DETERMINING NILPOTENCY CLASS OF LOOPS OF SMALL ORDER

by

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Abstract

The normality, the center and the derived notion of central nilpotency are important concepts in loops. Determining nilpotency class of loops of small orders is the main focus of this paper. Examples of some constructed commutative and non-commutative loops are shown. In order to determine the nilpotency class of the constructed loops using subnormal series method, we obtain the nuclei of the loops, the nucleus, the centrum, the center, the left and right cosets, the quotient set (factor loop), normal and proper subloops. Using subnormal series method, the nilpotency class of such loops were obtained and presented. A brief characterization of the constructed commutative loops of order 12 and of order 16 were obtained and presented.

Keywords: *Centrally nilpotent, Nilpotency class, Normal subloop, Proper subloop, Center of Quotient set, Subnormal series.*

Introduction

As it is well known, a group is of nilpotency class at most two if and only if its inner mapping group is abelian. In 1946 Bruck published a long paper that influenced the development of loop theory for decades, in which he proved that a loop of nilpotency class two possesses an abelian inner mapping group. The converse problem of Bruck's result: Is every finite loop (even infinite) with abelian inner mapping group of nilpotency class at most two? While working on this problem, Niemenmaa and Kepka (1994) proved that a finite loop with abelian inner mapping group must be nilpotent. However, Vesanen, as reported in Niemenmaa and Rytty (2011), found a nilpotent loop of order 18 with nilpotency class three such that the inner mapping group is not even nilpotent. This is also a partial counter of the converse of Bruck's result of 1946. Kepka (1998) and Niemenmaa (2009) later improved upon their result of 1994 and showed that if the inner mapping is abelian and finite, then the loop is nilpotent. But they did not establish an upper bound on the nilpotency class of the loop, and indeed, no such bound is presently known. For a long time, there was no example of a nilpotency class greater than two. Infact, for many years the prevailing opinion has been that all such loops have to be of nilpotency class two. This seems to have been well substantiated if the loop is a group, then we clearly get this restriction on the nilpotency class. Some well-behaved classes of loops fulfill this restriction, too. However, in 2004, Csörgö used the technique of H-connected transversals to construct a counter example of a loop of order 128 of nilpotency class three with abelian inner mapping groups. This was a counter example to this long-standing conjecture. This result was published in 2007. Nagy and Vojtechovsky (2008) used GAP, to analyze the loop structure of Csörgö counter example and they could construct by some algorithm another loop of order 128 with nilpotency class 3. Nagy and Vojtechovsky (2009) constructed a Moufang loop of order 2^{14} of nilpotency class 3, and with abelian inner mapping group and at the same time they showed that Moufang loops of odd order with abelian inner mapping groups have nilpotency class at most two.

Kinyon, Veroff and Vojtechovsky (2012) says that Bol loops of nilpotency class 3 does exist. Recently, Drapal and Vojtechovsky (2011) with the aid of LOOPS package in GAP constructed the multiplication table of Csörgö counter example loop. By analyzing the structure of the counter example, they developed a method by which they were able to construct a class of other examples. Drapal and Vojtechovsky (2011) showed using a new set up that there are many examples of loops of Csörgö type of order 128. They describe all 125 groups of order 128 that can be used in the construction as the starting point. These loops Q with commutative inner mapping groups and nilpotency class equal to 3 are called in literature, loops of Csörgö type. The focus of this paper is not on the nature of the inner mapping group of the constructed loops of small order (orders 12, 16 and 18) but using the subnormal series method to determine their nilpotency classes.

Definition 1.1

A loop $(L,\cdot)$ is a quasigroup with an identity element e such that ψ: $(L,\cdot) \times (L,\cdot) \rightarrow (L,\cdot)$ then the following hold

(i) $x \cdot y \in L \quad \forall\, x, y \in L$

(ii) $a \cdot x = b$ implies that $x = a^{-1} \cdot b$ and $y \cdot a = b$ implies that $y = b \cdot a^{-1}$

(iii) $a \cdot e = a = e \cdot a \quad \forall\, a \in (L,\cdot)$.

The uniqueness of equation (ii) shows the existence of inverses and the separation of the inverses into two is to show that Loops are not necessarily commutative under the operator.

Definition 1.2: Let $(L,\cdot)$ be a loop.

The Left nucleus $N_\lambda(L,\cdot)$ is denoted as $N_\lambda(L,\cdot) = \{x \in L\colon (x \cdot a) \cdot b = x \cdot (a \cdot b),\ \forall\, a, b \in L\}$

The Right nucleus $N\rho(L,\cdot)$ is denoted as $N\rho(L,\cdot) = \{x \in L\colon (a \cdot b) \cdot x = a \cdot (b \cdot x), \forall\, a, b \in L\}$

The Middle nucleus $N\mu(L,\cdot)$ is denoted as $N\mu(L,\cdot) = \{x \in L\colon (a \cdot x) \cdot b = a \cdot (x \cdot b), \forall\, a, b \in L\}$

The Nucleus of $N(L,\cdot)$ is denoted as $N(L,\cdot) = N_\lambda(L,\cdot) \cap N\rho(L,\cdot) \cap N\mu(L,\cdot)$.

The Centrum $C(L,\cdot)$ is denoted as $C(L,\cdot) = \{x \in L\colon x \cdot a = a \cdot x, \quad \forall\, a \in L\}$

The center $Z(L,\cdot)$ is denoted as $Z(L,\cdot) = C(L,\cdot) \cap N(L,\cdot)$.

Remark 1.1: Each of these nuclei is a subloop of $(L,\cdot)$.

Definition 1.3: Let $(L,\cdot)$ be a loop.

(a) A subloop N of a loop $(L,\cdot)$ is said to be normal in $(L,\cdot)$ if

$$xN = Nx, \quad x(yN) = (xy)N, \ N(xy) = (Nx)y, \quad \forall\, x, y \in (L,\cdot)$$

(b) A subnormal series of a loop $(L,\cdot)$ is a finite sequence of subloops

$(L,\cdot) = Z_n \supset Z_{n-1} \supset \ldots\ldots\ldots\ldots Z_1 \supset Z_o = 1$, Where Z_i is a normal subloop of Z_{i+1}. If each

Z_i is normal in $(L,\cdot)$, then the series is called a normal series.

Remark 1.2: (i) The center is always a normal subloop of $(L,\cdot)$.

(ii) The length of a subnormal series or a normal series is the number of proper inclusions.

(iii) The center of a quotient set given as $Z({(L,\cdot)}/{Z_i})$ is used to obtain the series of normal subloops.

Definition 1.4: Let $(L,\cdot)$ be a loop. The quotient set (factor loop) of Z_i in $(L,\cdot)$ denoted as ${(L,\cdot)}/{Z_i}$ is the left (right) cosets of Z_i in $(L,\cdot)$ that coincide.

Definition 1.5: If we write $Z_o = 1$, $Z_1 = Z(L,\cdot)$ and ${Z_{i+1}}/{Z_i} = Z({(L,\cdot)}/{Z_i})$, then we obtain series of normal subloops of the loop $(L,\cdot)$. If Z_{n-1} is a proper subloop of $(L,\cdot)$ but $Z_n = (L,\cdot)$, then we say that the loop $(L,\cdot)$ is (centrally) nilpotent of class n. The smallest such n is called the nilpotency class.

Definition 1.6: A loop $(L,\cdot)$ is called: (a) a weak inverse property loop $(WIPL)$ if and only if it obeys the identity: $x(yx)^{\rho} = y^{\rho}$ or $(xy)^{\lambda}x = y^{\lambda}$ $for\ all\ x, y \in (L,\cdot)$.
(b) a flexible loop if the flexibility property $xy \cdot x = x \cdot yx$ holds for all $x, y \in (L,\cdot)$.

In this paper, our interest is not on the nature of the inner mapping group of the constructed loops but determining their nilpotency class. In next section, we will present some examples of constructed loops and their nilpotency class

Main Results
In this section, loops of orders (orders 12, 16 and 18) are constructed and using the Subnormal
Series method, their nilpotency classes were determined.

Example 2.1: **A COMMUTATIVE LOOP OF ORDER 12**

*	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	3	6	5	8	7	10	9	12	11
3	3	4	1	2	8	7	6	5	12	11	10	9
4	4	3	2	1	7	8	5	6	11	12	9	10
5	5	6	8	7	10	9	12	11	1	2	4	3
6	6	5	7	8	9	10	11	12	2	1	3	4
7	7	8	6	5	12	11	10	9	4	3	1	2
8	8	7	5	6	11	12	9	10	3	4	2	1
9	9	10	12	11	1	2	4	3	8	7	6	5
10	10	9	11	12	2	1	3	4	7	8	5	6
11	11	12	10	9	4	3	1	2	6	5	8	7
12	12	11	9	10	3	4	2	1	5	6	7	8

Table 1 (commutative loop of order 12)

Theorem 2.1: Example 2.1 is a commutative loop of order 12 that is centrally nilpotent of Class 2, that is a weak inverse property and flexible loop.

Proof:
First, we show that $(L,\cdot)$ is a non-associative loop.

$(8 \cdot 8) \cdot 10 = 8$ but $8 \cdot (8 \cdot 10) = 6$

Thus, it is non-associative.

Next we show that $(L,\cdot)$ is a commutative loop.

$(x \cdot y) = (y \cdot x)$ holds for all $x, y \in (L,\cdot)$. Hence it is a commutative loop.

The nuclei of the loop are $N_\lambda(L,\cdot) = \{1, 2, 3, 4\}$, $N\rho(L,\cdot) = \{1, 2, 3, 4\}$, $N\mu(L,\cdot) = \{1, 2, 3, 4\}$. $N(L,\cdot) = \{1, 2, 3, 4\}$, The Centrum is obtained as: $C(L,\cdot) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

The Center follows as: $Z(L,\cdot) = \{1, 2, 3, 4\}$.

Let $Z(L,\cdot) = Z_1 = \{1, 2, 3, 4\}$, then $Z_2 = {}^{Z_2}/_{Z_1} = Z({}^{(L,\cdot)}/_{Z_1}) = (L,\cdot)$.

Now representing these in a finite sequence of subnormal series of the loop, we have

$Z_o = 1$, $Z_1 = Z(L,\cdot) = \{1, 2, 3, 4\}$, $Z_2 = Z({}^{(L,\cdot)}/_{Z_1}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} = (L,\cdot)$.

Since Z_1 is a proper subloop of the loop $(L,\cdot)$ and $Z_2 = (L,\cdot)$, then the loop $(L,\cdot)$ is centrally nilpotent of class 2. Next, to show that $(L,\cdot)$ is a weak inverse property loop, it is needed to show that

$$x(yx)^\rho = y^\rho \quad \text{or} \quad (xy)^\lambda x = y^\lambda \ \textit{holds in } (L,\cdot) \textit{ for all } x, y \in (L,\cdot).$$

LHS $= x(yx)^\rho = x(y^\rho x^\rho) = x(x^\rho y^\rho) = y^\rho =$ RHS

For all $x, y \in (L,\cdot)$, $x \cdot x^\rho = \varepsilon$ and $x^\rho y^\rho = y^\rho x^\rho$ since $(L,\cdot)$ is a commutative loop.

Since L.H.S = R.H.S, it is a weak inverse property loop.

Next, to show that $(L,\cdot)$ is a flexible loop, it is needed to show that $xy \cdot x = x \cdot yx$ holds for all $x, y \in (L,\cdot)$. $xy \cdot x = yx \cdot x = x \cdot yx$ since $(L,\cdot)$ is commutative. Thus, it is a flexible loop.

The proof for commutativity, weak inverse property and flexibility property were confirmed with GAP package version 3.1.0 available at www.math.dv.edu/loops. See appendix.

Example 2: A COMMUTATIVE LOOP OF ORDER 16

$\odot$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	8	7	6	5	12	11	10	9	16	15	14	13
4	4	3	2	1	7	8	5	6	11	12	9	10	15	16	13	14
5	5	6	8	7	10	9	12	11	14	13	16	15	1	2	4	3
6	6	5	7	8	9	10	11	12	13	14	15	16	2	1	3	4
7	7	8	6	5	12	11	10	9	16	15	14	13	4	3	1	2
8	8	7	5	6	11	12	9	10	15	16	13	14	3	4	2	1
9	9	10	12	11	14	13	16	15	4	3	1	2	8	7	6	5
10	10	9	11	12	13	14	15	16	3	4	2	1	7	8	5	6
11	11	12	10	9	16	15	14	13	1	2	4	3	6	5	8	7
12	12	11	9	10	15	16	13	14	2	1	3	4	5	6	7	8
13	13	14	16	15	1	2	4	3	8	7	6	5	12	11	10	9
14	14	13	15	16	2	1	3	4	7	8	5	6	11	12	9	10
15	15	16	14	13	4	3	1	2	6	5	8	7	10	9	12	11
16	16	15	13	14	3	4	2	1	5	6	7	8	9	10	11	12

Table 2 (Commutative loop of order 16)

Theorem 2.2: Example 2.2 is a commutative loop of order 16 that is centrally nilpotent of $Cl(L) = 2$, that is an Automorphic inverse property and a flexible loop.

Proof:

First, we show that $(L, \odot)$ is a non-associative loop.

$(7 \odot 7) \odot 11 = 2$ but $7 \odot (7 \odot 11) = 3$. Thus, it is non-associative.

Next we show that $(L, \odot)$ is a commutative loop.

$(x \odot y) = (y \odot x)$ holds for all $x, y \in (L, \odot)$. Hence it is a commutative loop.

The nuclei of the loop are $N_\lambda(L, \odot) = \{1, 2, 3, 4\}$, $N\rho(L, \odot) = \{1, 2, 3, 4\}$, $N\mu(L, \odot) = \{1, 2, 3, 4, 9, 10, 11, 12\}$. $N(L, \odot) = \{1, 2, 3, 4\}$, The centrum is obtained accordingly as: $C(L, \odot) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. The Center follows as: $Z(L, \odot) = \{1, 2, 3, 4\}$.

Let $Z(L, \odot) = Z_1 = \{1, 2, 3, 4\}$, then $Z_2 = {}^{Z_2}/_{Z_1} = Z({}^{(L, \odot)}/_{Z_1}) = (L, \odot)$

Now representing these in a finite sequence of subnormal series of the loop, we have

Example 2.3: A NON- COMMUTATIVE LOOP OF ORDER 16

*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	5	6	8	7	1	2	4	3	13	14	16	15	10	9	11	12
6	6	5	7	8	2	1	3	4	14	13	15	16	9	10	12	11
7	7	8	6	5	3	4	2	1	15	16	14	13	12	11	9	10
8	8	7	5	6	4	3	1	2	16	15	13	14	11	12	10	9
9	9	10	11	12	15	16	13	14	5	6	7	8	3	4	1	2
10	10	9	12	11	16	15	14	13	6	5	8	7	4	3	2	1
11	11	12	9	10	13	14	15	16	7	8	5	6	1	2	3	4
12	12	11	10	9	14	13	16	15	8	7	6	5	2	1	4	3
13	13	14	16	15	12	11	9	10	1	2	4	3	7	8	6	5
14	14	13	15	16	11	12	10	9	2	1	3	4	8	7	5	6
15	15	16	14	13	10	9	11	12	3	4	2	1	5	6	8	7
16	16	15	13	14	9	10	12	11	4	3	1	2	6	5	7	8

Table 3 (Non-Commutative loop of order 16)

Theorem 2.3: Example 2.3 is a non-commutative loop of order 16 that is centrally nilpotent of $Cl(L) = 3$.

Proof:

First, we show that $(L, *)$ is a non-associative loop.

$(10 * 10) * 11 = 16$ but $10 * (10 * 11) = 13$. Thus, it is non-associative.

Next we show that $(L, *)$ is a non-commutative loop.

$(8 * 9) = 16$ *but* $(9 * 8) = 14$. Thus, it is a non-commutative loop.
The nuclei of the loop are $N_\lambda(L,*) = \{1, 2, 3, 4\}$, $N\rho(L,*) = \{1, 2\}$, $N\mu(L,*) = \{1, 2, 3, 4\}$. $N(L,*) = \{1, 2\}$, The Centrum is obtained accordingly as: $C(L,*) = \{1, 2\}$. The Center follows as: $Z(L,*) = \{1, 2\}$.

Let $Z(L,*) = Z_1 = \{1, 2\}$, then $Z_2 = {}^{Z_2}/_{Z_1} = Z({}^{(L,*)}/_{Z_1}) = \{1, 2, 3, 4\}$ which is the next proper subloop that is normal in $(L,*)$. Continuing the subnormal series gives $Z_3 = {}^{Z_3}/_{Z_2} = Z({}^{(L,*)}/_{Z_2}) = (L,*)$. Now representing these in a finite sequence of subnormal series of the loop, we have $Z_o = 1$, $Z_1 = Z(L,*) = \{1, 2\}$, $Z_2 = {}^{Z_2}/_{Z_1} = Z({}^{(L,*)}/_{Z_1}) = \{1, 2, 3, 4\}$,

$Z_3 = {}^{Z_3}/_{Z_2} = Z({}^{(L,*)}/_{Z_2}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} = (L,*)$.

Since Z_2 is a proper subloop of the loop $(L,*)$ and $Z_3 = (L,*)$, then the loop $(L,*)$ is centrally nilpotent of class 3.

Example 2.4: A NON-COMMUTATIVE LOOP OF ORDER 18

•	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	2	3	1	5	6	4	8	9	7	11	12	10	14	15	13	17	18	16
3	3	1	2	6	4	5	9	7	8	12	10	11	15	13	14	18	16	17
4	4	5	6	1	2	3	11	12	10	8	9	7	17	18	16	14	15	13
5	5	6	4	2	3	1	12	10	11	9	7	8	18	16	17	15	13	14
6	6	4	5	3	1	2	10	11	12	7	8	9	16	17	18	13	14	15
7	7	8	9	11	12	10	14	15	13	17	18	16	1	2	3	5	6	4
8	8	9	7	12	10	11	15	13	14	18	16	17	2	3	1	6	4	5
9	9	7	8	10	11	12	13	14	15	16	17	18	3	1	2	4	5	6
10	10	11	12	8	9	7	17	18	16	14	15	13	5	6	4	1	2	3
11	11	12	10	9	7	8	18	16	17	15	13	14	6	4	5	2	3	1
12	12	10	11	7	8	9	16	17	18	13	14	15	4	5	6	3	1	2
13	13	14	15	17	18	16	5	6	4	1	2	3	8	9	7	11	12	10
14	14	15	13	18	16	17	6	4	5	2	3	1	9	7	8	12	10	11
15	15	13	14	16	17	18	4	5	6	3	1	2	7	8	9	10	11	12
16	16	17	18	14	15	13	1	2	3	5	6	4	11	12	10	8	9	7
17	17	18	16	15	13	14	2	3	1	6	4	5	12	10	11	9	7	8
18	18	16	17	13	14	15	3	1	2	4	5	6	10	11	12	7	8	9

Table 4 (Non-commutative loop of order 18)

Theorem 2.4: Example 2.4 is a non-commutative loop of order 18 that is centrally nilpotent of $Cl(L) = 3$.

Proof:

First, we show that $(L,\bullet)$ is a non-associative loop.

$(8 \bullet 8) \bullet 10 = 1$ but $8 \bullet (8 \bullet 10) = 5$. Thus, it is non-associative.

Next we show that $(L,\bullet)$ is a non-commutative loop.

$(9 \bullet 15) = 2$ *but* $(15 \bullet 9) = 6$. Thus, it is a non-commutative loop.

The nuclei of the loop are $N_\lambda(L,\bullet) = \{1, 2, 3\}$, $N\rho(L,\bullet) = \{1, 2, 3\}$, $N\mu(L,\bullet) = \{1, 2, 3\}$. $N(L,\bullet) = \{1, 2, 3\}$, The centrum is obtained accordingly as: $C(L,\bullet) = \{1, 2, 3, 4, 5, 6\}$.

The Center follows as: $Z(L,\bullet) = \{1, 2, 3\}$.

Let $Z(L,\bullet) = Z_1 = \{1, 2, 3\}$, then $Z_2 = {}^{Z_2}/_{Z_1} = Z({}^{(L,\bullet)}/_{Z_1}) = \{1, 2, 3, 4, 5, 6\}$ which is the next proper subloop that is normal in $(L,\bullet)$. Continuing the subnormal series gives $Z_3 = {}^{Z_3}/_{Z_2} = Z({}^{(L,*)}/_{Z_2}) = (L,\bullet)$. Now representing these in a finite sequence of subnormal series of the loop, we have $Z_0 = 1$, $Z_1 = \{1, 2, 3\}$, $Z_2 = {}^{Z_2}/_{Z_1} = Z({}^{(L,\bullet)}/_{Z_1}) = \{1, 2, 3, 4, 5, 6\}$,

$Z_3 = {}^{Z_3}/_{Z_2} = Z({}^{(L,\bullet)}/_{Z_2}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\} = (L,\bullet)$.

Since Z_2 is a proper subloop of the loop $(L,\bullet)$ and $Z_3 = (L,\bullet)$, then the loop $(L,\bullet)$ is centrally nilpotent of class 3.

3 Concluding Remarks

Determining nilpotency class of loops of small order was the focus of this paper. In achieving this, a simplified step by step guide to obtaining nilpotency classes of loops using the subnormal series method was presented. The result of the loops constructed in this paper shows that the commutative loops of order 12 and 16 are centrally nilpotent of class two, while the non-commutative loops of order 16 and 18 are centrally nilpotent of class three.
The characterizations of the constructed commutative loops showed that the loop of order 12 is a weak inverse property loop and a flexible loop, while the constructed commutative loop of order 16 is an automorphic inverse property loop and a flexible loop.

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