AN INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEM WITH TIME DEPENDENT QUADRATIC DEMAND AND LINEAR HOLDING COST UNDER TRADE CREDIT POLICY

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Abstract
In this article, an inventory model for non-instantaneous deteriorating item with two components demand and linear time holding cost under trade credit has been developed. The demand rate is assumed to be time dependent quadratic before deterioration sets in after which it is considered as constant. Optimal cycle length and order quantity are determined so as to minimise total variable cost. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions are provided. Numerical examples are given to demonstrate the application of the model. Finally, sensitivity analysis of some model parameters on the decision variables have been carried out and the implications are discussed. In the discussions, suggestions toward minimizing the total variable cost of the inventory system are given.

Keywords: *Economic Order Quantity, Non- instantaneous deteriorating item, Quadratic demand, Linear Holding, Trade Credit Policy.*

1. Introduction
In developing inventory models, two factors of deterioration of items and variation in the demand rate with time have been of growing interest to the researchers in the inventory modeling. Harris (1915) developed the classical Economic Order Quantity formula, in which the demand of items was assumed to be constant over time. But in the real life situations, demand rate of items is not always constant. Silver and Meal (1969) were the first to modify a simple classical square root formula for time-varying demand. Later, Silver and Meal (1973) developed a heuristic approach to determine EOQ in the general case of a time varying-demand pattern. Many inventory models with demand that change with respect to time have been developed in this direction. Some of these articles are developed by Dave and Patel (1981), Goswami and Chaudhuri (1991), Chang and Dye (1999), Khanra and Chaudhuri (2003), Ghosh and Chaudhuri (2006), Khanra *et al.* (2011), Sarkar*et al*. (2012), Mishra (2016), and so on.

The traditional EOQ models consider that the retailers should pay the purchasing cost for the items as soon as the items are received. In real practice, a supplier/wholesaler offers the retailera delay period in paying for purchasing cost, known as trade credit period. Retailer can accumulate revenues by selling items and by earning interests. The concept of trade credit was first introduced by Haley and Higgins (1973), while Goyal (1985) was the first to consider EOQ model under the conditions of permissible delay in payments. Some notable articles in this direction are those developed by Aggarwal and Jaggi (1995), Ouyang *et al.* (2006), Liao (2007), Sana and Chaudhuri (2008), Jaggi *et al.* (2008), Min et al. (2010), Babangida and Baraya (2018), and so on.

Most of the researches on inventory models consider holding cost to be constant. But in the real life situations, holding cost of any item may always be in a dynamic state. Roy (2008) developed an inventory model for deteriorating items with price dependent demand and time-varying holding cost. Baraya and Sani (2011) considered an economic production quantity (EPQ) model for delayed deteriorating items with stock-dependent demand rate and linear time dependent holding cost. Mishra and Singh (2011), Musa and Sani (2012) and Dutta and Kumar (2015) consider holding cost to be time-varying in their respective models. In this present model, an effort has been made to extend the work of Babangida and Baraya (2018) by considering time varying holding cost. The analytical solution of the model is obtained and the solution is illustrated with the help of numerical examples. Finally, sensitivity analysis is carried out to show the effect of changes in some model parameters on decision variables. This is followed by discussions and conclusion.

2. Model Description and Formulation

This section describes the proposed model notation, assumptions and formulation.

2.1 Notation and assumptions

The inventory system is developed based on the following assumptions and notation.

Notation:

C_0 The ordering cost per order.
C_p The purchasing cost per unitper unit time ($/unit/ year).
C_s The selling price per unit per unit time ($/unit/ year).
I_c The interest charged in stock by the supplier per Dollar per year ($/unit/year)($I_c \geq I_e$).
I_e The interest earned per Dollar per year ($/unit/year).
M The trade credit period (in year) for settling accounts.
ω The constant deterioration rates function($0 < \omega < 1$).
μ The length of time in which the product exhibits no deterioration.
T The length of the replenishment cycle time (time unit).
I_0 The number of items received at the beginning of the inventory system (units).
$I_1(t)$ The inventory level before deterioration sets in.
$I_2(t)$ The inventory level after deterioration begins.

Assumptions:

1. The replenishment rate is infinite.
2. The lead time is zero.
3. A single non-instantaneous deteriorating item is considered.
4. During the fixed period, μ, there is no deterioration and at the end of this period, the inventory item deteriorates at the constant rate.
5. There is no replacement or repair for deteriorated items.
6. Demand rate before deterioration begins is a quadratic function of time t and is given by $\alpha + \beta t + \gamma t^2$where $\alpha \geq 0,\ \beta \neq 0, \gamma \neq 0$.
7. Demand rate after deterioration sets in is assumed to be constant and is given byλ.
8. Holding cost $C_1(t)$per unit time is linear time dependent and is assumed to be $C_1(t) = h_1 + h_2 t$where$h_1 > 0$ and $h_2 > 0$.

9. During the trade credit period M $(0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.
10. Shortages are not allowed.

2.2 Formulation of the model

The inventory system is developed as follows. I_0 units of items of a single product from the manufacturer arrive at the inventory system at the beginning of each cycle (i.e., at time $t = 0$). During the time interval $[0, \mu]$, the inventory level $I_1(t)$ is depleting gradually due to market demand only and it is assumed to be a quadratic function of time t. At time interval $[\mu, T]$ the inventory level $I_2(t)$ is depleting due to combined effects of demand and deterioration and the demand rate at this time is reduced to a constant λ. At time $t = T$, the inventory level depletes to zero. The behaviour of the inventory system is described in figure 1 and 2 below.

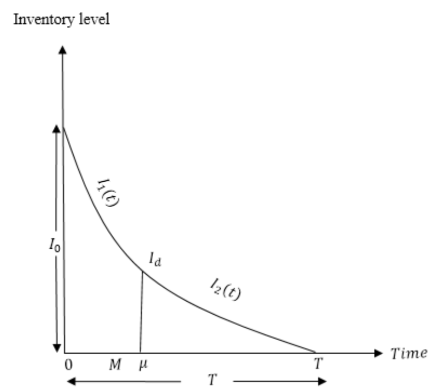

Figure 1: Inventory situation for the case $(0 < M \leq \mu)$.

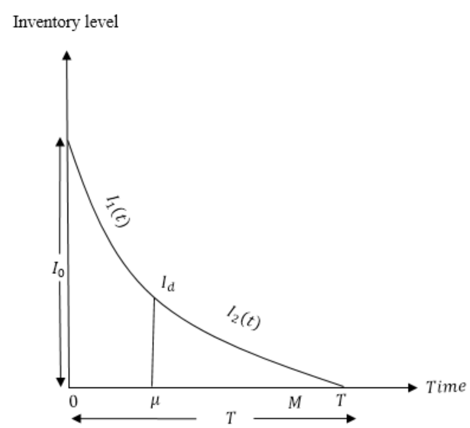

Figure 2: Inventory situation for the case $(\mu < M \leq T)$.

Based on the above description, the inventory level $I_1(t)$ at time $t \in [0, \mu]$ is given by

$$\frac{dI_1(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad 0 \leq t \leq \mu \qquad (1)$$

with boundary conditions $I_1(0) = I_0$and$I_1(\mu) = I_d$, the solution of equation (1) is as follows

$$I_1(t) = -\int(\alpha + \beta t + \gamma t^2)dt$$

$$= -\left(\alpha t + \beta\frac{t^2}{2} + \gamma\frac{t^3}{3}\right) + d_1 \quad (2)$$

where$d_1$is a constant of integration, from which we have

Applying the condition $I_1(0) = I_0$into equation (2), we obtain

$$d_1 = I_0 \quad (3)$$

Substitutingequation (3) intoequation (2), we obtain

$$I_1(t) = -\left(\alpha t + \beta\frac{t^2}{2} + \gamma\frac{t^3}{3}\right) + I_0 \quad (4)$$

Also applying the condition$I_1(\mu) = I_d$ at$t = \mu$ into equation (4), we obtain

$$I_d = -\left(\alpha\mu + \beta\frac{\mu^2}{2} + \gamma\frac{\mu^3}{3}\right) + I_0$$

$$I_0 = I_d + \left(\alpha\mu + \beta\frac{\mu^2}{2} + \gamma\frac{\mu^3}{3}\right) \quad (5)$$

Substituting equation (5) intoequation (4), we obtain

$$I_1(t) = -\left(\alpha t + \beta\frac{t^2}{2} + \gamma\frac{t^3}{3}\right) + I_d + \left(\alpha\mu + \beta\frac{\mu^2}{2} + \gamma\frac{\mu^3}{3}\right)$$

$$= \alpha(\mu - t) + \frac{\beta}{2}(\mu^2 - t^2) + \frac{\gamma}{3}(\mu^3 - t^3) + I_d \qquad 0 \le t \le \mu \quad (6)$$

During the second interval $[\mu,\ T]$, the inventory level decreases due to combined effects of demand and deterioration and is governed by the differential equation below

$$\frac{dI_2(t)}{dt} + \omega I_2(t) = -\lambda, \mu \le t \le T \quad (7)$$

Equation (7) is a first order linear differential equation whose integrating factor is $e^{\boldsymbol{\omega} t}$ and is reduces to

$$I_2(t)e^{\boldsymbol{\omega} t} = -\int e^{\boldsymbol{\omega} t}\lambda dt$$

$$= -\int e^{\boldsymbol{\omega} t}dt$$

$$= -\frac{\lambda}{\boldsymbol{\omega}}e^{\boldsymbol{\omega} t} + d_2$$

where$d_2$is a constant of integration, from which we have

$$I_2(t) = -\frac{\lambda}{\boldsymbol{\omega}} + d_2 e^{-\boldsymbol{\omega} t} \quad (8)$$

Applying the condition$I_2(T) = 0$ at $t = T$ into equation (8), we obtain

$$0 = -\frac{\lambda}{\boldsymbol{\omega}} + d_2 e^{-\boldsymbol{\omega} T}$$

$$d_2 = \frac{\lambda}{\boldsymbol{\omega}}e^{\boldsymbol{\omega} T} \quad (9)$$

Substituting equation (9) into equation (8), we obtain

$$I_2(t) = -\frac{\lambda}{\boldsymbol{\omega}} + \frac{\lambda}{\boldsymbol{\omega}}e^{\boldsymbol{\omega} T}e^{-\boldsymbol{\omega} t}$$

$$= \frac{\lambda}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-t)} - 1\right), \qquad \mu \leq t \leq T \qquad (10)$$

Also applying the condition $I_2(\mu) = I_d$ at $t = \mu$ into equation(10), we obtain

$$I_d = \frac{\lambda}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)} - 1\right) \qquad (11)$$

Substituting equation (11) into equations (5) and(6) respectively, we obtain

$$I_0 = \frac{\lambda}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)} - 1\right) + \left(\alpha\mu + \beta\frac{\mu^2}{2} + \gamma\frac{\mu^3}{3}\right) \qquad (12)$$

and

$$I_1(t) = \frac{\lambda}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)} - 1\right) + \alpha(\mu - t) + \frac{\beta}{2}(\mu^2 - t^2) + \frac{\gamma}{3}(\mu^3 - t^3) \qquad 0 \leq t \leq \mu \qquad (13)$$

(i) The total demand during the period $[\mu, T]$ is given by

$$D_i = \int_{\mu}^{T} \lambda \, dt$$

$$= \lambda(T - \mu) \qquad (14)$$

(ii) The total number of deteriorated items per cycle is given by

$N_i = I_d - D_i$

Substituting I_d and D_i fromequations (11) and (14) respectively intoN_i, we obtain

$$N_i = \frac{\lambda}{\omega}\left(e^{\omega(T-\mu)} - 1\right) - \lambda(T - \mu)$$

$$= \frac{\lambda}{\omega}\left[e^{\omega(T-\mu)} - 1 - \omega(T - \mu)\right] \qquad (15)$$

(iii) The deterioration cost is given by

$$D_C = C_p\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)} - 1 - \omega(T - \mu)\right)\right] \qquad (16)$$

(iv) The ordering cost per order is given by C_0

(v) The inventory holding cost during the period [0, T] is given by

$$C_H = \int_0^{\mu}(h_1 + h_2 t)I_1(t)dt + \int_{\mu}^{T}(h_1 + h_2 t)I_2(t)dt \qquad (17)$$

Substituting equations(10)and(13) into equation(17), we obtain

$$C_H = \int_0^{\mu}(h_1 + h_2 t)\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)} - 1\right) + \alpha(\mu - t) + \frac{\beta}{2}(\mu^2 - t^2) + \frac{\gamma}{3}(\mu^3 - t^3)\right]dt$$

$$+ \int_{\mu}^{T}(h_1 + h_2 t)\left[\frac{\lambda}{\omega}\left(e^{\omega(T-t)} - 1\right)\right]dt$$

$$= h_1\int_0^{\mu}\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)} - 1\right) + \alpha(\mu - t) + \frac{\beta}{2}(\mu^2 - t^2) + \frac{\gamma}{3}(\mu^3 - t^3)\right]dt$$

$$+h_2\int_0^{\mu} t\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)}-1\right)+\alpha(\mu-t)+\frac{\beta}{2}(\mu^2-t^2)+\frac{\gamma}{3}(\mu^3-t^3)\right]dt$$
$$+h_1\int_{\mu}^{T}\left[\frac{\lambda}{\omega}\left(e^{\omega(T-t)}-1\right)\right]dt+h_2\int_{\mu}^{T} t\left[\frac{\lambda}{\omega}\left(e^{\omega(T-t)}-1\right)\right]dt$$

$$=h_1\int_0^{\mu}\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)}-1\right)+\alpha(\mu-t)+\frac{\beta}{2}(\mu^2-t^2)+\frac{\gamma}{3}(\mu^3-t^3)\right]dt$$
$$+h_2\int_0^{\mu}\left[\frac{\lambda t}{\omega}\left(e^{\omega(T-\mu)}-1\right)+\alpha(\mu t-t^2)+\frac{\beta}{2}(t\mu^2-t^3)+\frac{\gamma}{3}(t\mu^3-t^4)\right]dt$$
$$+h_1\int_{\mu}^{T}\left[\frac{\lambda}{\omega}\left(e^{\omega(T-t)}-1\right)\right]dt+h_2\int_{\mu}^{T}\left[\frac{\lambda t}{\omega}e^{\omega(T-t)}-\frac{\lambda t}{\omega}\right]dt$$

$$=h_1\left[\frac{\lambda t}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)}-1\right)+\alpha\left(\mu t-\frac{t^2}{2}\right)+\frac{\beta}{2}\left(t\mu^2-\frac{t^3}{3}\right)+\frac{\gamma}{3}\left(t\mu^3-\frac{t^4}{4}\right)\right]_0^{\mu}$$
$$+h_2\left[\frac{\lambda t^2}{2\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)}-1\right)+\alpha\left(\frac{\mu t^2}{2}-\frac{t^3}{3}\right)+\frac{\beta}{2}\left(\frac{\mu^2t^2}{2}-\frac{t^4}{4}\right)+\frac{\gamma}{3}\left(\frac{\mu^3t^2}{2}-\frac{t^5}{5}\right)\right]_0^{\mu}$$
$$+h_1\left[\frac{\lambda}{\boldsymbol{\omega}}\left(-\frac{e^{\boldsymbol{\omega}(T-t)}}{\boldsymbol{\omega}}-t\right)\right]_{\mu}^{T}+h_2\left[-\frac{\lambda t}{\boldsymbol{\omega}^2}e^{\boldsymbol{\omega}(T-t)}+\frac{\lambda}{\boldsymbol{\omega}^2}\int e^{\boldsymbol{\omega}(T-t)}dt-\frac{\lambda t^2}{2\boldsymbol{\omega}}\right]_{\mu}^{T}$$

$$=h_1\left[\frac{\lambda t}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)}-1\right)+\alpha\left(\mu t-\frac{t^2}{2}\right)+\frac{\beta}{2}\left(t\mu^2-\frac{t^3}{3}\right)+\frac{\gamma}{3}\left(t\mu^3-\frac{t^4}{4}\right)\right]_0^{\mu}$$
$$+h_2\left[\frac{\lambda t^2}{2\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)}-1\right)+\alpha\left(\frac{\mu t^2}{2}-\frac{t^3}{3}\right)+\frac{\beta}{2}\left(\frac{\mu^2t^2}{2}-\frac{t^4}{4}\right)+\frac{\gamma}{3}\left(\frac{\mu^3t^2}{2}-\frac{t^5}{5}\right)\right]_0^{\mu}$$
$$+h_1\left[\frac{\lambda}{\boldsymbol{\omega}}\left(-\frac{e^{\boldsymbol{\omega}(T-t)}}{\boldsymbol{\omega}}-t\right)\right]_{\mu}^{T}+h_2\left[-\frac{\lambda t}{\boldsymbol{\omega}^2}e^{\boldsymbol{\omega}(T-t)}-\frac{\lambda}{\boldsymbol{\omega}^3}e^{\boldsymbol{\omega}(T-t)}-\frac{\lambda t^2}{2\boldsymbol{\omega}}\right]_{\mu}^{T}$$

$$=h_1\left[\frac{\lambda\mu}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)}-1\right)+\alpha\left(\mu^2-\frac{\mu^2}{2}\right)+\frac{\beta}{2}\left(\mu^3-\frac{\mu^3}{3}\right)+\frac{\gamma}{3}\left(\mu^4-\frac{\mu^4}{4}\right)\right]$$
$$+h_2\left[\frac{\lambda\mu^2}{2\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)}-1\right)+\alpha\left(\frac{\mu^3}{2}-\frac{\mu^3}{3}\right)+\frac{\beta}{2}\left(\frac{\mu^4}{2}-\frac{\mu^4}{4}\right)+\frac{\gamma}{3}\left(\frac{\mu^5}{2}-\frac{\mu^5}{5}\right)\right]$$
$$+h_1\left[\frac{\lambda}{\boldsymbol{\omega}}\left(-\frac{1}{\boldsymbol{\omega}}+\frac{e^{\boldsymbol{\omega}(T-\mu)}}{\boldsymbol{\omega}}-(T-\mu)\right)\right]$$
$$+h_2\left[-\frac{\lambda T}{\boldsymbol{\omega}^2}+\frac{\lambda\mu}{\boldsymbol{\omega}^2}e^{\boldsymbol{\omega}(T-\mu)}-\frac{\lambda}{\boldsymbol{\omega}^3}+\frac{\lambda}{\boldsymbol{\omega}^3}e^{\boldsymbol{\omega}(T-\mu)}-\frac{\lambda(T^2-\mu^2)}{2\boldsymbol{\omega}}\right]$$

$$=h_1\left[\frac{\lambda\mu}{\omega}\left(e^{\omega(T-\mu)}-1\right)+\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4+\frac{\lambda}{\omega^2}\left(e^{\omega(T-\mu)}-1-\omega(T-\mu)\right)\right]$$
$$+h_2\left[\frac{\lambda\mu^2}{2\omega}\left(e^{\omega(T-\mu)}-1\right)+\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu}{\omega^2}e^{\omega(T-\mu)}-\frac{\lambda T}{\omega^2}-\frac{\lambda}{\omega^3}+\frac{\lambda}{\omega^3}e^{\omega(T-\mu)}\right.$$
$$\left.-\frac{\lambda(T^2-\mu^2)}{2\omega}\right]$$

$$= h_1\left[\frac{\lambda\mu}{\omega}e^{\omega(T-\mu)} - \frac{\lambda\mu}{\omega} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 + \frac{\lambda}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda}{\omega^2} - \frac{\lambda T}{\omega} + \frac{\lambda\mu}{\omega}\right]$$
$$+h_2\left[\frac{\lambda\mu^2}{2\omega}e^{\omega(T-\mu)} - \frac{\lambda\mu^2}{2\omega} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda T}{\omega^2} - \frac{\lambda}{\omega^3} + \frac{\lambda}{\omega^3}e^{\omega(T-\mu)} - \frac{\lambda T^2}{2\omega} + \frac{\lambda\mu^2}{2\omega}\right]$$

$$= h_1\left(\frac{\lambda\mu}{\omega}e^{\omega(T-\mu)} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 + \frac{\lambda}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda}{\omega^2} - \frac{\lambda T}{\omega}\right)$$
$$+ h_2\left(\frac{\lambda\mu^2}{2\omega}e^{\omega(T-\mu)} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda T}{\omega^2} - \frac{\lambda}{\omega^3} + \frac{\lambda}{\omega^3}e^{\omega(T-\mu)} - \frac{\lambda T^2}{2\omega}\right) \quad (18)$$

(vi) The Interest Payable

This is the interest charged for the inventory not being sold after the expiration of trade credit period which are categorised into Case 1$(0 < M \le \mu)$and Case 2$(\mu < M \le T)$.

Case 1: $(\mathbf{0} < M \le \mu)$

This is the period before deterioration sets in, and payment for goods is settled with the capital opportunity cost rate I_c for the items in stock. Thus, the interest payable is given by

$$I_{P1} = C_p I_c\left[\int_M^\mu I_1(t)dt + \int_\mu^T I_2(t)dt\right]$$
$$= C_p I_c\left[\int_M^\mu \frac{\lambda}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)} - 1\right) + \alpha(\mu - t) + \frac{\beta}{2}(\mu^2 - t^2) + \frac{\gamma}{3}(\mu^3 - t^3)dt + \int_\mu^T \frac{\lambda}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-t)} - 1\right)dt\right]$$
$$= C_p I_c\left\{\left[\frac{\lambda t}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)} - 1\right) + \alpha\left(t\mu - \frac{t^2}{2}\right) + \frac{\beta}{2}\left(t\mu^2 - \frac{t^3}{3}\right) + \frac{\gamma}{3}\left(t\mu^3 - \frac{t^4}{4}\right)\right]_M^\mu + \left[\frac{\lambda}{\boldsymbol{\omega}}\left(-\frac{e^{\boldsymbol{\omega}(T-t)}}{\boldsymbol{\omega}} - t\right)\right]_\mu^T\right\}$$
$$= C_p I_c\left[\frac{\lambda(\mu - M)}{\boldsymbol{\omega}}\left(e^{\boldsymbol{\omega}(T-\mu)} - 1\right) + \alpha\left(\mu^2 - \frac{\mu^2}{2} - \mu M + \frac{M^2}{2}\right) + \frac{\beta}{2}\left(\mu^3 - \frac{\mu^3}{3} - \mu^2 M + \frac{M^3}{3}\right) + \frac{\gamma}{3}\left(\mu^4 - \frac{\mu^4}{4} - \mu^3 M + \frac{M^4}{4}\right) + \frac{\lambda}{\boldsymbol{\omega}}\left(-\frac{1}{\boldsymbol{\omega}} + \frac{e^{\boldsymbol{\omega}(T-\mu)}}{\boldsymbol{\omega}} - (T - \mu)\right)\right]$$
$$= C_p I_c\left[\frac{\lambda(\mu - M)}{\omega}\left(e^{\omega(T-\mu)} - 1\right) + \frac{\alpha}{2}(\mu^2 - 2\mu M + M^2) + \frac{\beta}{6}(2\mu^3 - 3\mu^2 M + M^3) + \frac{\gamma}{12}(3\mu^4 - 4\mu^3 M + M^4) + \frac{\lambda}{\omega^2}\left(e^{\omega(T-\mu)} - 1 - \omega(T - \mu)\right)\right]$$

$$= C_p I_c \left[\frac{\lambda(\mu - M)}{\omega} \left(e^{\omega(T-\mu)} - 1\right) + \frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 + \frac{\lambda}{\omega^2}\left(e^{\omega(T-\mu)} - 1 - \omega(T - \mu)\right) \right] \quad (19)$$

Case 2: $(\boldsymbol{\mu} < M \leq T)$
This is when the end point of credit period is greater than the period with no deterioration but shorter than or equal to the length of period with positive inventory. The interest payable is

$$I_{P2} = C_p I_c \left[\int_M^T I_2(t)dt \right]$$
$$= C_p I_c \left[\int_M^T \frac{\lambda}{\omega}\left(e^{\omega(T-t)} - 1\right)dt \right]$$
$$= C_p I_c \left[\frac{\lambda}{\omega^2}\left(e^{\omega(T-M)} - 1 - \omega(T - M)\right) \right] \quad (20)$$

(vii) **The interest Earned**

It is assume that during the period when the account is not settled, the retailer sells the goods and continues to accumulate sales revenue and earns the interest with rate I_e. Therefore, the interest earned per cycle for two different cases are given below

Case 1: $(\mathbf{0} < M \leq \mu)$
In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although, the retailer has to settle the accounts at period$M$, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to μ. The interest earn is

$$I_{E1} = C_s I_e \left[\int_0^M (\alpha + \beta t + \gamma t^2) t dt \right]$$

$$= C_s I_e \left[\alpha \frac{t^2}{2} + \beta \frac{t^3}{3} + \gamma \frac{t^4}{4} \right]_0^M$$
$$= C_s I_e \left(\alpha \frac{M^2}{2} + \beta \frac{M^3}{3} + \gamma \frac{M^4}{4} \right) \quad (21)$$

Case 2: $(\boldsymbol{\mu} < M \leq T)$
In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although, the retailer has to settle the accounts at period M, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to T. The interest earn is

$$I_{E2} = C_s I_e \left[\int_0^\mu (\alpha + \beta t + \gamma t^2) t dt + \int_\mu^M \lambda t dt \right]$$

$$= C_s I_e \left[\left(\alpha \frac{t^2}{2} + \beta \frac{t^3}{3} + \gamma \frac{t^4}{4}\right)\right]_0^{\mu} + \left[\frac{\lambda t^2}{2}\right]_{\mu}^{M}$$

$$= C_s I_e \left[\left(\alpha \frac{\mu^2}{2} + \beta \frac{\mu^3}{3} + \gamma \frac{\mu^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda \mu^2}{2}\right] \quad (22)$$

(viii) The Average Total Variable Cost per Unit Time

The average total variable cost per unit time for case 1 ($0 < M \leq \mu$) is given by

$Z_1(T) = \frac{1}{T}${Ordering cost +inventory holding cost + deterioration cost + interest payable during the permissible delay period – interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{C_0 + h_1\left(\frac{\lambda\mu}{\omega}e^{\omega(T-\mu)} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 + \frac{\lambda}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda}{\omega^2} - \frac{\lambda T}{\omega}\right)$$
$$+ h_2\Bigg(\frac{\lambda\mu^2}{2\omega}e^{\omega(T-\mu)} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda T}{\omega^2} - \frac{\lambda}{\omega^3} + \frac{\lambda}{\omega^3}e^{\omega(T-\mu)}$$
$$- \frac{\lambda T^2}{2\omega}\Bigg) + C_p\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)} - 1 - \omega(T-\mu)\right)\right]$$
$$+ C_p I_c\Bigg[\frac{\lambda(\mu - M)}{\omega}\left(e^{\omega(T-\mu)} - 1\right) + \frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2$$
$$+ \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 + \frac{\lambda}{\omega^2}\left(e^{\omega(T-\mu)} - 1 - \omega(T-\mu)\right)\Bigg]$$
$$- C_s I_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\} \quad (23)$$

The average total variable cost per unit time for case 2($\mu < M \leq T$) is given by

$Z_2(T) = \frac{1}{T}${Ordering cost +inventory holding cost + deterioration cost + interest payable during the permissible delay period – interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{C_0 + h_1\left(\frac{\lambda\mu}{\omega}e^{\omega(T-\mu)} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 + \frac{\lambda}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda}{\omega^2} - \frac{\lambda T}{\omega}\right)$$
$$+ h_2\Bigg(\frac{\lambda\mu^2}{2\omega}e^{\omega(T-\mu)} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu}{\omega^2}e^{\omega(T-\mu)} - \frac{\lambda T}{\omega^2} - \frac{\lambda}{\omega^3} + \frac{\lambda}{\omega^3}e^{\omega(T-\mu)}$$
$$- \frac{\lambda T^2}{2\omega}\Bigg) + C_p\left[\frac{\lambda}{\omega}\left(e^{\omega(T-\mu)} - 1 - \omega(T-\mu)\right)\right]$$
$$+ C_p I_c\left[\frac{\lambda}{\omega^2}\left(e^{\omega(T-M)} - 1 - \omega(T-M)\right)\right]$$
$$- C_s I_e\left[\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right]\Bigg\} \quad (24)$$

The Average total cost per unit time for a replenishment cycle is given by

$$Z(T) = \begin{cases} Z_1(T) & 0 < M \leq \mu \\ Z_2(T) & \mu < M \leq T \end{cases} \quad (25)$$

Since

$0 < \omega < 1$, by utilizing a quadratic approximation for the exponential terms as in Babangida and Baraya (2018), we have

$$
\begin{aligned}
Z_1(T) = \frac{1}{T}\Bigg\{ & C_0 + h_1\Bigg(\frac{\lambda\mu}{\omega}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}\Bigg)+\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4 \\
& +\frac{\lambda}{\omega^2}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}\Bigg)-\frac{\lambda}{\omega^2}-\frac{\lambda T}{\omega}\Bigg) \\
& +h_2\Bigg(\frac{\lambda\mu^2}{2\omega}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}\Bigg)+\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5 \\
& +\frac{\lambda\mu}{\omega^2}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}\Bigg)-\frac{\lambda T}{\omega^2}-\frac{\lambda}{\omega^3}+\frac{\lambda}{\omega^3}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}\Bigg) \\
& -\frac{\lambda T^2}{2\omega}\Bigg)+C_p\Bigg[\frac{\lambda}{\omega}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}-1-\omega(T-\mu)\Bigg)\Bigg] \\
& +C_pI_c\Bigg[\frac{\lambda(\mu-M)}{\omega}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}-1\Bigg)+\frac{\alpha}{2}(\mu-M)^2 \\
& +\frac{\beta}{6}(2\mu+M)(\mu-M)^2+\frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2 \\
& +\frac{\lambda}{\omega^2}\Bigg(1+\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}-1-\omega(T-\mu)\Bigg)\Bigg]-C_sI_e\Bigg(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}+\gamma\frac{M^4}{4}\Bigg)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ & C_0 + h_1\Bigg(\frac{\lambda\mu}{\omega}+\lambda\mu(T-\mu)+\frac{\lambda\mu\omega(T-\mu)^2}{2}+\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4+\frac{\lambda}{\omega^2}+\frac{\lambda(T-\mu)}{\omega}+\frac{\lambda(T-\mu)^2}{2}-\frac{\lambda}{\omega^2} \\
& -\frac{\lambda T}{\omega}\Bigg) \\
& +h_2\Bigg(\frac{\lambda\mu^2}{2\omega}+\frac{\lambda\mu^2(T-\mu)}{2}+\frac{\lambda\mu^2\omega(T-\mu)^2}{4}+\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu}{\omega^2}+\frac{\lambda\mu(T-\mu)}{\omega} \\
& +\frac{\lambda\mu(T-\mu)^2}{2}-\frac{\lambda T}{\omega^2}-\frac{\lambda}{\omega^3}+\frac{\lambda}{\omega^3}+\frac{\lambda(T-\mu)}{\omega^2}+\frac{\lambda(T-\mu)^2}{2\omega}-\frac{\lambda T^2}{2\omega}\Bigg)+C_p\Bigg(\frac{\lambda\omega(T-\mu)^2}{2}\Bigg) \\
& +C_pI_c\Bigg[\frac{\lambda(\mu-M)}{\omega}\Bigg(\omega(T-\mu)+\frac{\omega^2(T-\mu)^2}{2}\Bigg)+\frac{\alpha}{2}(\mu-M)^2+\frac{\beta}{6}(2\mu+M)(\mu-M)^2 \\
& +\frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2+\frac{\lambda}{\omega^2}\Bigg(\frac{\omega^2(T-\mu)^2}{2}\Bigg)\Bigg]-C_sI_e\Bigg(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}+\gamma\frac{M^4}{4}\Bigg)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ & C_0 + h_1\Bigg(\lambda\mu T-\lambda\mu^2+\frac{\lambda\mu\omega(T^2-2\mu T+\mu^2)}{2}+\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4+\frac{\lambda(T^2-2\mu T+\mu^2)}{2}\Bigg) \\
& +h_2\Bigg(\frac{\lambda\mu^2}{2\omega}+\frac{\lambda\mu^2T}{2}-\frac{\lambda\mu^3}{2}+\frac{\lambda\mu^2\omega(T^2-2\mu T+\mu^2)}{4}+\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu}{\omega^2} \\
& +\frac{\lambda\mu(T-\mu)}{\omega}+\frac{\lambda\mu(T^2-2\mu T+\mu^2)}{2}-\frac{\lambda T}{\omega^2}-\frac{\lambda}{\omega^3}+\frac{\lambda}{\omega^3}+\frac{\lambda(T-\mu)}{\omega^2}+\frac{\lambda(T^2-2\mu T+\mu^2)}{2\omega} \\
& -\frac{\lambda T^2}{2\omega}\Bigg)+C_p\Bigg(\frac{\lambda\omega(T^2-2\mu T+\mu^2)}{2}\Bigg) \\
& +C_pI_c\Bigg[\frac{\lambda(\mu-M)}{\omega}\Bigg(\omega(T-\mu)+\frac{\omega^2(T^2-2\mu T+\mu^2)}{2}\Bigg)+\frac{\alpha}{2}(\mu-M)^2+\frac{\beta}{6}(2\mu+M)(\mu-M)^2 \\
& +\frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2+\frac{\lambda(T^2-2\mu T+\mu^2)}{2}\Bigg]-C_sI_e\Bigg(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}+\gamma\frac{M^4}{4}\Bigg)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\bigg\{ & C_0 + h_1\bigg(\frac{\lambda\mu}{\omega} + \lambda\mu T - \lambda\mu^2 + \frac{\lambda\mu\omega(T^2-2\mu T+\mu^2)}{2} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu}{\omega} + \frac{\lambda(T^2-2\mu T+\mu^2)}{2}\bigg) \\
& + h_2\bigg(\frac{\lambda\mu^2}{2\omega} + \frac{\lambda\mu^2 T}{2} - \frac{\lambda\mu^3}{2} + \frac{\lambda\mu^2\omega(T^2-2\mu T+\mu^2)}{4} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu}{\omega^2} \\
& + \frac{\lambda\mu(T-\mu)}{\omega} + \frac{\lambda\mu(T^2-2\mu T+\mu^2)}{2} - \frac{\lambda T}{\omega^2} + \frac{\lambda(T-\mu)}{\omega^2} + \frac{\lambda(T^2-2\mu T+\mu^2)}{2\omega} - \frac{\lambda T^2}{2\omega}\bigg) \\
& + C_p\bigg(\frac{\lambda\omega(T^2-2\mu T+\mu^2)}{2}\bigg) \\
& + C_pI_c\bigg[\frac{\lambda(\mu-M)}{\omega}\bigg(\omega(T-\mu) + \frac{\omega^2(T^2-2\mu T+\mu^2)}{2}\bigg) + \frac{\alpha}{2}(\mu-M)^2 + \frac{\beta}{6}(2\mu+M)(\mu-M)^2 \\
& + \frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2 + \frac{\lambda(T^2-2\mu T+\mu^2)}{2}\bigg] - C_sI_e\bigg(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\bigg)\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\bigg\{ & C_0 + h_1\bigg(\frac{\lambda\mu}{\omega} + \lambda\mu T - \lambda\mu^2 + \frac{\lambda\mu\omega(T^2-2\mu T+\mu^2)}{2} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu}{\omega} + \frac{\lambda(T^2-2\mu T+\mu^2)}{2}\bigg) \\
& + h_2\bigg(\frac{\lambda\mu^2}{2\omega} + \frac{\lambda\mu^2 T}{2} - \frac{\lambda\mu^3}{2} + \frac{\lambda\mu^2\omega(T^2-2\mu T+\mu^2)}{4} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu}{\omega^2} \\
& + \frac{\lambda\mu(T-\mu)}{\omega} + \frac{\lambda\mu(T^2-2\mu T+\mu^2)}{2} - \frac{\lambda T}{\omega^2} + \frac{\lambda(T-\mu)}{\omega^2} + \frac{\lambda(T^2-2\mu T+\mu^2)}{2\omega} - \frac{\lambda T^2}{2\omega}\bigg) \\
& + C_p\bigg(\frac{\lambda\omega(T^2-2\mu T+\mu^2)}{2}\bigg) \\
& + C_pI_c\bigg[\frac{\lambda(\mu-M)}{\omega}\bigg(\omega(T-\mu) + \frac{\omega^2(T^2-2\mu T+\mu^2)}{2}\bigg) + \frac{\alpha}{2}(\mu-M)^2 + \frac{\beta}{6}(2\mu+M)(\mu-M)^2 \\
& + \frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2 + \frac{\lambda(T^2-2\mu T+\mu^2)}{2}\bigg] - C_sI_e\bigg(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\bigg)\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\bigg\{ & C_0 + h_1\bigg(\lambda\mu T - \lambda\mu^2 + \frac{\lambda\mu\omega T^2}{2} - \lambda\mu^2\omega T + \frac{\lambda\mu^3\omega}{2} + \frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 + \frac{\lambda T^2}{2} - \lambda\mu T + \frac{\lambda\mu^2}{2}\bigg) \\
& + h_2\bigg(\frac{\lambda\mu^2}{2\omega} + \frac{\lambda\mu^2 T}{2} - \frac{\lambda\mu^3}{2} + \frac{\lambda\mu^2\omega T^2}{4} - \frac{\lambda\mu^3\omega T}{2} + \frac{\lambda\mu^4\omega}{4} + \frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 \\
& + \frac{\lambda\mu}{\omega^2} + \frac{\lambda\mu T}{\omega} - \frac{\lambda\mu^2}{\omega} + \frac{\lambda\mu T^2}{2} - \lambda\mu^2 T + \frac{\lambda\mu^3}{2} - \frac{\lambda T}{\omega^2} + \frac{\lambda T}{\omega^2} - \frac{\lambda\mu}{\omega^2} + \frac{\lambda T^2}{2\omega} - \frac{\lambda\mu T}{\omega} + \frac{\lambda\mu^2}{2\omega} \\
& - \frac{\lambda T^2}{2\omega}\bigg) + C_p\bigg(\frac{\lambda\omega T^2}{2} - \lambda\omega\mu T + \frac{\lambda\omega\mu^2}{2}\bigg) \\
& + C_pI_c\bigg[\lambda\mu T - \lambda\mu^2 - \lambda MT + \lambda M\mu + \frac{\lambda(\mu-M)\omega T^2}{2} - \lambda(\mu-M)\omega\mu T \\
& + \frac{\lambda(\mu-M)\omega\mu^2}{2} + \frac{\alpha}{2}(\mu-M)^2 + \frac{\beta}{6}(2\mu+M)(\mu-M)^2 \\
& + \frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2 + \frac{\lambda T^2}{2} - \lambda\mu T + \frac{\lambda\mu^2}{2}\bigg] \\
& - C_sI_e\bigg(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\bigg)\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ C_0 &+ h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} + \frac{\lambda\mu\omega T^2}{2} + \frac{\lambda T^2}{2} - \lambda\mu^2\omega T\right) \\
&+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} + \frac{\lambda\mu^2\omega T^2}{4} + \frac{\lambda\mu T^2}{2} - \frac{\lambda\mu^3\omega T}{2} - \frac{\lambda\mu^2 T}{2}\right) \\
&+ \frac{C_p\lambda\omega}{2}(\mu^2 + T^2 - 2\mu T) \\
&+ C_p I_c \Bigg[\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 \\
&- \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2} + \frac{\lambda T^2}{2} + \frac{\lambda(\mu - M)\omega T^2}{2} - \lambda MT - \lambda(\mu - M)\omega\mu T\Bigg] \\
&- C_s I_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\} \qquad (26)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Z_2(T) = \frac{1}{T}\Bigg\{ C_0 &+ h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} + \frac{\lambda\mu\omega T^2}{2} + \frac{\lambda T^2}{2} - \lambda\mu^2\omega T\right) \\
&+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} + \frac{\lambda\mu^2\omega T^2}{4} + \frac{\lambda\mu T^2}{2} - \frac{\lambda\mu^3\omega T}{2} - \frac{\lambda\mu^2 T}{2}\right) \\
&+ \frac{C_p\lambda\omega}{2}(\mu^2 + T^2 - 2\mu T) \\
&+ C_p I_c \frac{\lambda}{\omega^2}\left[1 + \omega(T - M) + \frac{\omega^2(T - M)^2}{2} - 1 - \omega(T - M)\right] \\
&- C_s I_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ C_0 &+ h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} + \frac{\lambda\mu\omega T^2}{2} + \frac{\lambda T^2}{2} - \lambda\mu^2\omega T\right) \\
&+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} + \frac{\lambda\mu^2\omega T^2}{4} + \frac{\lambda\mu T^2}{2} - \frac{\lambda\mu^3\omega T}{2} - \frac{\lambda\mu^2 T}{2}\right) \\
&+ \frac{C_p\lambda\omega}{2}(\mu^2 + T^2 - 2\mu T) + C_p I_c \frac{\lambda}{2}[M^2 + T^2 - 2MT] \\
&- C_s I_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\Bigg\} \qquad (27)
\end{aligned}
$$

3. Optimal Decision

In order to find the optimal ordering policy that minimizes the total variable cost per unit time, we establish the necessary and sufficient conditions. The necessary condition for the total variable cost per unit time $Z_i(T)$ to be minimum is obtained by differentiating $Z_i(T)$ with respect T for $i = 1, 2$ and equates to zero. The optimum value of T for which the sufficient condition $\frac{d^2 Z_i(T)}{dT^2} > 0$ is satisfied gives a minimum for the total variable cost per unit time$Z_i(T)$.

Case 1: ($\mathbf{0} < M \leq \mu$)

The necessary and sufficient conditions that minimize $Z_1(T)$are respectively,$\frac{dZ_1(T)}{dT} = 0$ and

$$\frac{d^2Z_1(T)}{dT^2} > 0$$

The first derivatives of the total variable cost, in (26), with respect to T is as follows.

$$\begin{aligned}
\frac{dZ_1(T)}{dT} = -\frac{1}{T^2}\Bigg\{ & C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} + \frac{\lambda\mu\omega T^2}{2} + \frac{\lambda T^2}{2} - \lambda\mu^2\omega T\right) \\
& + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} + \frac{\lambda\mu^2\omega T^2}{4} + \frac{\lambda\mu T^2}{2} - \frac{\lambda\mu^3\omega T}{2} - \frac{\lambda\mu^2 T}{2}\right) \\
& + \frac{C_p\lambda\omega}{2}(\mu^2 + T^2 - 2\mu T) \\
& + C_p I_c\left[\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2\right. \\
& \left. - \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2} + \frac{\lambda T^2}{2} + \frac{\lambda(\mu - M)\omega T^2}{2} - \lambda MT - \lambda(\mu - M)\omega\mu T\right] \\
& - C_s I_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\} \\
& + \frac{1}{T}\Bigg\{h_1(\lambda\mu\omega T + \lambda T - \lambda\mu^2\omega) + h_2\left(\frac{\lambda\mu^2\omega T}{2} + \lambda\mu T - \frac{\lambda\mu^3\omega}{2} - \frac{\lambda\mu^2}{2}\right) \\
& + \frac{C_p\lambda\omega}{2}(2T - 2\mu) + C_p I_c[\lambda T + \lambda(\mu - M)\omega T - \lambda M - \lambda(\mu - M)\omega\mu]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
= -\frac{1}{T^2}\Bigg\{ & C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} + \frac{\lambda\mu\omega T^2}{2} + \frac{\lambda T^2}{2} - \lambda\mu^2\omega T - \lambda\mu\omega T^2 - \lambda T^2\right. \\
& \left. + \lambda\mu^2\omega T\right) \\
& + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} + \frac{\lambda\mu^2\omega T^2}{4} + \frac{\lambda\mu T^2}{2} - \frac{\lambda\mu^3\omega T}{2} - \frac{\lambda\mu^2 T}{2}\right. \\
& \left. - \frac{\lambda\mu^2\omega T^2}{2} - \lambda\mu T^2 + \frac{\lambda\mu^3\omega T}{2} + \frac{\lambda\mu^2 T}{2}\right) + \frac{C_p\lambda\omega}{2}(\mu^2 + T^2 - 2\mu T - 2T^2 + 2\mu T) \\
& + C_p I_c\left[\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2\right. \\
& - \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2} + \frac{\lambda T^2}{2} + \frac{\lambda(\mu - M)\omega T^2}{2} - \lambda MT - \lambda(\mu - M)\omega\mu T \\
& \left. - \lambda T^2 - \lambda(\mu - M)\omega T^2 + \lambda MT + \lambda(\mu - M)\omega\mu T\right] - C_s I_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
= \frac{1}{T^2}\Bigg\{ -C_0 - h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} - \frac{\lambda\mu\omega T^2}{2} - \frac{\lambda T^2}{2}\right) \\
- h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} - \frac{\lambda\mu^2\omega T^2}{4} - \frac{\lambda\mu T^2}{2}\right) - \frac{C_p\lambda\omega}{2}(\mu^2 - T^2) \\
- C_pI_c\Bigg[\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 \\
- \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2} - \frac{\lambda T^2}{2} - \frac{\lambda(\mu - M)\omega T^2}{2}\Bigg] \\
+ C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{T^2}\Bigg\{\frac{T^2}{2}\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c(\omega(\mu - M) + 1)\right] \\
- \Bigg[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right) \\
+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2 \\
+ C_pI_c\Bigg(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 \\
- \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2}\Bigg) - C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg]\Bigg\} \qquad (28)
\end{aligned}
$$

Therefore, $\frac{dZ_1(T)}{dT} = 0$ gives the following nonlinear equation in terms T

$$
\begin{aligned}
\frac{1}{T^2}\Bigg\{\frac{T^2}{2}\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c(\omega(\mu - M) + 1)\right] \\
- \Bigg[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right) \\
+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2 \\
+ C_pI_c\Bigg(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 \\
- \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2}\Bigg) - C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg]\Bigg\} \\
= 0 \qquad (29)
\end{aligned}
$$

From equation (29), let

$$X_1 = \lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c(\omega(\mu - M) + 1)\right]$$

and

$$X_2 = C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right) + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2$$
$$+ C_pI_c\left(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 - \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2}\right) - C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)$$

Substituting X_1 and X_2 intoequation (29) to obtain

$$\frac{1}{T^2}\left\{\frac{T^2}{2}X_1 - X_2\right\} = 0$$

which implies

$$T^2X_1 - 2X_2 = 0 \tag{30}$$

Let

$$\Delta_1 = C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right) - \Bigg[A + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \lambda\mu^2\right) + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 - \frac{\lambda\mu^3}{2}\right)$$
$$+ C_pI_c\left(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 - \lambda\mu^2 + \lambda M\mu\right)\Bigg]$$

Lemma 1. For $0 < M \leq \mu$, we have

(i) If $\Delta_1 \leq 0$, then the solution of $T \in [\mu, \infty)$ (sayT_1^*) which satisfies equation (30) not only exists but also is unique.
(ii) If $\Delta_1 > 0$, then the solution of $T \in [\mu, \infty)$ which satisfies equation (30) does not exist.

Proof of (i). From equation (29), we define a new function $F_1(T)$ as follows

$$F_1(T) = \frac{T^2}{2}\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c(\omega(\mu - M) + 1)\right] - \Bigg[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right)$$
$$+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2$$
$$+ C_pI_c\left(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 - \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2}\right) - C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg], T \in [\mu, \infty) \tag{31}$$

Taking the first-order derivative of $F_1(T)$with respect to $T \in [\mu, \infty)$, we have

$$\frac{F_1(T)}{dT} = T\lambda \left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c(\omega(\mu - M) + 1)\right]$$

$$= TX_1 > 0$$

We obtain that $F_1(T)$is an increasing function of T in the interval $[\mu, \infty)$. Moreover, we have

$$\lim_{T\to\infty} F_1(T) = \infty$$

and

$$\begin{aligned}F_1(\mu) = \frac{\mu^2}{2}\lambda \Big[&h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c(\omega(\mu - M) + 1)\Big] \\ &- \Bigg[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right) \\ &+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2 \\ &+ C_pI_c\Bigg(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 \\ &- \frac{\lambda\mu^2}{2} + \lambda M\mu + \frac{\lambda(\mu - M)\omega\mu^2}{2}\Bigg) - C_sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg]\end{aligned}$$

$$\begin{aligned}= C_sI_e&\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right) \\ &- \Bigg[A + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \lambda\mu^2\right) + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 - \frac{\lambda\mu^3}{2}\right) \\ &+ C_pI_c\Bigg(\frac{\alpha}{2}(\mu - M)^2 + \frac{\beta}{6}(2\mu + M)(\mu - M)^2 + \frac{\gamma}{12}(3\mu^2 + 2\mu M + M^2)(\mu - M)^2 \\ &- \lambda\mu^2 + \lambda M\mu\Bigg)\Bigg]\end{aligned}$$

$$= \Delta_1 \leq 0$$

Now$F_1(\mu) \leq 0$. Therefore, by applying intermediate value theorem, there exists a unique $T_1^* \in [\mu, \infty)$ such that$F_1(T_1^*) = 0$. Hence T_1^* isthe unique solution of equation (30). Thus, the value of T (denoted byT_1^*) can be found fromequation (30) and is given by

$$T_1^* = \sqrt{\frac{2X_2}{X_1}} \tag{32}$$

Proof of (ii). If $\Delta_1 > 0$, then fromequation (31), we have$F_1(T) > 0$. Since $F_1(T)$is an increasing function of $T \in [\mu, \infty)$,then$F_1(T) > 0$ for all$T \in [\mu, \infty)$. Thus, we cannot find a value of $T \in [\mu, \infty)$ such that$F_1(T) = 0$. This completes the proof.

Theorem 1.When $0 < M \leq \mu$, we have

(i) If $\Delta_1 \leq 0$, then the total variable cost $Z_1(T)$is convex and reaches its global minimum at the point$T_1^* \in [\mu, \infty)$, where $T_1^*$is the point which satisfiesequation (30).

(ii) If $\Delta_1 > 0$, then the total variable cost $Z_1(T)$has a minimum value at the point$T_1^* = \mu$.

Proof of (i). When $\Delta_1 \leq 0$,we see that $T_1^*$is the unique solution of (30) from Lemma l(i). Taking the second derivative of$Z_1(T)$with respect to T and then finding the value of the function at the pointT_1^*, we obtain

$$\begin{aligned}
\left.\frac{d^2 Z_1(T)}{dT^2}\right|_{T_1^*} = -\frac{2}{T^3}\Bigg\{&\frac{T^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c(\omega(\mu-M)+1)\right]\\
&-\Bigg[C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right)\\
&+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)+\frac{C_p\lambda\omega}{2}\mu^2\\
&+C_pI_c\left(\frac{\alpha}{2}(\mu-M)^2+\frac{\beta}{6}(2\mu+M)(\mu-M)^2+\frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2\right.\\
&\left.-\frac{\lambda\mu^2}{2}+\lambda M\mu+\frac{\lambda(\mu-M)\omega\mu^2}{2}\right)-C_sI_e\left(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}+\gamma\frac{M^4}{4}\right)\Bigg]\Bigg\}\\
&\left.+\frac{1}{T^2}\left\{T\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c(\omega(\mu-M)+1)\right]\right\}\right|_{T_1^*}
\end{aligned}$$

$$\begin{aligned}
= -\frac{2}{T^3}\Bigg\{&\frac{T^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c(\omega(\mu-M)+1)\right]\\
&-\Bigg[C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right)\\
&+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)+\frac{C_p\lambda\omega}{2}\mu^2\\
&+C_pI_c\left(\frac{\alpha}{2}(\mu-M)^2+\frac{\beta}{6}(2\mu+M)(\mu-M)^2+\frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2\right.\\
&\left.-\frac{\lambda\mu^2}{2}+\lambda M\mu+\frac{\lambda(\mu-M)\omega\mu^2}{2}\right)-C_sI_e\left(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}+\gamma\frac{M^4}{4}\right)\Bigg]\\
&\left.-\frac{T^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c(\omega(\mu-M)+1)\right]\Bigg\}\right|_{T_1^*}
\end{aligned}$$

$$\begin{aligned}
= \frac{2}{T^3}\Bigg\{\Bigg[&C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right)+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)\\
&+\frac{C_p\lambda\omega}{2}\mu^2\\
&+C_pI_c\left(\frac{\alpha}{2}(\mu-M)^2+\frac{\beta}{6}(2\mu+M)(\mu-M)^2+\frac{\gamma}{12}(3\mu^2+2\mu M+M^2)(\mu-M)^2\right.\\
&\left.\left.-\frac{\lambda\mu^2}{2}+\lambda M\mu+\frac{\lambda(\mu-M)\omega\mu^2}{2}\right)-C_sI_e\left(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}+\gamma\frac{M^4}{4}\right)\Bigg]\Bigg\}\right|_{T_1^*}
\end{aligned}$$

$$= \left.\frac{2X_2}{T^3}\right|_{T_1^*}$$

$$= \frac{2X_2}{\left(\frac{2X_2}{X_1}\right)T_1^*}$$

$$= \frac{X_1}{T_1^*} > 0 \qquad (33)$$

We thus conclude from equation (33) and Lemma 1 that $Z_1(T_1^*)$ is convex and T_1^* is the global minimum point of $Z_1(T)$. Hence the value of T in equation(32) is optimal.

Proof of (ii). When $\Delta_1 > 0$, then we know that $F_1(T) > 0$ for all $T \in [\mu, \infty)$. Thus, $\frac{dZ_1(T)}{dT} = \frac{F_1(T)}{T^2} > 0$ for all $T \in [\mu, \infty)$ which implies $Z_1(T)$ is an increasing function of T. Thus $Z_1(T)$ has a minimum value when T is minimum. Therefore, $Z_1(T)$ has a minimum value at the point $T = \mu$. This completes the proof.

Case 2: $(\boldsymbol{\mu} < M \le T)$

The necessary and sufficient conditions to minimize $Z_2(T)$ are respectively $\frac{dZ_2(T)}{dT} = 0$ and

$$\frac{d^2 Z_2(T)}{dT^2} > 0$$

Similarly the first derivatives of the total variable cost in equation(27) with respect to T is as follows.

$$\begin{aligned}\frac{dZ_2(T)}{dT} = -\frac{1}{T^2}\Bigg\{ & C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} - \frac{\lambda\mu\omega T^2}{2} - \frac{\lambda T^2}{2}\right) \\ & + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} - \frac{\lambda\mu^2\omega T^2}{4} - \frac{\lambda\mu T^2}{2}\right) \\ & + \frac{C_p\lambda\omega}{2}(\mu^2 - T^2) + C_p I_c \frac{\lambda}{2}[M^2 + T^2 - 2MT] \\ & - C_s I_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\Bigg\} + \frac{1}{T}\left\{C_p I_c \frac{\lambda}{2}[2T - 2M]\right\}\end{aligned}$$

$$\begin{aligned}= -\frac{1}{T^2}\Bigg\{ & C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} - \frac{\lambda\mu\omega T^2}{2} - \frac{\lambda T^2}{2}\right) \\ & + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} - \frac{\lambda\mu^2\omega T^2}{4} - \frac{\lambda\mu T^2}{2}\right) \\ & + \frac{C_p\lambda\omega}{2}(\mu^2 - T^2) + C_p I_c \frac{\lambda}{2}(M^2 + T^2 - 2MT - 2T^2 + 2MT) \\ & - C_s I_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\Bigg\}\end{aligned}$$

$$= \frac{1}{T^2}\left\{-C_0 - h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2} - \frac{\lambda\mu\omega T^2}{2} - \frac{\lambda T^2}{2}\right)\right.$$
$$- h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4} - \frac{\lambda\mu^2\omega T^2}{4} - \frac{\lambda\mu T^2}{2}\right) - \frac{C_p\lambda\omega}{2}[\mu^2 - T^2]$$
$$\left. - C_pI_c\frac{\lambda}{2}(M^2 - T^2) + C_sI_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\right\}$$

$$= \frac{1}{T^2}\left\{\frac{T^2}{2}\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c\right]\right.$$
$$- \left[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right)\right.$$
$$+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2 + C_pI_c\frac{\lambda}{2}M^2$$
$$\left.\left. - C_sI_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\right]\right\} \quad (34)$$

Therefore, $\frac{dZ_2(T)}{dT} = 0$ gives the following nonlinear equation in T

$$\frac{1}{T^2}\left\{\frac{T^2}{2}\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c\right]\right.$$
$$- \left[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right)\right.$$
$$+ h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2 + C_pI_c\frac{\lambda}{2}M^2$$
$$\left.\left. - C_sI_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\right]\right\} = 0 \quad (35)$$

From equation (35), let

$$Y_1 = \lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c\right]$$

and

$$Y_2 = C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right) + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right)$$
$$+ \frac{C_p\lambda\omega}{2}\mu^2 + C_pI_c\frac{\lambda}{2}M^2 - C_sI_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)$$

Substituting Y_1 and Y_2 intoequation (35) to obtain

$$\frac{T^2}{2}Y_1 - Y_2 = 0$$

which implies

$$T^2Y_1 - 2Y_2 = 0 \tag{36}$$

Let

$$\Delta_2 = C_sI_e\left(\alpha\frac{\mu^2}{2} + \beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right) - \left[A + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \lambda\mu^2\right) + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 - \frac{\lambda\mu^3}{2}\right) + C_pI_c\frac{\lambda}{2}(\mu^2 - M^2)\right]$$

Lemma 2. For $\mu < M \leq T$, we have

(i) If $\Delta_2 \leq 0$, then the solution $T \in [M, \infty)$ (sayT_2^*) which satisfies equation (36) not only exists but also is unique.

(ii) If $\Delta_2 > 0$, then the solution $T \in [M, \infty)$ which satisfiesequation (36) does not exist.

Proof of (i). Fromequation (35), we define a new function $F_2(T)$ as follows:

$$F_2(T) = \frac{T^2}{2}\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c\right] - \left[C_0 + h_1\left(\frac{\alpha}{2}\mu^2 + \frac{\beta}{3}\mu^3 + \frac{\gamma}{4}\mu^4 - \frac{\lambda\mu^2}{2} + \frac{\lambda\mu^3\omega}{2}\right) + h_2\left(\frac{\alpha}{6}\mu^3 + \frac{\beta}{8}\mu^4 + \frac{\gamma}{10}\mu^5 + \frac{\lambda\mu^4\omega}{4}\right) + \frac{C_p\lambda\omega}{2}\mu^2 + C_pI_c\frac{\lambda}{2}M^2 - C_sI_e\left(\alpha\frac{\mu^2}{2} + +\beta\frac{\mu^3}{3} + \gamma\frac{\mu^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda\mu^2}{2}\right)\right], \quad T \in [M, \infty). \tag{37}$$

The first-order derivative of $F_2(T)$with respect to $T \in [M, \infty)$, we have

$$\frac{F_2(T)}{dT} = T\lambda\left[h_1(\mu\omega + 1) + h_2\mu\left(\frac{\mu\omega}{2} + 1\right) + C_p\omega + C_pI_c\right]$$

$$= TY_1 > 0$$

We obtain that $F_2(T)$is an increasing function of T in the interval $[M, \infty)$. Moreover, we have

$$\lim_{T\to\infty} F_2(T) = \infty$$

and

$$
\begin{aligned}
F_2(M) = \frac{M^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c\right] \\
-\Bigg[C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right) \\
+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)+\frac{C_p\lambda\omega}{2}\mu^2+C_pI_c\frac{\lambda}{2}M^2 \\
-C_sI_e\left(\alpha\frac{\mu^2}{2}+\beta\frac{\mu^3}{3}+\gamma\frac{\mu^4}{4}+\frac{\lambda M^2}{2}-\frac{\lambda\mu^2}{2}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= -\Bigg[A+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\lambda\mu^2\right)+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5-\frac{\lambda\mu^3}{2}\right) \\
+C_pI_c\frac{\lambda}{2}(\mu^2-M^2)\Bigg]+C_sI_e\left(\alpha\frac{\mu^2}{2}+\beta\frac{\mu^3}{3}+\gamma\frac{\mu^4}{4}+\frac{\lambda M^2}{2}-\frac{\lambda\mu^2}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
= C_sI_e\left(\alpha\frac{\mu^2}{2}+\beta\frac{\mu^3}{3}+\gamma\frac{\mu^4}{4}+\frac{\lambda M^2}{2}-\frac{\lambda\mu^2}{2}\right) \\
-\Bigg[A+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\lambda\mu^2\right)+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5-\frac{\lambda\mu^3}{2}\right) \\
+C_pI_c\frac{\lambda}{2}(\mu^2-M^2)\Bigg]
\end{aligned}
$$

$$= \Delta_2 \leq 0$$

We have $F_2(M) \leq 0$. Therefore, by applying intermediate value theorem, there exists a unique $T_2^* \in [M,\infty)$ such that$F_2(T_2^*) = 0$. Hence T_2^* isthe unique solution of (36). Thus, the value of T (denoted byT_2^*) can be found from (38) and is given by

$$T_2^* = \sqrt{\frac{2Y_2}{Y_1}} \tag{38}$$

Proof of (ii).If $\Delta_2 > 0$, then from (37), we have$F_2(M) > 0$. Since $F_2(T)$is an increasing function of$T \in [M,\infty)$, we have $F_2(T) > 0$ for all$T \in [M,\infty)$. Thus, we cannot find a value of $T \in [M,\infty)$ such that$F_2(T) = 0$. This completes the proof.

Theorem 2.When $\mu < M \leq T$, we have

(i) If $\Delta_2 \leq 0$, then the total variable cost $Z_2(T)$is convex and reaches its global minimum at the point$T_2^* \in [M,\infty)$, where $T_2^*$is the point which satisfies equation (36)

(ii) If $\Delta_2 > 0$, then the total variable cost $Z_2(T)$has a minimum value at the point$T_2^* = M$.

Proof of (i). When $\Delta_2 \leq 0$,we see that $T_2^*$is the unique solution of equation (36) from Lemma 2(i).

the second derivative of $Z_2(T)$ with respect to T and finding the value of the function at the point of T_2^*, we obtain

$$\left.\frac{d^2 Z_2(T)}{dT^2}\right|_{T_2^*} = -\frac{2}{T^3}\Bigg\{\frac{T^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c\right]$$

$$-\Bigg[C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right)$$

$$+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)+\frac{C_p\lambda\omega}{2}\mu^2+C_pI_c\frac{\lambda}{2}M^2$$

$$-C_sI_e\left(\alpha\frac{\mu^2}{2}+\beta\frac{\mu^3}{3}+\gamma\frac{\mu^4}{4}+\frac{\lambda M^2}{2}-\frac{\lambda\mu^2}{2}\right)\Bigg]\Bigg\}$$

$$\left.+\frac{1}{T^2}\left\{T\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c\right]\right\}\right|_{T_2^*}$$

$$= -\frac{2}{T^3}\Bigg\{\frac{T^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c\right]$$

$$-\Bigg[C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right)$$

$$+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)+\frac{C_p\lambda\omega}{2}\mu^2+C_pI_c\frac{\lambda}{2}M^2$$

$$-C_sI_e\left(\alpha\frac{\mu^2}{2}+\beta\frac{\mu^3}{3}+\gamma\frac{\mu^4}{4}+\frac{\lambda M^2}{2}-\frac{\lambda\mu^2}{2}\right)\Bigg]$$

$$\left.-\frac{T^2}{2}\lambda\left[h_1(\mu\omega+1)+h_2\mu\left(\frac{\mu\omega}{2}+1\right)+C_p\omega+C_pI_c\right]\Bigg\}\right|_{T_2^*}$$

$$= \frac{2}{T^3}\Bigg\{\Bigg[C_0+h_1\left(\frac{\alpha}{2}\mu^2+\frac{\beta}{3}\mu^3+\frac{\gamma}{4}\mu^4-\frac{\lambda\mu^2}{2}+\frac{\lambda\mu^3\omega}{2}\right)+h_2\left(\frac{\alpha}{6}\mu^3+\frac{\beta}{8}\mu^4+\frac{\gamma}{10}\mu^5+\frac{\lambda\mu^4\omega}{4}\right)$$

$$\left.+\frac{C_p\lambda\omega}{2}\mu^2+C_pI_c\frac{\lambda}{2}M^2-C_sI_e\left(\alpha\frac{\mu^2}{2}+\beta\frac{\mu^3}{3}+\gamma\frac{\mu^4}{4}+\frac{\lambda M^2}{2}-\frac{\lambda\mu^2}{2}\right)\Bigg]\Bigg\}\right|_{T_2^*}$$

$$= \left.\frac{2Y_2}{T^3}\right|_{T_2^*}$$

$$= \frac{2Y_2}{\left(\frac{2Y_2}{Y_1}\right)T_1^*}$$

$$= \frac{Y_1}{T_2^*} > 0 \qquad (39)$$

We thus conclude from equation (39) that $Z_2(T_2^*)$ is convex and T_2^* is the global minimum point of $Z_2(T)$. Hence the value of T in equation (38) is optimal.

Proof of (ii). When $\Delta_2 > 0$, then we know that $F_2(T) > 0$,for all $T \in [M, \infty)$.Thus,$\frac{dZ_2(T)}{dT} = \frac{F_2(T)}{T^2} > 0$ which implies $Z_2(T)$ is an increasing function of T. Thus $Z_2(T)$ has a minimum value when T is minimum. Therefore, $Z_2(T)$ has a minimum value at the point $T = M$. This completes the proof.

Thus the EOQ corresponding to the best cycle length T^*will be computed as follows:
$EOQ^* =$Total demand before deterioration set in+total demand after deterioration set in
+total number of deteriorated items

$$= \int_0^{\mu} (\alpha + \beta t + \gamma t^2) dt + \int_{\mu}^{T^*} \lambda dt + \left[\frac{\lambda}{\omega}\left(e^{\omega(T^*-\mu)} - 1\right) - \lambda(T^* - \mu)\right]$$

$$= \alpha\mu + \beta\frac{\mu^2}{2} + \gamma\frac{\mu^3}{3} + \frac{\lambda}{\omega}\left(e^{\omega(T^*-\mu)} - 1\right) \qquad (40)$$

4. Numerical Examples

This section will provide some numerical examples to illustrate the application of models developed by considering the following examples.

Example 4.1 (Case 1)

The following parameters used in this example are adapted from Babangida and Baraya (2018) but adding the values for the parameter h_2 which was not considered in their model. Thus, we assume the following input parameters:$C_0 = \$250$/order,$C_p = \$80$/unit/year,$C_s = \$85$/unit/year,$h_1 = \$15$/unit/year,$h_2 = \$0.8$/unit/year,$\omega = 0.05$ units/year,$\alpha = 1000$ units, $\beta = 150$ units, $\gamma = 15$ units, $\lambda = 500$ units, $\mu = 0.5014$ year (183days), $M = 0.0548$ year (20days),$I_c = 0.15$, $I_e = 0.12$. We first check the condition $\Delta_1 = -244.147 < 0$. Substituting the above values into equations (32), (26) and(40), we obtain as follows the values of the optimal cycle length$T_1^* = 0.530917$ year (194days), the optimal average total cost$Z_1(T^*) = \$7003.169$ per year, and the Economic Order Quantity$EOQ_1^* = 535.6547$units per year respectively.

Example 4.2 (Case 1)

Consider an inventory system with the following input parameters:$C_0 = \$200$/order, $C_p = \$45$/unit/year,$C_s = \$50$/unit/year,$h_1 = \$4$/unit/year,$h_2 = \$0.5$/unit/year,$\omega = 0.02$ units/year,$\alpha = 1500$ units, $\beta = 400$ units, $\gamma = 50$ units, $\lambda = 800$ units, $\mu = 0.2190$ year (80days), $M = 0.1971$ year (72days),$I_c = 0.12$, $I_e = 0.08$. We first check the condition $\Delta_1 = -52.916 < 0$. Substituting the above values into equations (32), (26) and(40), we obtain as follows the values of the optimal cycle length $T_1^* = 0.244207$ year (89days), the optimal average total cost$Z_1(T_1^*) = \$1049.682$ per year, and the Economic Order Quantity$EOQ_1^* = 358.4381$units per year respectively.

Example 4.3 (Case 2)

The data are same as in Example 4.1 except that $\mu = 0.08904$ (33days) and $M = 0.1058$ year (39days). We first check the condition $\Delta_2 = -179.267 < 0$. Substituting the above values into equations $(38), (27)$ and(40), we obtain as follows the values of the optimal cycle length $T_2^* = 0.184262$ year (67days), the optimal average total cost$Z_2(T_2^*) = \$2055.5248$ per year, and the Economic Order Quantity$EOQ_2^* = 137.3625$units per year respectively.

Example 4.4 (Case 2)

The data are same as in Example 4.2 except that $M = 0.2327$ year (85 days). We first check the condition $\Delta_2 = -20.300 < 0$. Substituting the above values into equations $(38), (27)$ and(40), we obtain as follows the values of the optimal cycle length $T_2^* = 0.239080$ year (87days), the optimal average total cost$Z_2(T_2^*) = \$825.0691$ per year, and the Economic Order Quantity$EOQ_2^* = 354.3341$units per year respectively.

5. Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from$-15\%, -10\%, -5\%$, $+5\%, +10\%$ to $+15\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these changes on the decision variables are discussed.

Table 1: Percentage change in the decision variables with respect to the percentage change in parameters from $-15\%, -10\%, -5\%$, $5\%, 10\%$ to 15% for examples 4.2 & 4.4.

Parameter	% Change in Parameter	% Change in T^*		% Change in EOQ^*		% Change in $Z(T^*)$	
		T_1^*	T_2^*	EOQ_1^*	EOQ_2^*	$Z_1(T_1^*)$	$Z_2(T_2^*)$
λ	-15	7.327	6.009	2.553	2.078	-1.866	-1.533
	-10	4.673	3.824	1.731	1.405	-1.131	-0.926
	-5	2.240	1.829	0.879	0.712	-0.512	-0.418
	$+5$	-2.071	-1.684	-0.904	-0.728	0.414	0.336
	$+10$	-3.993	-3.241	-1.832	-1.471	0.739	0.599
	$+15$	-5.781	-4.684	-2.781	-2.228	0.980	0.793
ω	-15	0.121	0.099	0.066	0.053	-0.020	-0.008
	-10	0.080	0.066	0.044	0.035	-0.013	-0.005
	-5	0.040	0.033	0.022	0.018	-0.007	-0.003
	$+5$	-0.040	-0.032	-0.022	-0.017	0.007	0.003
	$+10$	-0.079	-0.065	-0.043	-0.035	0.013	0.005
	$+15$	-0.118	-0.097	-0.064	-0.052	0.020	0.008
C_p	-15	1.686	0.348	0.919	0.188	-0.443	-0.052
	-10	1.089	0.224	0.594	0.121	-0.289	-0.035
	-5	0.528	0.109	0.288	0.059	-0.141	-0.017
	$+5$	-0.498	-0.102	-0.272	-0.055	0.136	0.017
	$+10$	-0.969	-0.198	-0.528	-0.107	0.267	0.033
	$+15$	-1.414	-0.288	-0.771	-0.156	0.393	0.050
C_s	-15	3.295	4.789	1.797	2.586	6.949	12.563
	-10	2.209	3.217	1.204	1.737	4.657	8.440
	-5	1.110	1.621	0.606	0.876	2.341	4.253
	$+5$	-1.123	-1.648	-0.612	-0.890	-2.368	-4.323

	$+10$	-2.258	-3.324	-1.232	-1.795	-4.763	-8.720
	$+15$	-3.408	-5.030	-1.858	-2.716	-7.186	-13.195
I_c	-15	1.539	0.239	0.839	0.129	-0.425	-0.040
	-10	0.998	0.155	0.544	0.084	-0.278	-0.027
	-5	0.486	0.075	0.265	0.041	-0.137	-0.013
	$+5$	-0.462	-0.071	-0.252	-0.038	0.132	0.013
	$+10$	-0.901	-0.139	-0.492	-0.075	0.259	0.026
	$+15$	-1.320	-0.203	-0.720	-0.120	0.383	0.039
I_e	-15	3.295	4.789	1.797	2.586	6.949	12.563
	-10	2.209	3.217	1.204	1.737	4.657	8.440
	-5	1.110	1.621	0.606	0.876	2.341	4.253
	$+5$	-1.123	-1.648	-0.612	-0.890	-2.368	-4.323
	$+10$	-2.258	-3.324	-1.232	-1.795	-4.763	-8.720
	$+15$	-3.408	-5.030	-1.858	-2.716	-7.186	-13.195

6. Results and Discussion

Based on the computational results shown in Tables 1 and 2 the following managerial insights are obtained.

(i) It can be seen that when the demand rate after deterioration sets in (λ) increases, the optimal cycle length(T^*) and economic order quantity (EOQ^*)decrease while total variable cost$(Z(T^*))$ increase. In this case the retailer should order less to shorten the cycle length.

(ii) As the rate of deterioration (ω) increases, the optimal cycle length(T^*) and economic order quantity (EOQ^*) decrease while total variable cost$(Z(T^*))$ increase.Hence the retailer will order less quantity to avoid the items being deteriorating when the deterioration rate.

(iii) As the unit purchasing cost (C_p) increases, the optimal cycle length(T^*), and the economic order quantity (EOQ^*) decrease while the total variable cost$(Z(T^*))$ increase. In real market situation the higher the cost of an item, the higher the total variable cost. This result implies that the retailer will order a smaller quantity to enjoy the benefits of permissible delay in payments more frequently in the presence of an increased unit purchasing price and consequently shortening cycle length.

(iv) As the unit selling price (C_s) increases, the optimal cycle length(T^*), the economic order quantity (EOQ^*) and the total variable cost $(Z(T^*))$decrease. In real market situation the higher the selling price of an item, the lower the demand.This means that when the unit selling price is increasing, the retailer will order less quantity to take the benefits of the trade credit more frequently.

(v) As the interest payable(I_c) increases, the optimal cycle length(T^*) and the economic order quantity (EOQ^*) decrease. The total variable cost $(Z(T^*))$increase when interest payable is high. This means that when interest payable is high the retailer should order less amount of inventory.

(vi) As the interest earn (I_e) increases, the optimal cycle length(T^*), the economic order quantity (EOQ^*) and the total variable cost$(Z(T^*))$ decrease. This implies that when the interest earned is high, the optimal cycle length, the economic order quantity and the total variable cost are low.

3 Conclusion

In this paper, we develop an economic order quantity model for non-instantaneous deteriorating items with quadratic demand function of time and linear time dependent holding cost under trade credit. The optimal cycle length and economic order quantity that minimise total variable cost are determined. Some numerical examples are presented to illustrate the application of the model. Sensitivity analysis is also carried out to show the effect of changes in system parameters on decision variables. The results show that the retailer can reduce total variable cost by ordering less to shorten the cycle length when deterioration sets in, unit purchasing price increases, unit selling price increases, interest charges increases and interest earn decreases respectively.

The model developed in this paper is a generalisation of the work Babangida and Baraya (2018). If$h_2 = 0$, the results obtained in examples 1and 3 above correspond to that in Babangida and Baraya (2018). The proposed model can be extended to allow for shortages, variable deterioration rate, quantity discounts, inflation rates, finite time horizon and so on.

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