r(x) = \int \frac{1}{\sqrt[4]{f(x)}}dx \qquad 4.20$$

Corollary

In general we conclude that given any RLDEF of the form $f_n(x)\frac{d^n y}{dx^n} + f_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + ay = 0$, where a is a constant, is reducible to a LDE by the substitution $z = \int \frac{1}{\sqrt[n]{f_n(x)}}dx$

5.0 Illustrative examples

5.1 Solve the differential equation with algebraic coefficient

$$\left(1+x^2\right)^2 \frac{d^2y}{dx^2} + \left(1+x^2\right)(7+2x)\frac{dy}{dx} + 10y = 0 \qquad 5.0$$

Solution

First we check if it is reducible,

By (2.11), we have

$$f(x) = \left(1+x^2\right)^2$$

$$g(x) = \left(1+x^2\right)(2x+b) \qquad 5.1$$

Where $b = 7$

Which satisfies $g(x)$ in (5.0) and hence (5.0) is reducible,

To reduce (5.0) we apply (2.12) to obtain

$$z = \tan^{-1} x \qquad 5.2$$

From (4.2), we have

$$\left(1+x^2\right)(7+2x)\frac{dy}{dx} = (7+2x)\frac{dy}{dz} \qquad 5.3$$

$$\left(1+x^2\right)^2 \frac{d^2y}{dx^2} = -2x\frac{dy}{dz} + \frac{d^2y}{dz^2} \qquad 5.4$$

Substituting (5.3) and (5.4) into (5.0) and simplify

$$\frac{d^2y}{dz^2} + 7\frac{dy}{dz} + 10y = 0 \qquad 5.5$$

$$y = Ae^{-2z} + Be^{-5z} \qquad 5.6$$

$$y = Ae^{-2\tan^{-1} x} + Be^{-5\tan^{-1} x} \qquad 5.7$$

5.2 Solve the differential equation with logarithmic coefficient

$$(x \ln x)^2 \frac{d^2y}{dx^2} + x(\ln x)^2 \frac{dy}{dx} - 2y = 0 \qquad 5.8$$

SOLUTION

To check if the equation is reducible,

From (2.11),

$$f(x) = (x\ln x)^2$$

$$g(x) = (x\ln x)(1 + \ln x + b)$$

Where $b=-1$

Which satisfies $g(x)$ in (5.8) and hence (5.8) is reducible,

By (2.12), we reduce (5.8) by the substitution

$$z(x)=\ln(\ln x) \qquad 5.9$$

$$x(\ln x)^2\frac{dy}{dx}=(\ln x)\frac{dy}{dz} \qquad 5.10$$

$$(x\ln x)^2\frac{d^2y}{dx^2}=-(1+\ln x)\frac{dy}{dz}+\frac{d^2y}{dz^2} \qquad 5.11$$

Substituting (5.10) and (5.11) into (5.8) and simplify, we have

$$\frac{d^2y}{dz^2}-\frac{dy}{dz}-2y=0 \qquad 5.12$$

$$y=A(\ln x)^2+\frac{B}{\ln x} \qquad 5.13$$

5.3 Solve the differential equation with trigonometry coefficient

$$\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}+4y\cos^3 x=8\cos^5 x \qquad 5.14$$

SOLUTION

$$\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}+4y\cos^3 x=8\cos^5 x$$

Divide through by $\cos^3 x$

$$\sec^2 x\frac{d^2y}{dx^2}+\tan x\sec^2 x\frac{dy}{dx}+4y=8\cos^2 x$$

To check if the equation is reducible,

From (2.11),

$$f(x)=\sec^2 x$$

$$g(x)=\sec^2 x\tan x+b\sec x$$

Where $b=0$

Which satisfies $g(x)$ in (4.14) and hence (4.14) is reducible

To reduce (4.14) we apply (2.12) to obtain

$$z=\sin x \qquad 5.15$$

$$\tan x\sec^2 x\frac{dy}{dx}=\tan x\sec x\frac{dy}{dz} \qquad 5.16$$

$$\sec^2 x\frac{d^2y}{dx^2}=-\tan x\sec x\frac{dy}{dz}+\frac{d^2y}{dz^2} \qquad 5.17$$

Substituting (5.16) and (5.17) into (5.14) and simplify, we have

$$\frac{d^2y}{dz^2}+4y=8(1-z^2)$$

$$y=A\cos(2\sin x)+B\sin(2\sin x)+\frac{1}{8}(3-2\sin^2 x) \qquad 5.18$$

5.4 Solve the differential equation with trigonometry coefficient

$$2\cos ecx\frac{d^2y}{dx^2}-\cos ecx\cot x\frac{dy}{dx}+25y=0 \qquad 5.19$$

SOLUTION

To check if the equation is reducible,
From (2.11),

$$f(x)=2\cos ecx$$

$$g(x)=-\cos ecx\cot x+b\sqrt{2\cos ecx}$$

Where $b=0$

Which satisfies $g(x)$ in (5.19) and hence (5.19) is reducible,
By (2.12), we reduce (5.19) by the substitution

$$z=\int\sqrt{\frac{\sin x}{2}}dx \qquad 5.20$$

$$-\cos ecx\cot x\frac{dy}{dx}=-\cot x\sqrt{\frac{\cos ecx}{2}}\frac{dy}{dz} \qquad 5.21$$

$$2\cos ecx\frac{d^2y}{dx^2}=-\cot x\sqrt{\frac{\cos ecx}{2}}\frac{dy}{dz}+\frac{d^2y}{dz^2} \qquad 5.22$$

Substituting (5.21) and (5.22) into (5.19) and simplify, we have

$$\frac{d^2y}{dz^2}+25y=0 \qquad 5.23$$

$$y=A\cos\left(\frac{5}{\sqrt{2}}\int\sqrt{\sin x}dx\right)+B\sin\left(\frac{5}{\sqrt{2}}\int\sqrt{\sin x}dx\right) \qquad 5.24$$

5.5 Solve the third order linear differential equation with algebraic coefficients

$$x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{1}{9x}\frac{dy}{dx}+y=0 \qquad 5.25$$

SOLUTION

$x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{1}{9x}\frac{dy}{dx}+y=0$ $\qquad f(x)=x,\ g(x)=1,\ p(x)=\frac{-1}{9x}$

If the equation is reducible then from (3.13) and (3.14)

$g(x)=1+b\sqrt[3]{x}$ and $p(x)=\frac{-1}{9x}+b\frac{1}{\sqrt[3]{x}}+c\sqrt[3]{x}$

Where $b=0$ and $a=0$

$g(x)=1,$ and $p(x)=\frac{-1}{9x}$

Implies (5.25) is reducible, using (3.15) to get the substitution

$$z=\frac{3}{2}x^{\frac{2}{3}} \qquad 5.26$$

By substituting (5.26) into (5.25) and simplify we have

$$\frac{d^3y}{dz^3}+y=0$$

$$y=Ae^{-z}+B\cos\frac{(1+\sqrt{3})z}{2}+C\sin\frac{(1+\sqrt{3})z}{2} \qquad 5.27$$

Put (5.26) into (5.27)

$$y=Ae^{-z}+B\cos\frac{3(1+\sqrt{3})x^{\frac{2}{3}}}{4}+C\sin\frac{3(1+\sqrt{3})x^{\frac{2}{3}}}{4} \qquad 5.28$$

5.6 Solve the fourth order differential equation with algebraic coefficient

$$x^2\frac{d^4y}{dx^4}+3x\frac{d^3y}{dx^3}+\frac{3}{4}\frac{d^2y}{dx^2}+\frac{45-70x}{16x^2}\frac{dy}{dx}-81y=0 \qquad 5.29$$

SOLUTION

Compare (5.29) and (4.0), we have

$f(x)=x^2$, $g(x)=3x$, $p(x)=\frac{3}{4}$, $h(x)=\frac{45-70x}{16x^2}$

If the equation is reducible then from (4.17), (4.18) and (4.19)

$g(x)=3x+b\sqrt[4]{x^3}$, $p(x)=\frac{3}{4}+b\frac{2x}{\sqrt{x}}+cx$

$$h(x)=-\frac{280}{64x}+\frac{45}{16x^2}-\frac{5}{4}x+b\left(4\frac{2}{\sqrt{x}}-\frac{4x^2}{\sqrt{x^5}}\right)+$$
$$2c+d\sqrt{x}$$

Where $b=0,\ c=0,\ d=0$

Implies (5.29) is reducible, using (4.20) to get the substitution

$$z=\sqrt[3]{x} \qquad 5.30$$

$$D^4-81=0$$

$$D=\pm 3,\ \pm i3$$

$$y=A\cosh 3z+B\sinh 3z+C\cos 3z+D\sin 3z$$

$$y=A\cosh 3\sqrt[3]{x}+B\sinh 3\sqrt[3]{x}+C\cos 3\sqrt[3]{x}+D\sin 3\sqrt[3]{x}$$

6.0 Derivation of the substitution for the Cauchy-Euler's homogeneous differential equation using the proposed method.

The Cauchy-Euler's homogeneous differential equation is of the form

$$x^2\frac{d^2y}{dx^2}+a_1x\frac{dy}{dx}+a_0y=0 \qquad 6.0$$

where $f(x)=x^2$ and $g(x)=a_1x$

(6.0) is reducible if

$$g(x)=\frac{1}{2}f'(x)+b\sqrt{f(x)}$$

$$g(x)=(1+b)x$$

Where $b=a_1-1$

The substitution is $z=\int\frac{1}{\sqrt{x^2}}dx$

$$z = \ln x \qquad 6.1$$

Substituting (6.1) into (6.0) we have,

$$\frac{d^2y}{dz^2} + (a_1 - 1)\frac{dy}{dz} + a_0 y = 0 \qquad 6.2$$

(6.1) is the substitution given by Cauchy-Euler to reduce (6.0) to a differential equation with constant coefficient as seen in (6.2).

7.0 Deriving the substitution for the Legendre's homogeneous differential equation using the proposed method.

The Legendre's homogeneous differential equation is of the form

$$(a_1x + c)^2 \frac{d^2y}{dx^2} + (a_1x + c)\frac{dy}{dx} + a_0 y = 0 \qquad 7.0$$

where $f(x) = (a_1x + c)^2$ and $g(x) = (a_1x + c)$

(7.0) is reducible if

$$g(x) = \frac{1}{2} f'(x) + b\sqrt{f(x)}$$

$$g(x) = (a_1 + b)(a_1x + c)$$

Where $b = 1 - a_1$

The substitution is $z = \int \frac{1}{\sqrt{(a_1x + c)^2}} dx$

$$z = \ln(a_1x + c) \qquad 7.1$$

Substituting (7.1) into (7.0) we have,

$$\frac{d^2y}{dz^2} + (1 - a_1)\frac{dy}{dz} + a_0 y = 0 \qquad 7.2$$

(7.1) is the substitution proposed by Legendre to reduce (7.0) to a differential equation with constant coefficient as seen in (7.2).

8.0 Conclusion

The proposed method has able to reduce linear ordinary differential equation with function coefficients to linear ordinary differential equation with constant coefficient as illustrated in section three. Section 3, ordinary differential equation with algebraic, logarithmic and trigonometry coefficients were treated.

The Cauchy-Euler's and Legendre substitution for reducing equations of specific form were derived using our proposed method. The derived substitution is the same as the ones proposed by Cauchy-Euler and Legendre to solve their form of equations as seen sections four and five.

The proposed method can be use to solve problems that even the popular Frobenius method will be unable to solve.

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