AN EPQ MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH STOCK-DEPENDENT DEMAND RATE UNDER TWO-PHASE PRODUCTION RATE AND SHORTAGES

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Abstract

This paper proposes a production inventory model for non-instantaneous deteriorating items in which two-phase production rates are considered. The demand during production time is constant while it is stock-dependent after production stops. Shortages are allowed and completely backlogged. In reality, not all kinds of items such as meat, bread, cassava, and so on, deteriorate as soon as they are produced, but they maintain their freshness for some period before they begin to deteriorate. Demands for such items is constant at the initial stage of the products life cycle; at the end stage of life cycle and/or after production stopping time, the demand rate is sometimes influenced by the stock level. During the shortage period, the backlogging rate is complete. The purpose is to determine the optimal cycle length and optimal inventory level in each cycle so that the total cost is minimized. The necessary and sufficient conditions are provided to show the existence and uniqueness of the optimal solution. Also, the decision rule of Newton-Raphson method has been used to determine the optimal solutions and maple software version 13.0 were also used in plotting graph in order to show the convexity of the propose model. Then numerical example, sensitivity analysis and graphical presentation are provided to illustrate the application of the proposed model.

Keywords: *EPQ, two-phase production rate, non-instantaneous, deterioration, shortages.*

1 INTRODUCTION

Many inventory modelers have studied inventory models for deteriorating items. In fact, in daily life, deterioration of items becomes a common factor in inventory analysis. Generally, deterioration is the physical depletion/decay of products over time which prevents item from being used for its original purpose. A model for deteriorating items with constant and varying rate of deterioration was initially proposed by Misra (1975) and others are Goyal and Gunasekaran (1995), Jiang and Du (1998), Gontg and Wang (2005), Maity *et al*. (2007) and so on, with assumption that demand rate, production rate and deterioration rate are all constant. In general, almost all products are found to be deteriorating over time. Sometimes the rate of deterioration is too low, for items such as hardware, glassware, metals and toys, however some items have significant rate of deterioration, such as food grains, vegetables, medicines gasoline and radioactive chemicals and so on, which cannot be ignored in the decision making process of production lot size.

In all the models stated above, shortages are not allowed. However, Lin *el al*. (2007) established a production inventory model with constant production rate, demand rate and deterioration rate and allowed shortages. Zhou *et al*. (2003) also considered production inventory problem in which each cycle of a production inventory scheduled starts with replenishment and ends with a shortage. Sana *et al*. (2004) and Zhou and Gu (2007), shortages are allowed and occur at the end of the cycle. Sugapriya and Jeyaraman (2008a) developed a model to determine a common production cycle time for an economic production quantity model of non- instantaneous deteriorating items allowing price discount and permissible delay in payments. Sugapriya and Jeyaraman (2008b) also developed an

EPQ model for non-instantaneous deteriorating item in which production and demand rate are constant, holding cost varies with time, completely deteriorated units are discarded, partially deteriorated items are allowed to carry discount and no shortage is allowed. Baraya and Sani (2011) developed an economic production model for delayed deterioration items with stock-dependent demand rate and linear time holding cost, the model was developed as a single product with delayed deterioration in which the production rate is constant, demand rate is inventory level dependent in a linear functional form before and after production and the holding cost is a linear function of time. Baraya and Sani (2012) developed An economic production model for delayed deterioration items with stock-dependent demand rate and time dependent deterioration rate, the model was develop as a single product with delayed deterioration in which the production rate is constant, demand rate is inventory level dependent in a linear functional form before and after production and deteriorating rate is linear increasing function of time. Sivashankari and panayappan (2013) integrated a cost reduction delivery policy in to production inventory model with defecting item in which three different rate of production are considered. Sivashankari and Panayappan (2014) Production inventory model for two levels of production with defecting items and incorporating multi-delivery policy. Sivashankari and panayappan (2014) considered production inventory model for two levels of production and deteriorating items and shortages. Viji and Karthikeyan developed an economic production quantity model for three levels of production with Weibull distribution deterioration and shortage. Krishnamoorthi and Sivashankari (2016) developed production inventory models for deteriorating items with three levels of production and shortages, where they consider constant demand and deterioration in both during and after production. In reality, not all kinds of items deteriorate as soon as they are produced, but they maintain their freshness for some period before they begin to deteriorate and usually constant demand rate is valid in the (production period) mature stage of product's life cycle. In the end stage life cycle and / or after production stopping time the demand rate is sometime influenced by the stock level. It is usually observed that a large bunch of goods displayed on shelves in a shop will lead to a higher demand.

This paper is aimed to proposing a production inventory model for non-instantaneous deteriorating items in which two-phase of production rate are considered under constant demand during production time and stock dependent demand rate after production. Lower rate of production in the first phase is to avoid a large amount of inventory items, which helps in reducing holding cost and therefore provides a way of attaining consumer satisfaction and earning maximum profit. The necessary and sufficient conditions are provided to show the existence and uniqueness of the optimal solution. Then numerical example, sensitivity analysis and graphical presentation are also provided to illustrate the application of the proposed model.

2 Mathematical Description and Formulation

2.1 Assumptions and Notation

The inventory model is proposed under the following notation and assumptions

Notation

λ Constant production rate for the first phase of production in units per unit time

$r\lambda$ Constant production rate for the second phase of production in units per unit time

	where $r > 0$ is constant increase of production
$\alpha + \beta I(t)$	The linear stock dependent demand, where β $(0 < \beta < 1)$ is the stock dependent parameter, $\alpha > 0$
C_1	Production cost per unit
C_3	Holding cost per unit per unit time
c_2	Shortage cost per unit per unit time
θ	Deterioration rate θ $(\ 0 < \theta < 1)$ is constant
α	Constant demand rate during production period
t_1	The time at which first production stops
t_2	The time at which second production stops and deterioration sets in
t_3	The time at which inventory depletes to zero and shortage sets in
t_4	The time at which shortage stops and production restart again to recover both the shortages and to satisfy the demand in the interval $[t_4, T]$
Q_1	Inventory level at time t_1
Q_2	Inventory level at time t_2
P	Total production per cycle
B	Maximum shortage level
A	Setup cost per production cycle
T	Length of the production cycle
TC	Total variable cost of the inventory system

Assumptions

1. Two rates of production are considered and are both known and constant
2. The demand rate is known, constant and positive during productions period and linear stock dependent during depletion period
3. Items are produced and added to the inventory
4. The production rate is always greater than the demand rate
5. The inventory item is single product
6. Shortages are allowed and completely backlogged
7. Lead time is zero

2.2 Mathematical formulation of the Model

The production starts with zero stock level at time $t = 0$. In the first and second production time intervals $[0, t_1]$ and $[t_1, t_2]$ respectively, there is no deterioration. During time interval $[0, t_1]$, the production is $\lambda > \alpha$. Thus inventory accumulates at the rate of $\lambda - \alpha$ units. Therefore, the maximum inventory level equal to $(\lambda - \alpha)t_1$. During the time interval $[t_1, t_2]$, the production rate is $r\lambda > r\alpha$. Thus inventory accumulates at the rate of $r(\lambda - \alpha)$ units. Amount produced at time interval $[0, t_1]$ are λt_1 and also amount produced at time interval $[t_1, t_2]$ are $r\lambda(t_2 - t_1)$. The inventory level starts to deplete due to deterioration and demand with demand rate $(\alpha + \beta I(t))$ and ultimately falls to zero at time t_3. In the shortage period, unfulfilled demands start to accumulate at a rate of α up to timet_4. Production restarts again at time t_4 at a rate of λ to recover both the previous shortages in the time interval $[t_3, t_4]$ and to satisfy the demand in the interval$[t_4, T]$. The process is repeated and the behavior of the inventory model for one cycle is depictured in figure 1.

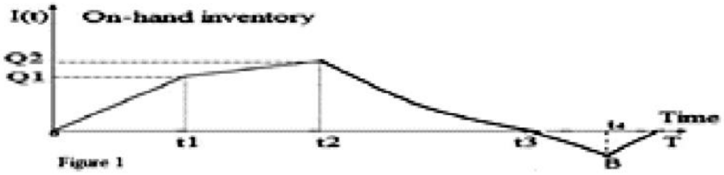

Figure 1

The differential equations describing the system in the interval are given in equation (1), (2), (3), (4) and (5).

$$\frac{dI(t)}{dt} = \lambda - \alpha, \qquad 0 \le t \le t_1 \qquad (1)$$

with conditions $I(0) = 0, and\ \ I(t_1) = Q_1$

$$\frac{dI(t)}{dt} = r(\lambda - \alpha), \qquad t_1 \le t \le t_2 \qquad (2)$$

with condition $I(t_2) = Q_2$

$$\frac{dI(t)}{dt} + \theta I(t) = -\big(\alpha + \beta I(t)\big), \qquad t_2 \le t \le t_3 \qquad (3)$$

with condition $I(t_3) = 0$

$$\frac{dI(t)}{dt} = -\alpha, \qquad t_3 \le t \le t_4 \qquad (4)$$

with condition $I(t_4) = B$

$$\frac{dI(t)}{dt} = (\lambda - \alpha), \qquad t_4 \le t \le T \qquad (5)$$

with condition $I(T) = 0$

From equation (1), we have

$$I(t) = \int (\lambda - \alpha)dt$$

$$= (\lambda - \alpha)t + k_1, \text{ where } k_1 \text{ is the constant of integration}$$

Using the condition $I(0) = 0$*, gives* $k_1 = 0$

$$I(t) = (\lambda - \alpha)t, \qquad 0 \le t \le t_1 \qquad (6)$$

Applying the condition $I(t_1) = Q_1$ *in equation* (1)*, we obtain*

$$Q_1 = (\lambda - \alpha)t_1 \qquad (7)$$

From equation (2), we have

$$I(t) = r(\lambda - \alpha)\int dt$$

$$= r(\lambda - \alpha)t + k_2, \text{ where } k_2 \text{ is the constant of integration} \qquad (8)$$

with condition $I(t_1) = Q_1$

$$Q_1 = r(\lambda - \alpha)t_1 + k_2 \qquad (9)$$

Using the condition $I(t_1) = Q_1$ in (6), we have

$$I(t_1) = (\lambda - \alpha)t_1 = Q_1 \qquad (10)$$

equating (8)and (9) leads to

$$(\lambda - \alpha)t_1 = r(\lambda - \alpha)t_1 + k_2$$

$$k_2 = (\lambda - \alpha)t_1 - r(\lambda - \alpha)t_1$$

Substituting $k_2 = (\lambda - \alpha)t_1 - r(\lambda - \alpha)t_1$ into above equation (7)

$$I(t) = r(\lambda - \alpha)t + (\lambda - \alpha)t_1 - r(\lambda - \alpha)t_1$$

$$= r(\lambda - \alpha)(t - t_1) + (\lambda - \alpha)t_1, \qquad t_1 \le t \le t_2 \qquad (11)$$

Using the condition $I(t_2) = Q_2$ from equation (2), we get

$$Q_2 = r(\lambda - \alpha)(t_2 - t_1) + (\lambda - \alpha)t_1 \tag{12}$$

equation (3) is first order linear differential equation whose integrating factor is $e^{(\theta+\beta)t}$ and so, we have

$$\frac{I(t)}{dt} + (\theta + \beta)I(t) = -\alpha$$

$$I(t)e^{(\theta+\beta)t} = -\alpha \int e^{(\theta+\beta)t}\, dt + k_3$$

$$I(t) = e^{-(\theta+\beta)t}\left[\frac{-\alpha e^{(\theta+\beta)t}}{\theta + \beta} + k_3\right]$$

Using the condition, $I(t_3) = 0$, we get

$$k_3 = \frac{\alpha}{\theta+\beta} e^{(\theta+\beta)t_3}$$

$$I(t) = \frac{\alpha}{\theta+\beta}\left(e^{(\theta+\beta)(t_3-t)} - 1\right), \qquad t_2 \le t \le t_3 \tag{13}$$

From equation (4)

$$\frac{dI(t)}{dt} = -\alpha$$

$$I(t) = -\int \alpha\, dt$$

$$= -\alpha t + k_4$$

Using the condition, $I(t_3) = 0$, we get

$$k_4 = \alpha t_3$$

$$I(t) = -\alpha(t - t_3), \qquad t_3 \le t \le t_4 \tag{14}$$

From equation (5)

$$\frac{dI(t)}{dt} = (\lambda - \alpha)$$

$$I(t) = (\lambda - \alpha)t + k_5$$

Using the condition, $I(T) = 0$, we get

$$k_5 = -(\lambda - \alpha)T$$

$$I(t) = -(\lambda - \alpha)(T - t), \qquad t_4 \le t \le T \tag{15}$$

Total amount of items produced for the entire cycle is given by

$$P = \int_0^{t_1} \lambda\, dt + \int_{t_1}^{t_2} r\lambda\, dt + \int_{t_4}^{T} \lambda\, dt$$

$$= \lambda t_1 + r\lambda(t_2 - t_1) + \lambda(T - t_4) \tag{16}$$

Holding cost per unit time is given by

$$Hc = \frac{C_3}{T}\left(\int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)dt + \int_{t_2}^{t_3} I(t)dt\right)$$

$$= \frac{C_3}{T}\left(\int_0^{t_1} (\lambda - \alpha)t\, dt + \int_{t_1}^{t_2} (r(\lambda - \alpha)(t - t_1) + (\lambda - \alpha)t_1)dt + \int_{t_2}^{t_3} \frac{\alpha}{(\theta + \beta)}\left[e^{(\theta+\beta)(t_3-t)} - 1\right]dt\right)$$

$$= \frac{\mathrm{C}_3}{T}\left(\left[\frac{(\lambda-\alpha)t^2}{2}\right]_0^{t_1} + r(\lambda-\alpha)\left[\frac{t^2}{2} - t_1 t\right]_{t_1}^{t_2} + [(\lambda-\alpha)t_1 t]_{t_1}^{t_2} - \frac{\alpha}{(\theta+\beta)}\left[\frac{e^{(\theta+\beta)(t_3-t)}}{(\theta+\beta)} + t\right]_{t_2}^{t_3}\right)$$

$$= \frac{\mathrm{C}_3}{T}\left(\frac{(\lambda-\alpha)t_1^2}{2} + \frac{r(\lambda-\alpha)(t_2-t_1)^2}{2} + (\lambda-\alpha)t_1(t_2-t_1) + \frac{\alpha}{(\theta+\beta)}\left[\frac{e^{(\theta+\beta)(t_3-t_2)}}{(\theta+\beta)} - (t_3-t_2) - \frac{1}{(\theta+\beta)}\right]\right)$$

$$= \frac{\mathrm{C}_3}{2T}\left((\lambda-\alpha)t_1^2 + r(\lambda-\alpha)(t_2-t_1)^2 + 2(\lambda-\alpha)t_1(t_2-t_1) + \frac{2\alpha}{(\theta+\beta)^2}\left[e^{(\theta+\beta)(t_3-t_2)} - (\theta+\beta)(t_3 - t_2) - 1\right]\right)$$ Type equation here.(17)

Deterioration cost per unit time is given by

$$Dc = \frac{\theta}{T}\int_{t_2}^{t_3} I(t)dt$$

$$= \frac{\theta}{T}\left[\int_{t_2}^{t_3} \frac{\alpha}{(\theta+\beta)}\left(e^{(\theta+\beta)(t_3-t)} - 1\right) dt\right]$$

$$= \frac{\theta}{T}\left(\frac{\alpha}{(\theta+\beta)}\left[\frac{e^{(\theta+\beta)(t_3-t_2)}}{(\theta+\beta)} - (t_3-t_2) - \frac{1}{(\theta+\beta)}\right]\right)$$

$$= \frac{\theta\alpha}{(\theta+\beta)^2 T}\left[e^{(\theta+\beta)(t_3-t_2)} - (\theta+\beta)(t_3-t_2) - 1\right] \tag{18}$$

Shortage cost per unit time is given as

$$Sc = \frac{C_2}{T}\left[\int_{t_3}^{t_4} -I(t)dt + \int_{t_4}^{T} -I(t)dt\right]$$

$$= \frac{C_2}{T}\left[\int_{t_3}^{t_4} \alpha(t-t_3)dt + \int_{t_4}^{T}(\lambda-\alpha)(T-t)dt\right]$$

$$= \frac{C_2}{T}\left[\alpha\left(\frac{t^2}{2} - t_3 t\right)_{t_3}^{t_4} + (\lambda-\alpha)\left(Tt - \frac{t^2}{2}\right)_{t_4}^{T}\right]$$

$$= \frac{C_2}{T}\left[\alpha\left(\frac{t_4^2}{2} - t_4 t_3 + \frac{t_3^2}{2}\right) + (\lambda-\alpha)\left(\frac{T^2}{2} - Tt_4 + \frac{t_4^2}{2}\right)\right]$$

$$= \frac{C_2}{2T}\left[\alpha(t_4-t_3)^2 + (\lambda-\alpha)(T-t_4)^2\right]$$

$$= \frac{C_2}{2T}\left[\alpha\left(\frac{\lambda-\alpha}{\lambda}T - \frac{\lambda-\alpha}{\lambda}t_3\right)^2 + \frac{\alpha(\lambda-\alpha)}{\lambda}(T-t_3)^2\right]$$

$$= \frac{\alpha(\lambda-\alpha)C_2}{2\lambda T}(T-t_3)^2 \tag{19}$$

Using condition $I(t_4) = B = -\alpha(t_4 - t_3)$ from equation (4) and (14) and $I(t_4) = B = -(\lambda-\alpha)(T-t_4)$ from equation (4) and (15) this leads to

$\alpha(t_4 - t_3) = (\lambda-\alpha)(T-t_4)$ (20)

now $t_4 = \frac{\lambda-\alpha}{\lambda}T + \frac{\alpha}{\lambda}t_3$ (21)

Total production cost $= \alpha\mathrm{C}_1$ (22)

Setup cost per set= $\frac{A}{T}$ (23)

Total variable cost per unit time is given by

$$\mathrm{TC} = [production\ cost/unit\ time + (setup\ cost + holding\ cost + deterioration\ cost + Shortage\ cost\ )/T]$$

$$TC = \alpha C_1 + \frac{A}{T} + \frac{C_3}{2T}\Big((\lambda-\alpha)t_1^2 + r(\lambda-\alpha)(t_2-t_1)^2 + 2(\lambda-\alpha)t_1(t_2-t_1)\Big) + \frac{C_3}{2T}\Big(\frac{2\alpha}{(\theta+\beta)^2}\big[e^{(\theta+\beta)(t_3-t_2)} - (\theta+\beta)(t_3-t_2) - 1\big]\Big) + \frac{\theta\alpha}{(\theta+\beta)^2 T}\big[e^{(\theta+\beta)(t_3-t_2)} - (\theta+\beta)(t_3-t_2) - 1\big] + \frac{\alpha(\lambda-\alpha)C_2}{2\lambda T}(T-t_3)^2 \tag{24}$$

$$TC = \alpha C_1 + \frac{A}{T} + \frac{C_3}{2T}\big((\lambda-\alpha)t_1^2 + r(\lambda-\alpha)(t_2-t_1)^2 + 2(\lambda-\alpha)t_1(t_2-t_1)\big) + \frac{(C_3+\theta)\alpha}{(\theta+\beta)^2 T}\big[e^{(\theta+\beta)(t_3-t_2)} - (\theta+\beta)(t_3-t_2) - 1\big] + \frac{\alpha(\lambda-\alpha)C_2}{2\lambda T}(T-t_3)^2 \tag{25}$$

$$Let\ t_1 = \xi_1 t_3\ and\ t_2 = \xi_2 t_3\ such\ that\ 0 < \xi_1, \xi_2 < 1\ \ and\ T > t_4 > t_3 > t_2 > t_1 \tag{26}$$

$$TC = \alpha C_1 + \frac{A}{T} + \frac{C_3}{2T}\Big((\lambda-\alpha)\xi_1^2 + r(\lambda-\alpha)(\xi_2-\xi_1)^2 + 2(\lambda-\alpha)\xi_1(\xi_2-\xi_1)\Big)t_3^2 + \frac{(C_3+\theta)\alpha}{(\theta+\beta)^2 T}\big[e^{(\theta+\beta)(1-\xi_2)t_3} - (\theta+\beta)(1-\xi_2)t_3 - 1\big] + \frac{\alpha(\lambda-\alpha)C_2}{2\lambda T}(T-t_3)^2 \tag{27}$$

The problem now is to minimize the cost function TC represented by equation (27) subject to $T > t_3 > t_2 > t_1 > 0$

3 Optimal Decision

The necessary conditions for the optimum values are

$$\frac{\partial TC}{\partial T} = 0\ and\ \frac{\partial TC}{\partial t_3} = 0 \tag{28}$$

The sufficient conditions for the existence of minima are

$$\frac{\partial^2 TC}{\partial T^{*2}} > 0\ , \frac{\partial^2 TC}{\partial t_3^{*2}} > 0\ and\ \frac{\partial^2 TC(T^*, t_3^*)}{\partial T^* \partial t_3^*} > 0 \tag{29}$$

$from\ (27)\ \frac{\partial TC}{\partial T} = 0\ \ gives\ ,$

$$-\frac{A}{T^2} - \frac{C_3}{2T^2}\big((\lambda-\alpha)\xi_1^2 + r(\lambda-\alpha)(\xi_2-\xi_1)^2 + 2(\lambda-\alpha)\xi_1(\xi_2-\xi_1)\big)t_3^2 - \frac{(C_3+\theta)\alpha}{(\theta+\beta)^2 T^2}\big[e^{(\theta+a)(1-\xi_2)t_3} - (\theta+a)(1-\xi_2)t_3 - 1\big] + \frac{\alpha(\lambda-\alpha)C_2}{2\lambda T^2}(T^2 - t_3^2) \quad = 0 \tag{30}$$

Multiplying equation (30) by T^2 we get,

$$-A-\frac{\mathrm{C}_3}{2}\Big((\lambda-\alpha)\xi_1^2+r(\lambda-\alpha)(\xi_2-\xi_1)^2+2(\lambda-\alpha)\xi_1(\xi_2-\xi_1)\Big)t_3^2 \\ -\frac{(\mathrm{C}_3+\theta)\alpha}{(\theta+\beta)^2}\left[e^{(\theta+a)(1-\xi_2)t_3}-(\theta+a)(1-\xi_2)t_3-1\right]+\frac{\alpha(\lambda-\alpha)C_2}{2\lambda}(T^2-t_3^2) \quad =0 \tag{31}$$

$from\ (27)\ \frac{\partial TC}{\partial t_3}=0\ gives,$

$$\frac{\mathrm{C}_3}{T}\Big((\lambda-\alpha)\xi_1^2+r(\lambda-\alpha)(\xi_2-\xi_1)^2+2(\lambda-\alpha)\xi_1(\xi_2-\xi_1)\Big)t_3 \\ +\frac{(\mathrm{C}_3+\theta)(1-\xi_2)\alpha}{(\theta+\beta)T}\left[e^{(\theta+\beta)(1-\xi_2)t_3}-1\right]-\frac{\alpha(\lambda-\alpha)C_2}{\lambda T}(T-t_3) \ =0 \tag{32}$$

Multiplying equation (32) by T and substituting the constant values and solved for T, we have

$$T=\frac{(\Delta_1+\Delta_4)t_3+\Delta_3(e^{\mu t_3}-1)}{\Delta_4} \tag{33}$$

where,

$\Delta_1=\mathrm{C}_3\big((\lambda-\alpha)\xi_1^2+r(\lambda-\alpha)(\xi_2-\xi_1)^2+2(\lambda-\alpha)\xi_1(\xi_2-\xi_1)\big),$

$\Delta_2=\frac{(\mathrm{C}_3+\theta)\alpha}{(\theta+\beta)^2},$

$\Delta_3=\frac{(\mathrm{C}_3+\theta)(1-\xi_2)\alpha}{(\theta+\beta)},$

$\Delta_4=\frac{\alpha(\lambda-\alpha)C_2}{\lambda},$

$\mu=(\theta+\beta)(1-\xi_2),$

$\Delta_3=\mu\Delta_2,$

Substituting equation (33) $into$ (31) and called it equation (34)

$$-2A-\Delta_1 t_3^2-2\Delta_2(e^{\mu t_3}-\mu t_3-1)-\Delta_4 t_3^2+\frac{\left[(\Delta_1+\Delta_4)t_3+\Delta_3(e^{\mu t_3}-1)\right]^2}{\Delta_4}\ =0 \tag{34}$$

Theorem 1. If we let $G(t_3)=-2A-\Delta_1 t_3^2-2\Delta_2(e^{\mu t_3}-\mu t_3-1)-\Delta_4 t_3^2+\frac{\left[(\Delta_1+\Delta_4)t_3+\Delta_3(e^{\mu t_3}-1)\right]^2}{\Delta_4}$ then the solution of $G(t_3)=0$, which satisfies equation (34), exist and unique.

Proof.
Suppose
$G(t_3)=2\Delta_4(\Delta_2-A)+\Delta_3^2(e^{\mu t_3}-1)^2+2e^{\mu t_3}[(\Delta_1+\Delta_4)\Delta_3 t_3-\Delta_2\Delta_4]+\Delta_1 t_2[(\Delta_1+\Delta_4)t_3-2\Delta_3]$
from equation (34)
$G(0)=-2\Delta_4 A<0$
and
$\lim_{t_3\to\infty} G(t_3)=+\infty$

Differentiating equation (34) yields
$G'(t_3) = 2\Delta_3^2\mu e^{kt_3}(e^{\mu t_3} - 1) + 2\Delta_3 e^{\mu t_3}[(\Delta_1 + \Delta_4 t_3)\mu + \Delta_1] + 2\Delta_1[(\Delta_1 + \Delta_4)t_3 - 2\Delta_3] > 0$
Thus implies G is increasing and continuous on [0,∞), hence by intermediate value theorem, there exists a unique solution $t_3^* \in$[0,∞) such that $G(t_3^*) = 0$. This completes the proof.

Theorem 2. $TC(t_3, T)$ is global minimum at the optimal point (t_3^*, T^*) which satisfied equation (31) and (32).

Proof.
For the global minimum point of $TC(t_3, T)$, we show that the principal minors are strictly positive at the optimum point (t_3^*, T^*).
i.e

$$\left.\frac{\partial^2 TC(T,t_3)}{\partial T^2}\right|_{(T,\, t_3)=(T^*,t_3^*)} > 0 \text{ and } \frac{\partial^2 TC(T^*,t_3^*)}{\partial T^{*2}}\frac{\partial^2 TC(T^*,t_3^*)}{t_3^{*2}} - \left(\frac{\partial^2 TC(T^*,t_3^*)}{\partial T^*\partial t_3^*}\right)^2 > 0$$

Now, we will obtain the first, second and the mixed derivatives of $TC(t_3, T)$ with respect to t_3 and T.
From equation (30) we have,

$$TC(t_3,T) = \alpha C_1 + \frac{A}{T} + \frac{\Delta_1}{2T}t_3^2 + \frac{\Delta_2}{T}(e^{\mu t_3} - \mu t_3 - 1) + \frac{\Delta_4}{2T}(T - t_3)^2$$

$$\frac{\partial TC(t_3,T)}{\partial T} = -\frac{A}{T^2} - \frac{\Delta_1}{2T^2}t_3^2 - \frac{\Delta_2}{T^2}(e^{\mu t_3} - \mu t_3 - 1) + \frac{\Delta_4}{2T^2}(T^2 - t_3^2) \qquad = 0,$$

$$\frac{\partial^2 TC(T,t)}{\partial T^2} = \frac{2A}{T^3} + \frac{\Delta_1}{T^3}t_3^2 + \frac{2\Delta_2}{T^3}(e^{\mu t_3} - \mu t_3 - 1) + \frac{\Delta_4}{T^3}t_3^2,$$

$$\frac{\partial TC(t_3,T)}{\partial t_3} = \frac{\Delta_1}{T}t_3 + \frac{\Delta_3}{T}(e^{\mu t_3} - 1) - \frac{\Delta_4}{T}(T - t_3),$$

$$\frac{\partial^2 TC(T,t)}{\partial {t_3}^2} = \frac{\Delta_1}{T} + \frac{\Delta_3\mu}{T}e^{\mu t_3} + \frac{\Delta_4}{T}$$

and

$$\frac{\partial^2 TC(T,t_3)}{\partial T\partial t_3} = -\left(\frac{\Delta_1}{T^2}t_3 + \frac{\Delta_3}{T^2}(e^{\mu t_3} - 1) + \frac{\Delta_4}{T^2}t_3\right)$$

Now

$$\frac{\partial^2 TC(T,t)}{\partial T^2}\frac{\partial^2 TC(T,t)}{\partial {t_3}^2} = \left(\frac{2A}{T^3} + \frac{\Delta_1}{T^3}t_3^2 + \frac{2\Delta_2}{T^3}(e^{\mu t_3} - \mu t_3 - 1) + \frac{\Delta_4}{T^3}t_3^2\right)\left(\frac{\Delta_1}{T} + \frac{\Delta_3\mu}{T}e^{\mu t_3} + \frac{\Delta_4}{T}\right)$$

$$= \frac{2A\Delta_1}{T^4} + \frac{\Delta_1^2}{T^4}t_3^2 + \frac{2\Delta_1\Delta_2}{T^4}(e^{\mu t_3} - \mu t_3 - 1) + \frac{2\Delta_3^2}{T^4}e^{\mu t_3}(e^{\mu t_3} - \mu t_3 - 1) + \frac{2\Delta_2\Delta_4}{T^4}(e^{\mu t_3} - \mu t_3 - 1) + \frac{2\Delta_1\Delta_4}{T^4}t_3^2 + \frac{2A\Delta_3\mu}{T^4}e^{\mu t_3} + \frac{\Delta_1\Delta_3\mu}{T^4}e^{\mu t_3}t_3^2 + \frac{\Delta_3\Delta_4\mu}{T^4}e^{\mu t_3}t_3^2 + \frac{2A\Delta_4}{T^4} + \frac{\Delta_4^2}{T^4}t_3^2$$

and

$$\left(\frac{\partial^2 TC(T,t_3)}{\partial T\partial t_3}\right)^2 = \left(\frac{\Delta_1}{T^2}t_3 + \frac{\Delta_3}{T^2}(e^{\mu t_3} - 1) + \frac{\Delta_4}{T^2}t_3\right)\left(\frac{\Delta_1}{T^2}t_3 + \frac{\Delta_3}{T^2}(e^{\mu t_3} - 1) + \frac{\Delta_4}{T^2}t_3\right)$$

$$= \left(\frac{\Delta_1}{T^2}t_3 + \frac{\Delta_3}{T^2}(e^{\mu t_3} - 1) + \frac{\Delta_4}{T^2}t_3\right)^2$$

Therefore,

$$
\begin{aligned}
|H| &= \frac{\partial^2 TC(T,t)}{\partial T^2}\frac{\partial^2 TC(T,t)}{\partial {t_3}^2} - \left(\frac{\partial^2 TC(T,t_3)}{\partial T \partial t_3}\right)^2 \\
&= \frac{2\Delta_3\Delta_4}{T^4}t_3 + \frac{2\Delta_1\Delta_2}{T^4}(e^{\mu t_3} - \mu t_3 - 1) + \frac{\Delta_3^2}{T^4}((e^{\mu t_3})^2 - 1) + \frac{2\Delta_2\Delta_4}{T^4}(e^{\mu t_3} - \mu t_3 - 1) \\
&\quad + \frac{2A\Delta_3\mu}{T^4}e^{\mu t_3} + \frac{\Delta_1\Delta_3}{T^4}e^{\mu t_3}(\mu t_3^2 - 2) + \frac{\Delta_3\Delta_4}{T^4}e^{\mu t_3}\big(t_3(\mu t_3 - 2)\big) + \frac{2A\Delta_1}{T^4} + \frac{2A\Delta_4}{T^4} \\
&\quad + \frac{2\Delta_1\Delta_3}{T^4} > 0
\end{aligned}
$$

Lemma 1. $e^{\mu t_3} - \mu t_3 - 1$ Is an increasing function with respect to t_3.

Proof.

Let $q(x) = e^x - x - 1$

where,

$x = \mu t_3 \; \forall \, x \geq 0$

$q(x) = e^x - x - 1$

$q'(x) = e^x - 1$

$q'(0) = 0$

again,

$q''(x) = e^x$

$q'(x) \geq q'(0) = 0 \;\; \forall \, x \geq 0$

Hence $q(x)$ is an increasing function of $x, \forall \, x \geq 0$

Thus $q(x) \geq q(0) = 0, \forall \, x \geq 0$

Implies $e^{\mu t_3} - \mu t_3 - 1$ is an increasing function.

lemma 2. $\mu t_3 - 2$ is an increasing function.

Proof.

Let $p(t_3) = \mu t_3 - 2$

$p'(t_3) = \mu > 0$

Hence $p(t_3) > 0$

Thus, from the above lemmas, the principal minors are strictly greater than zero.

Therefore,

$$
\begin{aligned}
|H| = \left.\frac{\partial^2 TC(T,t_3)}{\partial T^2}\right|_{(T,\,t_3)=(T^*,t_3^*)} &\times \left.\frac{\partial^2 TC(T,t_3)}{\partial t_3^2}\right|_{(T,\,t_3)=(T^*,t_3^*)} \\
&- \left[\left.\frac{\partial^2 TC(T^*,t_3^*)}{\partial T \partial t_3}\right|_{(T,\,t_3)=(T^*,t_3^*)}\right]^2 > 0
\end{aligned}
$$

Hence, the Hessian matrix H at point (t_3^*, T^*) is positive definite. Consequently, we can conclude that the stationary point for our optimization problem is a global minimum.

4 Solution Procedure and Algorithm

We can now use equation (34) to solve for t_3 using Newton-Raphson method since is highly non-linear, substituting the solution of (34) in to (33) to compute T. These solutions $T^* and\ t_3^*$ will together make the optimal solution of $(31) and$ (32) provided equation (28) are satisfied.

Step I: From equation (34) a unique optimum value t_3^* is obtained

Step II: We used solution of step I above in equation (33) to compute T^*

Step III: We used the solution of step I and step II in equation (21) and compute t_4^*, and the

same step I and step II in to equation (26) and compute t_1^* and t_2^*
Step IV: The values of t_3^* obtained in step I and values of T^* obtained in step II and also t_1^* and t_2^* obtained in step III are substituted in to equation (25) to get TC^*. Moreover, using step I and step III above, we obtained the maximum inventory level for the first and second phase of production and total produced items per cycle.

5 Numerical example

Let us consider the cost parameters $\lambda = 500\, units, \alpha = 200\, units, C_1 = 10, C_3 = 3, C_2 = 10\ \theta = 0.7, A = 1500, r = 5, \beta\ = 0.8, \xi 1 = 0.2\ \xi 2 = 0.3$.

From equations $(25), (30) and\ (32)$ above Cycle Time $T^* = 2.293419, t_1^* = 0.262411, t_2^* = 0.393617; t_3^* = 1.312057; t_4^* = 1.900874$ Total item produced $P^* = 655.4923$, $Q_1 = 78.72341$, $Q_2 = 275.5319$; Production cost= 2000, Setup cost= 654.0453, holding cost= 228.5499, Deterioration cost= 43.08231, shortage cost =251.95702 Total cost= 3177.635.

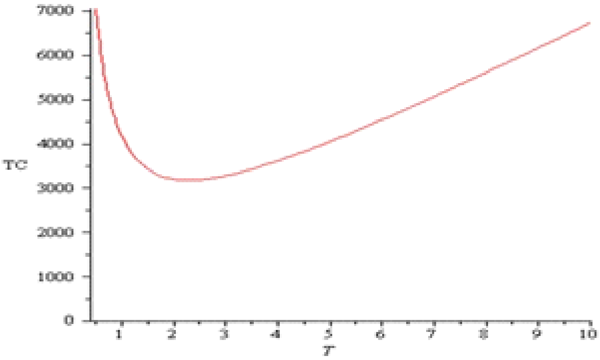

figure 2. The graphical representation of cost function versus total cycle length for the model with complete shortages

6 Sensitivity Analysis

Table 1. Sensitivity analysis based on the example.

		Optimum values					
Parameters	Percentage change %	T^*	t_3^*	Q_1^*	Q_2^*	P^*	TC^*
θ	-20	2.081	4.826	4.825	4.825	2.922	-0.676
	-10	0.873	2.068	2.068	2.068	1.235	-0.284
	10	-0.985	-2.413	-2.413	-2.413	-1.409	0.322
	20	-1.934	-4.827	-4.827	-4.827	-2.786	0.634
β	-20	1.742	3.849	3.849	3.849	2.385	-0.450
	-10	0.856	1.919	1.919	1.919	1.178	-0.222
	10	-0.828	-1.908	-1.908	-1.908	-1.150	0.216
	20	-1.629	-3.806	-3.806	-3.806	-2.273	0.427
	-20	0.114	0.412	0.412	-16.186	-10.921	-0.107

r	-10	0.057	0.206	0.206	-7.470	-5.166	-0.053
	10	-0.057	-0.207	-0.207	6.474	4.664	0.053
	20	-0.114	-0.414	-0.414	12.138	8.895	0.106
A	-20	-9.876	-7.544	-7.544	-7.544	-9.165	-4.503
	-10	-4.541	-3.463	-3.463	-3.463	-4.215	-2.149
	10	3.930	2.988	2.988	2.988	3.649	1.977
	20	7.377	5.602	5.602	5.602	6.851	3.807
c_3	-20	2.800	7.570	7.570	7.570	4.283	-1.588
	-10	1.341	3.742	3.742	3.742	2.074	-0.754
	10	-1.239	-3.667	-3.667	-3.667	-1.956	0.688
	20	-2.391	-7.269	-7.269	-7.269	-3.808	1.318
C_2	-20	6.019	-2.983	-2.983	-2.983	3.486	-1.86
	-10	2.821	-1.351	-1.351	-1.351	1.605	-0.856
	10	-2.512	1.138	1.138	1.138	-1.388	0.739
	20	-4.765	2.110	2.110	2.111	-2.601	1.384
α	-20	5.324	6.320	17.341	17.341	-0.276	-17.181
	-10	2.423	3.141	9.195	9.195	-0.329	-7.823
	10	-1.970	-3.138	-10.505	-10.505	0.710	6.640
	20	-3.450	-6.308	-22.663	-22.663	1.809	12.344
λ	-20	5.339	-0.481	-50.721	-50.721	-11.286	-1.946
	-10	2.290	-0.055	-20.067	-20.067	-5.684	-0.847
	10	-1.812	-0.166	14.144	14.144	5.392	0.693
	20	-3.304	-0.475	24.644	24.644	10.366	1.283

7 RESULT OBTAINED FROM THE MODEL

The sensitivity analysis is performed by changing the value of each of the parameters by –20%, –10%, 10%, and 20%, taking one parameter at a time and keeping the remaining parameters unchanged. Using the numerical example given above, a sensitivity analysis is performed to explore the sensitiveness of the decision variables to the model parameters. On the basis of the results, the following observations can be made.

(i) If the rate of deterioration decreases (increases), then $T^*, t_3^*, P^*, Q_1^*\ and\ Q_2^*$ increase (decrease) but TC^* decreases (increases). This result is expected since when deterioration cost increases, total variable cost per unit time will increase.

(ii) If the stock dependent demand rate decreases (increase), then TC^* decreases (increases) but $T^*,\ t_3^*,\ P^*,\ Q_1^*\ and\ Q_2^*$ increase (decrease) at higher percentage and in this case, stock dependent demand rate is highly sensitive.

(iii) If the set-up cost per production cycle is increases (decreases) then $TC^*, T^*, t_3^*,\ P^*$, $Q_1^*\ and\ Q_2^*$ increase (decrease). Then total variable cost per unit time is therefore expected to increase due to increase in stocking cost.

(iv) If the holding cost per unit / unit time increases (decreases) then $T^*, t_3^*,\ P^*, Q_1^*\ and\ Q_2^*$ will decrease (increase) but TC^* will increases (decreases).

This results is also expected since higher holding cost of the items discourage production and so reduces the profit

(v) If the constant increase of production parameter increases (decreases) then T^*, t_3^* and Q_1 decreases (increase) but P^* and Q_2^* increase (decrease) at higher percentage and TC^* is increases (decreases) at lower percentage.

(vi) If the constant demand rate during production period increases (decreases) then, T^*, t_3^*, Q_1^* and Q_2^* decrease (increase) but TC^* and P^* increases (decreases)

(vii) If the constant production rate is increases (decreases) then, T^* and t_3^* decreases (increases) but TC^*, P^*, Q_1^* and Q_2^* increases (decreases).

(viii) If the shortage cost per unit / unit time increases (decreases) then, T^* and P^* decreases (increases), but TC^*, t_3^*, Q_1^* and Q_2^* increases (decreases)

Concluding remarks

The proposed model extends the model of existing literature with two phases of production rate, constant demand rate, constant deterioration and / or weibull deterioration. In reality, not all kinds of items deteriorate as soon as they are produced, but they maintain their originality for some period before they begin to deteriorate such items are meat, bread, cloths, cassava, and so on, we also observed that constant demand rate is valid in the mature stage of product's life cycle. In the end stage life cycle and / or after production stopping time the demand rate is sometimes influenced by the stock level. It is usually observed that a large bunch of goods displayed on shelves in a shop will lead to a higher demand. In this model, a production inventory model for non-instantaneous deteriorating items in which two phases of production and complete shortages are considered, with constant demand during production and stock dependent demand rate after production, existence and uniqueness of the solution were obtained. Numerical example and sensitivity analysis to validate the model were also obtained. Furthermore, the model may be extended by considering time varying deterioration, quadratic demand rate, reliability of the items, inflations, etc.

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