Abacus (Mathematics Science Series) Vol. 44, No 1, Aug. 2019

A NOTE ON STRONGLY INVARIANT SUBGROUP

by

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Abstract.

In this paper,we formulate some results on strongly invariant subgroup. We show that diagonal subgroups are not strongly invariant, the union of a strongly invariant subgroup of a group G and a direct factor of $G$is not strongly invariant.We establish that every subgroup A of a torsion group G is strongly invariant if $A[n]$ is strongly invariant in A. The union and the intersection of a torsion part of a mixed group are strongly invariant. We finally show that every cyclic group of prime order is strongly invariant simple.

1. Introduction

The concept of fully invariant subgroup was introduced by F. Levi under the German namevollinvariant in[1].Fully invariant extending property (FI-extending property) for abelian groups was studied in[2], where it was proved thata torsion group has the FI-extending property if it is a direct sum of a divisible group and separable p-groups, every summand of a group with theFI-extending property enjoys theFI-extending property, a mixed abelian group has theFI-extending property if it is a direct sum of torsion and torsion-free Abelian group, both with the FI-extending property.

Chekhlov in[3], described the intermediately fully invariant subgroup (ifi-subgroup) of divisible, torsion and torsion-free groups, it is shown thatsum of ifi-subgroups is againifi-subgroup.The intersection of the subgroup N with the subgroup H of a group $G = H \oplus K$ such that$N = (N \cap H)\oplus(N \cap K)$ is an ifi-subgroup in H. Furthermore, in a torsion group Ga subgroup H is intermediately inert in G if every p-component of H is intermediately inert in p-component of G, and finally, every homogeneous separable torsion-free group ofrank ≥ 2 isifi-simple.

The notion of strongly invariant subgroups ofAbelian groupswas introduced and studied in [4] as an extension of fully invariant subgroups, and therein, it was shown that ina torsion group Ga subgroup A is strongly invariant if p-componenet of A is strongly invariant in$p$-component of G. For a reduced p-group the only strongly invariant subgroups are the subgroups $G[p^n]$. The intersection of the strongly invariant subgroup N and the subgroup H of a group$G = H \oplus K$ such that $N = (N \cap H) \oplus (N \cap K)$ is strongly invariant in H.For atorsion-free group, a subgroup A is not strongly invariant if it contains free direct summand. It was also discovered that rank 2 torsion-free group has no cyclic strongly invariant subgroup. The torsion part of subgroup N of a mixed group G is strongly invariant in torsion part of G if it is strongly invariant in G and infinite cyclic subgroups of G and a subgroup that contains a free direct summand are not strongly invariant.More results like; a p-group is fully invariant simple if it is elementary, a torsion group is strongly invariant simple if it is an elementary$p$-group, genuine mixed groups are not strongly invariant simple and any torsion-free divisible group is strongly invariant simple are established.

The strongly invariant subgroup of torsion-free groups was studied in [5].Some results like; in a divisible torsion-free group every fully invariant subgroup is strongly invariant, every homogeneous separable torsion-free group is strongly invariant simple, every strongly invariant subgroup coincides with some direct summand of the group, and the sum of strongly invariant subgroups is again strongly invariant subgroup.

This paper extend some of the results in[4]. We study the strongly invariant subgroup of the direct product of two subgroups, the strongly invariant subgroup of torsion group, mixed group and in particular, those of strongly invariant simple group and obtain some results.

2. Basic Definitions

Definition 1. (Fully invariant subgroup)

A subgroup B of a group A that is carried into itself by every endomorphism of A is said to be a fully invariant subgroup of A. [6].

*Example 2.1.*Commutator subgroups are fully invariant, in a cyclic group every subgroup is fully invariant, and every group is fully invariant as subgroup of itself.

Definition 2. (Strongly invariant subgroup)

A subgroup N of a group Gwill be called strongly invariant in G, if$f(N) \leq N$ for every group homomorphism $f : N \longrightarrow G$. [4].

*Example 2.2.*Normal Sylow-subgroup, normal Hall subgroup are strongly invariant and the center of the quaternion groupQ_8 is a strongly invariant subgroup ofQ_8.

Definition 3. (Torsion group)

An Abelian group is called a torsion or periodic group if every element of A is of finite order.[6].

*Example 2.3.*Every finite group is periodic.

Definition 4. (Torsion-free group)

An Abelian group is called a torsion-free if all its elements, except for 0, are of infinite order. [6].

*Example 2.4.*The set of integers under addition is a torsion-free group.

Definition 5.(Mixed group) An Abelian group is called mixed group if it contain both nonzeroelementsof finite order and elements of infinite order.[6].

*Example 2.5.*The group□ = □$_□$ ⊕ □ = {(□′, □), □′ ∈ □$_□$ and □ ∈ □} is a mixed group.

Definition 6. (Strongly invariant simple group)

An Abelian group is said to be strongly invariant simple if it has non nontrivial strongly invariant subgroup.[5].

*Example 2.6.*The group □ = □$_2$ ⊕ □$_2$ is strongly invariant simple group.

Definition 7. (Direct factor)

A subgroup □ of a group □ is called direct factor of □ if there is a subgroup □ of a group □ such that □ is the internal direct product of □ and □. [7].

*Example 2.7.*Let □ = □$_6$ = {0, 1, 2, 3, 4, 5} and □ = {0, 2, 4}be a subgroup of □ then □ is a direct factor of □ since there is □ = {0, 3}in □ with □ = □ □ and □ ∩ □ = {□}.

Definition 8. (Diagonal subgroup)

Let □ = □ ⊕ □ be a direct product of two isomorphic groups □ and □ then a subgroup □ of □is called diagonal subgroup if □ □= □ = □ □and □ ∩ □ = □ = □ ∩ □. [7].

*Example 2.8*Let □ = {□, □, □, □} be a Klein four-group and let□ = {□, □}and □ = {□, □}be two subgroups of □ such that □ = □ ⊕ □, then a subgroup □ = {□, □} is a diagonal subgroup of □.

□. Some Existing Results

Theorem □. □.Fully invariant direct factors are strongly invariant.[*4*].

Theorem □. □.If a group □ = □ ⊕ □ and □ is fully invariant subgroup of □, then□ = (□ ∩ □) ⊕ (□ ∩ □). [*6*].

Theorem □. □. Any sum of strongly invariant subgroups is a strongly invariant subgroup.[*5*].

Theorem □. □.Let □ be strongly invariant subgroup of a group □ = □ ⊕ □, then□ = (□ ∩ □) ⊕ (□ ∩ □) and □ ∩ □,□ ∩ □are strongly invariant in □ and □ respectively. Conversely, if $□_1$ and $□_1$are strongly invariant subgroups of □ and □ respectively, then$□_1$ ⊕ $□_1$is strongly invariant in □ if and only if for every □ : $□_1$ ⟶ □, □($□_1$) ≤ $□_1$and for every □ : $□_1$ ⟶ □, □($□_1$) ≤ $□_1$. [*4*].

Lemma □. □. In any group □, for any positive integer □, the subgroup□[□] = {□ ∈ □ : □ □= *0*} is strongly invariant in □. [*4*].

Theorem □. □.Let □ be a subgroup of a torsion group □. Then □ is strongly invariant subgroup of □ if and only if for every prime *p*, $□_□$ is strongly invariant in$□_□$.[*4*].

Theorem □. □.Let □ be a subgroup of a mixed group □. Then □(□) is strongly invariant subgroup of □(□) if and only if □(□) is strongly invariant subgroup of □.[*4*].

Theorem □. □. [*4*].

i. A □-group □ is fully invariant simple if and only if it is elementary.
ii. Genuine mixed groups are not strongly invariant simple.
iii. A torsion group is strongly invariant simple if and only if it is an elementary □-group.

□. Main Results

Theorem □. □.Let □ be strongly invariant subgroup of a group □ = □ ⊕ □, and□ = (□ ∩ □) ⊕ (□ ∩ □) then □ ∪ □ and□ ∪ □are not strongly invariant in □ and □ respectively if and only if □ ∩ □ ≠ {□} and □ ∩ □ ≠ {□}.

Proof. Suppose □ ∪ □ is strongly invariant in □ then □ ∪ □ ≤ □ this implies □ ⊆ □. But since □ ∩ □ ≠ {□} there exists □ ∈ □ ∩ □ such that □ ∉ □ and so □ ⊈ □ which is a contradiction. Therefore, □ ∪ □is not strongly invariant in □. Similarly, □ ∪ □is not strongly invariant in □.Conversely,if □ ∩ □ = {□} and □ ∩ □ = {□}then □ ∪ □ and□ ∪ □are strongly invariant in □ and □ respectively, which is a contradiction. Hence, □ ∩ □ ≠ {□} and □ ∩ □ ≠ {□}.

Theorem □. □.Let □ = □ ⊕ □ be the direct product of two isomorphic groups, then the diagonal subgroup □ of □ is not strongly invariant.

Proof. Suppose □ is strongly invariant subgroup of □ then □ = (□ ∩ □) ⊕ (□ ∩ □) since□ is a diagonal subgroup of □, we have □ □= □ = □ □and □ ∩ □ = □ = □ ∩ □ and so□ ≠ (□ ∩ □) ⊕ (□ ∩ □).

Theorem □. □.Let □ be a subgroup of a torsion group □. Then □ is strongly invariant subgroup of □ if and only if for any positive integer *n*, □[□] is strongly invariant in □[□].

Proof. *Necessity*. For any positive integer □, let □ ∶ □[□] ⟶ □[□] be a homomorphism then □(□[□]) ≤ □[□]. Since □[□] ≤ □ and □[□] ≤ □ we can extend the mapping□ ∶ □ ⟶ □ and by hypothesis □(□) ≤ □. Thus, we have □(□[□]) ≤ □ and so□(□[□]) ≤ □ ∩ □[□] = □[□].

Sufficiency. Let □ ∶ □ ⟶ □ be a group homomorphism. Since □[□] is a subgroup of □ and □[□] is a subgroup of □ we can restrict the mapping to □ ∶ □[□] ⟶ □[□] and by hypothesis□(□[□]) ≤ □[□] for any positive integer □. Therefore, □(□) ≤ □.

Theorem □. □.Let □ and □ be subgroups of a mixed group □ such that □ ≤ □ ≤ □. If □(□)is strongly invariant in □(□), then □(□) ∩ □(□) is strongly invariant in □.

Proof. Let□(□)be strongly invariant subgroup of □(□) then □(□) ≤ □(□) and so□(□) ∩ □(□) = □(□). From theorem 3.7 we infer that □(□) ∩ □(□) is strongly invariant in □.

Theorem □. □.Let □ and □ be subgroups of a mixed group □. If □(□) and □(□) are strongly invariant in □ then □(□) ∪ □(□) is strongly invariant in □ if and only if □(□)and □(□) are comparable.

Proof. Let □(□) ∪ □(□)be strongly invariant in □ then □(□) ∪ □(□) ≤ □.If □(□)and □(□) are not comparable we have □(□) ∪ □(□) ≰ □, which is a contradiction. Therefore,□(□)and □(□) are comparable. Conversely, Suppose □(□)and □(□) are comparable such that □(□) ⊆ □(□) then □(□) ∪ □(□) = □(□) and so □(□) ∪ □(□) is strongly invariant in □. Similarly, if □(□) ⊆ □(□) also□(□) ∪ □(□) is strongly invariant in □.

Theorem □. □.Every cyclic group of prime order □ is strongly invariant simple.

Proof. Let □ be a cyclic group with a generator □ ∈ □ then ∀ □, □ ∈ □, ∃ □, □ ∈ □ such that □ = □ □ and y = m□ it follows that

□ + □ = □ □+ □ □= □(□ + □) = □(□ + □) = □ □+ □ □= □ + □.

Hence, □ is abelian.

Let □ be any subgroup of □. Since □ is cyclic for every homomorphism □ ∶ □ ⟶ □, there exists □ ∈ □ such that ∀ □ ∈ □, □(□) = □□ ∈ □ this implies □is strongly invariant.But, since □(□) = □ by Lagrange's theorem □(□)|□ and so □ is a trivial subgroup. Hence □ is strongly invariant simple.

□. Conclusion and Recommendation

In this note we discussed the conditions for a subgroup □ of a group □ which is the direct product of its two subgroups to be strongly invariant. The concept of torsion Abelian group, mixed Abelian group and strongly invariant simple Abelian group have been discussed and some related results were obtained. We recommend for the investigationof theunion and intersection of strongly invariant subgroups of torsion-free group, torsion group and strongly invariant subgroups of non Abelian groups.

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