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THE EQUIGRADIENT LINES OF $\mathbb{Z}^{2}(d)$

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A. Abstract

Lines in near-linear finite geometry were studied. Comparison were made between slope of each lines of $\mathbb{G}(d)$, Lines $\mathbb{L}$ (aξ, aι_) were observed to have identical slope with $\mathbb{L}$ (ξ, ι) for ξ, and ι, a relatively prime.

Keywords: *Gradient, near-linear finite geometries*

I. INTRODUCTION

Finite geometry had received a lot of attention since time immemorial [1]. This could not be uncon- nected to its wide area of application discovered in recent past. In more recent times non-near-linear finite geometry started receiving audience from researcher this could be linked to its duality with the weak mutually unbiased bases in finite quantum systems with variables in $\mathbb{Z}(d)$ [10-11].

Lines in $\mathbb{Z}^{2}(d)$ had received much attention in recent times. The existence of the common axiom which states that two lines have at most one point in common. In this work, we studied each lines point-by- point to compare their gradients. We divide the whole work into the following parts; preliminaries of this work is discussed in section II. Here, we define the notation we used in the discourse. Section III covers finite geometry $\mathbb{G}(d)$. We discuss Symplectic on $\mathbb{G}(d)$ with numerical example was in section IV. Section V concludes our work.

II. PRELIMINARIES

(i) $\mathbb{Z}^{2}(d)$ denotes a field of integer modulo d for d prime number.

(ii) $|\mathbb{Z}^{*}(d)| = \varphi(d)$, where

$$\varphi(d) = d\prod\left(1-\frac{1}{d}\right);\ d = \text{prime} \tag{1}$$

(iii) $\psi(d)$ denotes the Euler psi function, it is defined as;

$$\psi(d) = d\prod\left(1+\frac{1}{d}\right);\ d = \text{prime} \tag{2}$$

(iv) The greatest common divisor of two element α and γ in $\mathbb{Z}(d)$ is denoted by GCD(α,γ).

(i) The notation g ≅h means g maps h.

III. FINITE GEOMETRY $\mathbb{G}(d)$

Let

$$\mathbb{G}(d) = (\mathbb{L}_{d}, \mathbb{P}_{d}) \tag{3}$$

represents a finite geometry, it is defined as the pair where $\mathbb{P}_d$ and $\mathbb{L}_d$ denote the set of points and lines in $\mathbb{G}(d)$ respectively.

$\mathbb{P}_d = \{(p,q) \mid p,q \in \mathbb{Z}(d)\}$ (4)

and a line through the point (0; 0) is defined as

$\mathbb{L}(\xi,\iota) = \{(p\xi, p\iota) \mid \xi,\iota \in \mathbb{Z}(d)\}, p \in \mathbb{Z}(d)$ (5)

Near-linear finite geometry is contains finite number of lines and points. In this type of geometry two lines has only one point in common. Near-linear geometry is connected to field of integer modulo d.

This work centres on it. From earlier work of [10] we confirm the following propositions;

(i) Let $\mathbb{L}(\xi,\iota)$ represents lines in a finite geometry $\mathbb{G}(d)$,

then $\mathbb{L}(\xi,\iota) = \mathbb{L}(l\,\xi, l\,\iota), \forall l \in \mathbb{Z}(d)$ (6)

and

$\mathbb{L}(l\xi, l\,\iota) \subset \mathbb{L}(l\,\xi, l\,\iota)\ \forall l \in \mathbb{Z}(d) \setminus \mathbb{Z}^*(d)$ (7)

A line $\mathbb{L}(l\xi, l\,\iota)$ is a maximal line in $\mathbb{G}(d)$ if GCD$(\xi,\iota) \in \mathbb{Z}(d)$ and $\mathbb{L}(\xi,\iota)$ is a subline in $\mathbb{G}(d)$ if GCD GCD$(\xi,\iota) \in \mathbb{Z}(d) \setminus \mathbb{Z}^*(d)$

(ii) An existence of $\psi(d)$ maximal lines in finite geometry $\mathbb{G}(d)$ with exactly d points each.

(iii) A line $\mathbb{L}(\xi,\iota)$ in $\mathbb{G}(d)$ is the same as

$\mathbb{L}(s\xi, s\iota) = \{(s\not{f}\xi, s\not{f}\iota) \mid \not{f} \in \mathbb{Z}(d)\}$ (8)

$\mathbb{L}(\xi,\iota)$ is also at the same time the line $\mathbb{L}(\zeta\xi, \zeta\iota)$ in $\mathbb{G}(\zeta d)$ is a subline of

$\mathbb{L}(\xi,\iota) = \{(s'\xi, s'\iota) \mid s' = 0,1, \ldots \not{f}d - 1\}$ (9)

(iv) There exists a duality between maximal lines in $\mathbb{G}(d)$ and weak mutually unbiased bases in $H(d)$, where $H(d)$ represents finite dimensional Hilbert space.

A. Example

From equation (5) above, Lines in geometry $\mathbb{G}(5) = \mathbb{Z}(5) \times \mathbb{Z}(5)$ is obtained as follows:

$\mathbb{L}(0,1) \cong \mathbb{L}(0,2) \cong \mathbb{L}(0,3) \cong \mathbb{L}(0,4)$

$\mathbb{L}(1,0) \cong \mathbb{L}(2,0) \cong \mathbb{L}(3,0) \cong \mathbb{L}(4,0)$

$\mathbb{L}(1,1) \cong \mathbb{L}(2,2) \cong \mathbb{L}(3,3) \cong \mathbb{L}(4,4)$

$\mathbb{L}(1,2) \cong \mathbb{L}(2,4) \cong \mathbb{L}(3,1) \cong \mathbb{L}(4,3)$

$\mathbb{L}(1,3) \cong \mathbb{L}(2,1) \cong \mathbb{L}(3,4) \cong \mathbb{L}(4,2)$

$\mathbb{L}(1,4) \cong \mathbb{L}(2,3) \cong \mathbb{L}(3,2) \cong \mathbb{L}(4,1)$

where

$\mathbb{L}(1,2) = \{(0,0), (1,2), (2,4), (3,1), (4,3)\}$ (10)

and finding the gradient of any two arbitrary points of $\mathbb{L}(1,2)$ in equation (10), it yields the following results

$$\frac{2-1}{1-2} = 2 \equiv \frac{2-1}{1-3} = \frac{1}{-2} \equiv \frac{1}{3} = 4\big(i.e\ 3^{-1}(mod\ 5)\big) = 2 \equiv \frac{2-3}{1-4} = \frac{4}{2}\ (mod\ 5) \equiv \frac{4-1}{2-3}$$
$$= 3\big(i.e\ 4^{-1}\ (mod\ 5)\big) \equiv \frac{4-3}{2-4} = 3^{-1}(mod\ 5) \equiv \frac{1-3}{3-4} \equiv \frac{3-1}{4-3} \equiv \frac{3-4}{4-2}$$
$$\equiv \frac{3-2}{4-1} \equiv \frac{1-4}{3-2} \equiv \frac{1-2}{3-1} \equiv \frac{4-2}{2-1} = 2(mod\ 5)$$

IV. SYMPLECTIC TRANSFORMATION ON $\mathbb{G}(\boldsymbol{d})$

An operator

$\mathbb{S}(\pi, \vartheta \mid \iota, \xi)$ (11)

where $\mathbb{S}(\pi,\vartheta \mid \iota,\xi) \equiv \begin{pmatrix}\pi & \vartheta \\ \iota & \xi\end{pmatrix}$, $\left|\mathbb{S}\right| = 1\ (mod\ d), \pi,\vartheta,\iota,\xi \in \mathbb{Z}(d)$

This transformation $\mathbb{S}$ form a Symplectic Sp$(2,\mathbb{Z}(d))$ group.

Acting $\mathbb{S}$ on all points of line $\mathbb{L}(a,b)$ in $\mathbb{Z}(5)\times\mathbb{Z}(5)$ generates all the points of the line $\mathbb{L}(\pi a+\vartheta b, \iota a+\xi b)$, This

is represented here in this work as $\mathbb{S}(\pi,\vartheta \mid \iota,\xi)\ \mathbb{L}(a,b)$.

Example suppose d = 5 if we replace π,ϑ,ι,ξ with

2,1,1, and 1 respectively into $\mathbb{S}$ and act it on $\mathbb{L}(a,b)$ where $a = 0, b \in \mathbb{Z}(5)$, it yields $\mathbb{L}(1,1)$.

For d is a prime, $\mathbb{S}(0,1|-1,\beth)\mathbb{L}(0,1)$produces all the lines $\mathbb{G}(5)$ through the origin. For

$\beth = 0,1,\dots,d-1 \rightarrow \Theta(\beth) = \mathbb{S}(0,1|-1,\beth)\mathbb{L}(0,1) = \mathbb{L}(1,\beth)$ (12)

Here if $\beth = -1$ then $\mathbb{S}(0,1|-1,\beth)$ is represented by an identity matrix that is to $\mathbb{S}(1,0|0,1)$.

If we substitute the value of $\beth = -1,0,\dots,d-1$, we obtain all the lines of $\mathbb{G}(d)$,for $a = 0, b \in \mathbb{Z}(d)$.

V. CONCLUSION

Lines in finite geometry $\mathbb{G}(d)$ where d is a prime integer were studied. We confirm that for $\mathbb{L}(\xi,\iota) \cong \mathbb{L}(p\,\xi, p\,\iota)$ for $p \in \mathbb{Z}(d)$.

VI. References

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