DESCENT ON Γ_1 NON-DERANGED PERMUTATION GROUP

by

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Abstract

Some further theoretic properties of the scheme called Γ_1 non-deranged permutation Group, especially in relation to Descent were identified and studied in this paper. This was done by first computations on this scheme using prime numbers $p \geq 5$. A recursion formula for generating the Descent number, union of Descent set, Descent bottom sum and Descent top sum was developed and used these numbers to identify theoretic consequences.

Keywords: *Descent Numbers; Descent set; Γ_1 non-deranged permutation Group.*

1 Introduction

A Permutation f of the Γ_1-non deranged permutation group presents as descent in i whenever $f(i) > f(i+1)$. Permutation statistics were first introduced by Euler(1913) and then extensively studied by MacMahon (1915).in the last decades much progress has made, both in the discovery and the study of new statistics, and in extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group S_n. Garba and Ibrahim (2010) used the concept to develop a scheme for prime numbers $P \leq 5$ and $\Omega \subseteq N$ which generate the cycles of permutations (derangements) using $\omega_i = ((1)(1+i)_{mp}(1+2i)_{mp}...(1+(p-1)i)_{mp})$ to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non derangement).The construction of the generated set of permutations from the work of Garba and Ibrahim (2010) as a permutation group was done by Usman and Ibrahim (2011). hey achieved this by embedding an identity element into the generated set of permutation(strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as G_p) there is no doubt, the patterns in permutations have been well studied for over a century. it seem to be the case, these patterns were studied on permutations arbitrary. he symmetric group S_n is the set of all permutations of a set Γ of cardinality n. There are several types of other smaller permutation groups (subgroup of S_n) of set Γ,a notable one among them is the alternating group A_n. On the other hand, Γ_1-non deranged permutations has been established as a group by Aremu *et al.* (2016), n their work, they studied the abstract theoretical properties of this Γ_1-non deranged permutation group and further established that the permutation group $G_P{}^{\Gamma_1}$ is a subgroup of S_p (p is a prime number),Afterwards, Ibrahim *et al.*(2016) studied the representation of Γ_1-non deranged permutation group $G_P{}^{\Gamma_1}$ via group character, hence established that the character of every $\omega_i \in G_P{}^{\Gamma_1}$ is never zero. Also the non standard Young tableaux of Γ_1-non deranged permutation group $G_P{}^{\Gamma_1}$ has been studied by Garba *et al*.(2017), they established that the Young tableaux of this permutation group is non standard. Aremu *et al.*(2017a) studied pattern popularity in Γ_1-non deranged permutations they establish algebraically that pattern τ_1 is the most popular and pattern τ_3, τ_4 *and* τ_5 are equipopular in $G_P{}^{\Gamma_1}$ they further provided efficient algorithms and some results on popularity of patterns of length-3 in $G_P{}^{\Gamma_1}$.Aremu *et al.*(2017b) studied Fuzzy on Γ_1-non deranged permutation group $G_P{}^{\Gamma_1}$ and discover that, it is a one sided fuzzy ideal (only right fuzzy but not left) also the $\alpha-$level cut of f coincides with $G_P{}^{\Gamma_1}$ if $\alpha = \frac{1}{p}$ Recently, Ibrahim *et al.*(2017) studied Ascent on Γ_1-non deranged permutation group $G_P{}^{\Gamma_1}$ and discover that, the union of Ascent of all Γ_1-non derangement is equal to identity also observed at the difference between $Asc(\omega_i)$ and $Dsc(\omega_{p-1})$ is one. Hence we will in this paper developed a recursion formula for generating the Descent number, union of descent set and intersection of descent set and used these numbers to identify theoretic consequences.

2.0 PRELIMINARIES

Definition 2.1

Let Γ be a non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subset \square$ bijection ω on Γ of the form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & . & . & . & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & & & & (1+(p-1)i)_{mp} \end{pmatrix}$$

is called a Γ_1-non deranged permutation. We denoted G_p to be the set of all Γ_1-non deranged permutations.

Definition 2.2

The pair G_p and the natural permutation composition forms a group which is denoted as $G_P{}^{\Gamma_1}$.This is a special permutation group which fixes the first element of Γ.

Definition 2.3

A descent of permutation $f = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ 1 & f(1) & f(2) & . & . & . & f(n) \end{pmatrix}$ is any positive $i > n$ (where i and n are positive integers) where the current value is less than the next, that is i is an descent of a permutation $f(i) > f(i+1)$.The descent set of f, denoted as $Des(f)$, is given by $Des(f) = \{i : f(i) > f(i+1)\}$ the descent number of f, denoted as $des(f)$, is defined as the number of descent and is given by $des(f) = |Des(f)|$.

Lemma 2.4

It follows from the properties of integer modulo that for $n \in \square$ an integer $n(\bmod)n = 0$ also implies $n = 0$.

Proof

From definition of integer modulo, we know that

$n(\bmod)n = 0$ (1)

It also follows from property of additive identity of integers hat

$0(\bmod)n = 0$ (2)

Now, equation (1) implies $\frac{n}{n} = 1+0$

equation (2) implies $\frac{0}{n} = 0+0$

That is $\frac{n}{n}$

Is modulo equivalent to $\frac{0}{n}$

That is $\frac{n}{n} \equiv \frac{0}{n} \equiv 0$

where $\equiv$ represents a notation modulo equivalence.

It follows $\equiv$ is an equivalence relation as we shall establish in the following claim.

Claim

If $\frac{n}{n} \equiv \frac{0}{n}$

Then $\equiv$ is an equivalence relation.

Reflexive

$\frac{n}{n} \equiv \frac{n}{n}$

By definition i.e uses $1+0 \equiv 1+0$

If $\frac{n}{n} \equiv \frac{0}{n}$

Then using modulo $\frac{n}{n} = 1+0$

And $\frac{0}{n} = 0+0$

In any case both $\frac{n}{n}$ and $\frac{0}{n} \equiv 0$

i.e $\frac{0}{n} \equiv \frac{n}{n}$

Transitive

If $\frac{n}{n} \equiv \frac{0}{n}$

And $\frac{0}{n} \equiv \frac{m}{n}$

It implies that $\frac{n}{n} \equiv \frac{m}{n}$

$$\frac{n}{n} \equiv \frac{0}{n}$$

It implies that $\frac{0}{n} = 0$

And $\frac{0}{n} \equiv \frac{m}{n}$

It implies that $\frac{m}{n} = 0$

It follows $\frac{n}{n} = 0$

From the claim it follows $n = 0$ whenever n is the integer modulo.
For the remainder in this paper we shall adopt the notation established in lemma 2.4 that $n = 0$ for all n representing the modulus of sub sequences in given permutation

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & . & . & . & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & & & & (1+(p-1)i)_{mp} \end{pmatrix}$$

In particular we shall let $n = p$ for some prime integer $p \geq 5$.

3.0 MAIN RESULTS

In this section, we present some descent results of subgroup $G_P{}^{\Gamma_1}$ of S_p (Symmetry group of prime order with $p \geq 5$).

Theorem 3.1

Let $G_P{}^{\Gamma_1}$ be a Γ_1-non derangement permutations, The total number of descent is

$$des(G_p{}^{\Gamma_1}) = \frac{1}{2}(p-1)(p-2)$$

Proof

The order of $G_P{}^{\Gamma_1}$ and the number of positions with ascent or descent are both $p-1$.We know that total number of ascent is greater than the total number of descent by $p-1$ in $G_P{}^{\Gamma_1}$ from the proposition that says $asc(G_p{}^{\Gamma_1}) - des(G_p{}^{\Gamma_1}) = p-1$ and from the theorem that say $asc(G_p{}^{\Gamma_1}) = \frac{1}{2}p(p-1)$ .Then

$$\begin{aligned} des(G_p{}^{\Gamma_1}) &= \frac{1}{2}p(p-1) - (p-1) \\ &= \frac{1}{2}(p-1)(p-2) \end{aligned}$$

Proposition 3.2

If the ascent set of a Γ_1-non derangement ω_i is $X \subset \Gamma$, then the descent set of $\omega_{p-1} = X - \{1\}$

Proof

If $\theta_i, \theta_j \in S_p$ (where $i \neq j$ and $i, j \in \square$) then the ascent of θ_j is equal to the descent set of θ_j .By the non derangement property of $G_P^{\Gamma_1}$ .1 is not contained in the descent set of any arbitrary $\omega_i \in G_p^{\Gamma_1}$ and so there exist a set $X - \{1\}$ that will be descent set of an arbitrary Γ_1-non derangement ω_{p-1}

Lemma 3.3

Suppose that $G_P^{\Gamma_1}$ is Γ_1-non deranged permutations. Then

$$Des(\omega_i) \cap Des(\omega_{p-i}) = \varnothing$$

Proof.

Suppose $\omega_i = a_1a_2...a_{p-1}a_p$ and $\omega_{p-i} = a_1a_pa_{p-1}...a_3a_2$. By restricting a_1 since it is the least of all functions and it is at the first position of ω_i and ω_{p-i} , that is, it has no effect on the descent, so we have that

$$Des(\omega_i) = Asc(\omega_{p-i}), \qquad (1)$$

And

$$Asc(\omega_i) = Des(\omega_{p-i}). \qquad (2)$$

It is obvious from the definition of descent ascent that

$$Des(\omega_i) \cap Asc(\omega_i) = \varnothing \qquad (3)$$

Substituting (2) into (3) we have

$$Des(\omega_i) \cap Des(\omega_{p-i}) = \varnothing$$

Lemma 3.4

Suppose that $G_P^{\Gamma_1}$ is Γ_1-non deranged permutations. Then

$$Des(\omega_i) \cup Des(\omega_{p-i}) = Des(\omega_{p-1})$$

Proof.

Given $\omega_i = a_1a_2...a_{p-1}a_p$, then $\omega_{p-i} = a_1a_pa_{p-1}...a_3a_2$ and $\omega_{p-1} = 1p(p-1)...2$,By restricting a_1 because it has no effect on the descent since it is the least of all functions and it is at the first position in ω_i and ω_{p-i} .Since ω_{p-1} is a strictly decreasing sequence when a_1 is restricted.

$$\begin{aligned} Des(\omega_{p-1}) &= \{i : f(i) > f(i+1)\} \cup \{i : f(i) < f(i+1)\} \\ &= Des(\omega_i) \cup Asc(\omega_i) \\ &= Des(\omega_i) \cup Des(\omega_{p-i}) \end{aligned}$$

By this we can see that $Des(\omega_i) \cup Des(\omega_{p-i}) = Des(\omega_{p-1})$

Corollary 3.5

Suppose that $G_p^{\Gamma_1}$ is Γ_1-non deranged permutations. Then $des(\omega_i) + des(\omega_{p-i}) = des(\omega_{p-1})$

Proof

Since by lemma 3.4 $Des(\omega_i)\cup Des(\omega_{p-i})=Des(\omega_{p-1})$ and by lemma 3.3 $Des(\omega_i)\cap Des(\omega_{p-i})=\varnothing$, the result follows

Theorem 3.6

Let ${G_P}^{\Gamma_1}$ be a Γ_1-non derangement permutations then $\bigcup_{i=1}^{p-1} Des(\omega_i)=Des(\omega_{p-1})$

Proof

Since $\omega_{p-1}=1p(p-1)...2$ so it takes all possible descent in ω_i, but by lemma 3.4 $Des(\omega_i)\cup Des(\omega_{p-i})=Des(\omega_{p-1})$, we want to show that for any ${G_P}^{\Gamma_1}$ there exist ω_i and ω_{p-i} where $i\neq 1$. Since $P\geq 5$ and any ${G_P}^{\Gamma_1}$ consist of non-deranged permutations $\{\omega_1,\omega_2,...\omega_{p-1}\}$, from this at least ω_1 and ω_{p-1}

Theorem 3.7

Let ${G_P}^{\Gamma_1}$ be a Γ_1-non derangement permutations then $\bigcap_{i=1}^{p-1} Des(\omega_i)=\varnothing$

Proof

From lemma 3.3 $Des(\omega_i)\cap Des(\omega_{p-i})=\varnothing$,we want to show that for any ${G_P}^{\Gamma_1}$ there exist ω_i and ω_{p-i} since $P\geq 5$ and any ${G_P}^{\Gamma_1}$ consist of non-deranged permutations $\{\omega_1,\omega_2,...\omega_{p-1}\}$, does not have any element in common the result follows

□

4.0 CONCLUSION

This paper has provided very useful theoretical properties of this scheme called Γ_1-non deranged permutations in relation to the Descent. We have shown that the intersection of Descent set of all Γ_1-non derangement is empty, we also observed that the descent number is strictly less than Ascent number by $P-1$.

5.0 RECOMMENDATION

Further researches should be conducted on Γ_1-non deranged permutations in relation to the other permutation statistics such as Excedance, Inversion, Major index, Rise. In order find new algebraic and combinatorial results.

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