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ON SEMIGROUPS OF PARTIAL CONTRACTIONS OF A FINITE CHAIN

by

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Abstract

Let $X_n = \{1, 2, \ldots, n\}$ be the finite n-elements set under its natural order. A partial map α of X_n is said to be a contraction if, for all $x, y \in dom(\alpha)$, $|x\alpha - y\alpha| \leq |x - y|$ and an order-preserving if, $x, y \in dom(\alpha)$, $x \leq y \Rightarrow x\alpha \leq y\alpha$. Let CP_n and OCP_n denote, respectively, semigroups of all partial contraction maps and of all order preserving partial contraction maps. In this paper, we study the structure of CP_n and OCP_n and present characterisations of starred Green's relations in these semigroups.

Keywords. *Order-preserving map, Contraction map, Green's relations, Starred Green's relations.*

1. Introduction

A *semigroup* is a non-empty set together with an associative binary operation. The commonest example of semigroups are semigroups of transformations of a set *X*. Let the finite set $\{1,2, \ldots, n\}$ be denoted by X_n, then a mapping $\alpha : A \to B$, where A and B are subsets of X_n, is called a *partial transformation* of X_n. If $A = X_n$, the mapping α is called a *full transformation* of X_n. The sets of all full, partial and partial one-to-one transformations of X_n form semigroups, under composition of mappings and are respectively denoted by T_n, P_n and I_n. These semigroups are often referred to as *full transformation*, *partial transformation* and *symmetric inverse* semigroups respectively. These semigroups provide interesting sources of examples for semigroups and they are worth studying as naturally occurring semigroups. This is recognised from the fact that every finite semigroup can be embedded in a full transformation semigroup and the fact that every finite inverse semigroup can be embedded in a symmetric inverse semigroup.

The concept of ideals led to the study of certain equivalence relations on a semigroup known as Green's relations. These equivalences denoted by $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*, \mathcal{J}^*$ and $\mathcal{D}^*$ have played a fundamental role in the development of semigroup theory. Since their introduction, they became standard tools for investigating the structure of semigroup; see also Howie (1995). Green's relations proved to be useful in studying classes of semigroup, especially regular and inverse semigroup, but still there are other important classes of semigroup that are not regular and Fountain (1979) introduced the starred Green's relation which generalises Green's relations. These relations are $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*, \mathcal{J}^*$ and $\mathcal{D}^*$.

The relation $\mathcal{L}^*$ on a semigroup *S* is defined by the rule that for all $\alpha, \beta \in S$, $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* α, β are related by the Green's relation $\mathcal{L}$ in some over semigroup of *S*. The relation $\mathcal{R}^*$ is defined dually. These relations also have the following characterisations (see Fountain (1979, 1982)).

$$\mathcal{L}^* (S) = \{(a, b): (\forall x, y \in S1)ax = ay \Leftrightarrow bx = by\} \text{ And} \qquad (1)$$

$\mathcal{R}^*(S) = \{(a,b) : (\forall x, y \in S1)\, xa = ya \Leftrightarrow xb = yb\}$. (2)

The join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. These relations have also played a fundamental role in the study of many important classes of semigroups, see for example the work of (Fountain (1979, 1982), Sun (2013), Sun and Han (2016), etc). Many other authors described the $\mathcal{L}^*$ and $\mathcal{R}^*$ in certain subsemigroup of T_n *and* P_n preserving order and equivalence relations. In this paper we present the structure of partial contractions mapping and characterisations of starred Green's relations in CP_n and OCP_n.

2. The Structure of Partial Contractions

For any two subsets A and B of X_n, we write $A < B$ to mean that, for all $x \in A$, $y \in B$, $x < y$.

Lemma 2.1 *Let* $\alpha \in P_n$ *and* $im(\alpha) = \{a_1 < a_2 < \ldots < a_r\}$. *Then* α *is an order-preserving map if and only if, for each* $1 \leq i \leq r-1$, $a_i < a_{i+1}$ *and* $a_i\alpha^{-1} < a_{i+1}\alpha^{-1}$.

Proof.

Suppose that for each $1 \leq i \leq r-1$, $a_i < a_{i+1}$ *and* $a_i\alpha^{-1} < a_{i+1}\alpha^{-1}$. Let $x, y \in dom(\alpha)$ with $x \leq y$. Then, since $dom(\alpha) = \cup\, a_i\alpha^{-1}$, for some $i, j, x \in a_i\alpha^{-1}$ and $y \in a_i\alpha^{-1}$. Thus, by assumption $a_i\alpha^{-1} < a_{i+1}\alpha^{-1}$ for all $1 \leq i \leq r-1$, we see that $i \leq j$. Therefore, $x\alpha = a_i < a_j = y\alpha$. Hence, α is an order-preserving partial map.

Conversely, if for some i, $a_i > a_{i+1}$ while $a_i\alpha^{-1} < a_{i+1}\alpha^{-1}$ for all i, then, clearly, for each $x \in a_i\alpha^{-1}$ and each $y \in a_{i+1}\alpha^{-1}$, $x < y$. But $x\alpha = a_i > a_{i+1} = y\alpha$. Thus, α is not order-preserving. On the other hand, if $a_i < a_{i+1}$ and $a_i\alpha^{-1} > a_{i+1}\alpha^{-1}$ for some i. Then, there exist $x \in a_i\alpha^{-1}$ and $y \in a_{i+1}\alpha^{-1}$ such that $x > y$ and $x\alpha = a_i\alpha^{-1} > a_{i+1}\alpha^{-1} = y\alpha$.

Thus, again α is not order-preserving.

Lemma 2.2 *Let* $\alpha \in OP_n$ and $im(\alpha) = \{a_1 < a_2 < \ldots < a_r\}$. *Then* α *is a contraction if and only if, for all* $1 \leq i \leq r-1$, $max(a_i\alpha^{-1}) + a_{i+1} \leq min(a_{i+1}\alpha^{-1}) + a_i$.

Proof.

Suppose that $\alpha \in OP_n$ is a contraction map. Then, for each $1 \leq i \leq r-1$, $a_{i+1} - a_i \leq min(a_{i+1}\alpha^{-1}) - max(a_i\alpha^{-1})$. And so, $max(a_i\alpha^{-1}) + a_{i+1} \leq min(a_{i+1}\alpha^{-1}) + a_i$ as required.

Conversely, suppose that, for some i, $max(a_i\alpha^{-1}) + a_{i+1} > min(a_{i+1}\alpha^{-1}) + a_i$. Then, clearly, $a_{i+1} - a_i > min(a_{i+1}\alpha^{-1}) - max(a_i\alpha^{-1})$, so that α is not a contraction.

Lemma 2.3 *Let* $\alpha \in CP_n$. *Then* $F(\alpha) = \{x \in dom(\alpha) | x\alpha = x\}$ *is a convex subset of* $dom(\alpha)$.

Proof.

Let $\alpha \in CP_n$ and $x, y, z \in dom(\alpha)$ such that $x < y < z$ and $x, z \in F(\alpha)$. Then, $x\alpha = x$ and $z\alpha = z$. If $y\alpha = y$, there is nothing to prove. Thus, we consider two cases as follows:

Case 1. $y\alpha < y$. Here $|z - y| = |z\alpha - y| = z\alpha - y < z\alpha - y\alpha = |z\alpha - y\alpha|$. This contradicts the fact that α is a contraction.

Case 2. $y\alpha > y$. Here $|y - x| = |y - x\alpha| = y - x\alpha < y\alpha - x\alpha = |y\alpha - x\alpha|$. Which, again contradicts the fact that α is a contraction.

Lemma 2.4 *Let $\alpha \in OCP_n$. Then, for each $x \in dom(\alpha)$,*

i. *if $x < min(F(\alpha))$, then $x\alpha > x$;*

ii. *if $x > max(F(\alpha))$, then $x\alpha < x$.*

Proof.

i. Let $x < min(F(\alpha))$. First, we note that, $x\alpha \neq x$. If $x\alpha < x$, then $x\alpha < y\alpha = y$ and so, $|y - x| = |y\alpha - x| = y\alpha - x < y\alpha - x\alpha = |y\alpha - x\alpha|$. And so, α is not a contradiction. This is a contraction. Thus, $x\alpha > x$ as required.

ii. Let $x > max(F(\alpha))$ and note that, $x\alpha \neq x$. If $x\alpha < x$, then $x\alpha > z\alpha = z$ and so, $|x - z| = x - z = x - z\alpha < x\alpha - z\alpha = |x\alpha - z\alpha|$. This contradict the fact that α is a contraction and so $x\alpha < x$ as required.

Theorem 2.5 *Let $\alpha \in OCP_n$. Then α can be decomposed as a product of three factors in OCP_n as $\alpha = \alpha_1\alpha_2\alpha_3$, where α_1 is an order-increasing partial map, α_2 is a partial identity and α_3 is an order-decreasing partial map.*

Proof.

Let $dom(\alpha) = \{x_1, x_2, \ldots, x_k\}$. Then, by Lemma 2.3 and 2.4 $dom(\alpha)$ partitioned into three classes $\{x_1, x_2, \ldots, x_{i-1}\}$, $\{x_i, x_{i+1}, \ldots, x_{j-1}\}$ and $\{x_j, x_{j+1}, \ldots, x_k\}$ consisting of increasing points, fixed points and decreasing points of α respectively. Defined α_1, α_2 and α_3 as follows:

$dom(\alpha_1) = dom(\alpha)$ and

$$x\alpha_1 = \begin{cases} x\alpha & if\ x \in \{\mathrm{x_1, x_2, \ldots, x_{i-1}}\}, \\ x & if\ x \in dom(\alpha)\backslash\{\mathrm{x_1, x_2, \ldots, x_{i-1}}\}. \end{cases}$$

$dom(\alpha_3) = im(\alpha_1)$ and $x\alpha_2 = x$ for all $x \in dom(\alpha_2)$;

$dom(\alpha_3) = im(\alpha_1)$ and

$$x\alpha_3 = \begin{cases} x & if\ x \notin \{\mathrm{x_j, x_{j+1}, \ldots, x_k}\}, \\ x\alpha & if\ x \in \{\mathrm{x_j, x_{j+1}, \ldots, x_k}\}. \end{cases}$$

It is then clear that α_1 is an order increasing map, α_2 is a partial identity and α_3 is an order-decreasing map in OCP_n. Also, $\alpha_1\alpha_2\alpha_3 = \alpha$.

3. Starred Green's Relations

On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ *if and only if* a, b are related by the Green's relation $\mathcal{L}$ in some over semigroup of S. The relation $\mathcal{R}^*$ is defined dually. These relations also have the following characterisations (see Fountain (1979))

$$\mathcal{L}^*(S) = \{(a, b) : (\forall x, y \in S1)ax = ay \Leftrightarrow bx = by\} \quad \text{And} \qquad (3)$$

$$\mathcal{R}^*(S) = \{(a, b) : (\forall x, y \in S1)xa = ya \Leftrightarrow xb = yb\}. \qquad (4)$$

The join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$.

Theorem 3.1 *Let $S = CP_n$ or OCP_n and let $(\alpha,\beta) \in S$. Then*

(i) $(\alpha,\beta) \in \mathcal{L}^*(S)$ *if and only if* $im(\alpha) = im(\beta)$,

(ii) $(\alpha,\beta) \in \mathcal{R}^*(S)$ *if and only if* $ker(\alpha) = ker(\beta)$,

(iii) $(\alpha,\beta) \in \mathcal{H}^*(S)$ *if and only if* $im(\alpha) = im(\beta)$ *and* $ker(\alpha) = ker(\beta)$.

Proof.

(i) Suppose that $(\alpha,\beta) \in \mathcal{L}^*(S)$. Then by equation (3)

$$\alpha\delta = \alpha\gamma \Leftrightarrow \beta\delta = \beta\gamma \ (for\ all\ \delta,\gamma \in S^1). \qquad (5)$$

Let $im(\alpha) = \{a_1, \ldots, a_r\}$. Then,

$$\alpha \cdot \begin{pmatrix} \{1,\ldots,a_1\} & a_2 & \cdots & a_{r-1} & \{a_r,\ldots,n\} \\ a_1 & a_2 & \cdots & a_{r-1} & a_r \end{pmatrix} = \alpha \cdot 1_{X_n}$$

and, by equation (5), if and only if

$$\beta \cdot \begin{pmatrix} \{1,\ldots,a_1\} & a_2 & \cdots & a_{r-1} & \{a_r,\ldots,n\} \\ a_1 & a_2 & \cdots & a_{r-1} & a_r \end{pmatrix} = \beta \cdot 1_{X_n}$$

which implies that $im(\beta) \subseteq \{a_1, \ldots, a_r\} = im(\alpha)$. Similarly, if $im(\beta) = \{b_1, \ldots, b_r\}$, we can show that $im(\alpha) \subseteq \{b_1, \ldots, b_r\} = im(\beta)$.

Conversely, suppose that $im(\alpha) = im(\beta)$. Then $(\alpha,\beta) \in \mathcal{L}(P_n)$ and, since P_n, the semigroup of all partial maps is an oversemigroup of S, the semigroup of all partial contraction maps, then it follows from definition that $(\alpha,\beta) \in \mathcal{L}^*(S)$.

(ii) Suppose that $(\alpha,\beta) \in \mathcal{R}^*(S)$. Then, by equation (4),

$$\delta\alpha = \gamma\alpha \Leftrightarrow \delta\beta = \gamma\beta \ (for\ all\ \delta,\gamma \in S^1). \qquad (6)$$

Now,

$$\begin{aligned} (x,y) \in Ker(\alpha) &\iff x\alpha = y\alpha \\ &\iff \begin{pmatrix} X_n \\ x \end{pmatrix} \cdot \alpha = \begin{pmatrix} X_n \\ y \end{pmatrix} \cdot \alpha \\ &\iff \begin{pmatrix} X_n \\ x \end{pmatrix} \cdot \beta = \begin{pmatrix} X_n \\ y \end{pmatrix} \cdot \beta \quad \text{(by (6))} \\ &\iff x\beta = y\beta \\ &\iff (x,y) \in Ker(\beta). \end{aligned}$$

Hence $Ker(\alpha) = Ker(\beta)$.

Conversely, suppose that $Ker(\alpha) = Ker(\beta)$. Then $(\alpha,\beta) \in \mathcal{R}(P_n)$ and, sineP_n is an oversemigroup of S, it follows from definition that $(\alpha,\beta) \in \mathcal{R}^*(S)$.

(iii) This follows from parts (i) and (ii) above.

Theorem 3.2 *Let $S = CP_n$ or OCP_n and $\alpha,\beta \in S$. Then $(\alpha,\beta) \in \mathcal{D}^*(S)$ if and only if $|im(\alpha)| = |im(\beta)|$.*

Proof.

Suppose $(\alpha,\beta) \in \mathcal{D}^*(S)$. Then, by [3, Proposition 1.5.11], for some $n \in N$, there exist elements $\delta_1, \delta_2, \ldots, \delta_{2n-1} \in S$ such that

$(\alpha,\delta_1) \in \mathcal{L}^*(S), (\delta_1,\delta_2) \in \mathcal{R}^*(S), \ldots, (\delta_{2n-1},\beta) \in R*(S)$.

Now, by Theorem 3.1, we have

$$|im(\alpha)| = |im(\delta_1)| = \left|\frac{X_n}{Ker(\delta_1)}\right| = \left|\frac{X_n}{Ker(\delta_2)}\right| = |im(\delta_2)| = |im(\delta_3)| = \cdots$$
$$= |X_n/Ker(\delta_{2n-1})| = |X_n/Ker(\beta)| = |im(\beta)|.$$

Conversely, suppose that $|im(\alpha)| = |im(\beta)|$ and let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}, \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}, \delta_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

$$\delta_2 = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ 1 & 2 & \cdots & r \end{pmatrix} \quad \text{and} \quad \delta_3 = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \\ 1 & 2 & \cdots & r \end{pmatrix},$$

where $a_1 > a_2 > a_3 > \cdots > a_r$ and $b_1 > b_2 > b_3 > \cdots > b_r$. Then,
$(\alpha, \delta_1) \in \mathcal{L}^*(S), (\delta_1, \delta_2) \in \mathcal{R}^*(S), (\delta_2, \delta_3) \in \mathcal{L}^*(S)$ (δ_3, δ_3) and $(\delta_3, \beta) \in \mathcal{R}^*(S)$ $(\delta_3,)$
and by Theorem 3.1 and [Howie (1995), Proposition 1.5.11], $(\alpha, \beta) \in \mathcal{D}^*(S)$.
The L^**−class* containing an element a is denoted by L_a^* and corresponding notations are used for the remaining starred relations. We define a *left*(*right*) $*-$ *ideal* of a semigroup S to be a left(right) ideal I of S for which $L_a^* \subseteq I (R_a^* \subseteq I)$ for all $a \in I$. A subset I of S is a $*-$*ideal* if it is both left and right $*-$*ideals* of S. The principal $*-$*ideal*, $\mathcal{J}^*(a)$, generated by $a \in S$ is the intersection of all $*-$ *ideals* of S to which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that: $a\mathcal{J}^*b$ if and only if $\mathcal{J}^*(a)) = \mathcal{J}^*(b)$.

Now we are going to show that on the semigroup $S = CP_n$ *or* OCP_n, $\mathcal{D}^* = \mathcal{J}^*$but first we record the following Lemma from [2].

Lemma 3.3 *Let a, b be elements of a semigroup S. Then $b \in \mathcal{J}^*(a)$ if and only if there are elements $a_0, a_1, \ldots, a_n \in S$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}^*(S)$ for $i = 1, \ldots, n$.*
Immediately we adopt the method used in Umar (1993) to have

Lemma 3.4 *Let S be any transformation semigroup. Then for each $\alpha, \beta \in S$, $\alpha \in \mathcal{J}^*(\beta)$ implies $|im(\alpha)| \leq |im(\beta)|$.*

Proof.
Let $\alpha \in \mathcal{J}^*(\beta)$, then by Lemma 3.3, there exist $\beta_0, \ldots, \beta_n \in S$, $\delta_1, \ldots, \delta_n$, $\gamma_1, \ldots, \gamma_n \in S^1$ such that $\beta = \beta_0$, $\alpha = \beta_n$ and $(\beta_i, \delta_i \beta_{i-1} \gamma_i) \in \mathcal{D}^*(S)$, *for* $i = 1, \ldots, n$. However, by Theorem 3.2, this implies that
$|im(\beta_i)| = |im(\delta_i \beta_{i-1} \gamma_i)| \leq |im(\beta_{i-1})|$,
for all $i = 1, \ldots, n$, which implies $|im(\alpha)| \leq |im(\beta)|$ as required.
Lemma 3.2 together with Lemma 3.4 give the following result.

Theorem 3.5 *On the semigroup $S = CP_n$ or OCP_n, $\mathcal{D}^*(S) = \mathcal{J}^*(S)$.*
Proof.
Let $(\alpha, \beta) \in \mathcal{J}^*$, then by definition we have,
$\mathcal{J}^*(\alpha) = \mathcal{J}^*(\beta)$ if and only if $\alpha \in \mathcal{J}^*(\beta)$ and $\beta \in \mathcal{J}^*(\alpha)$
if and only if $|im(\alpha)| \leq |im(\beta)|$ and $|im(\beta)| \leq |im(\alpha)|$

if and only if $|im(\alpha)| = |im(\beta)|$

if and only if $(\alpha, \beta) \in \mathcal{D}^*$.

Hence, $\mathcal{D}^* = \mathcal{J}^*$.

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SIQR MODEL FOR TRANSMISSION OF LASSA FEVER CONTROL DYNAMICS

by

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Abstract
In this article,SIQR model is proposed, the transmission of Lassafever control dynamics is analyzed and studied using stability theory of differential equations at both theoretical level and using numerical simulation, the sufficient conditions for disease free equilibrium is obtained.The infection-free stability is investigated. Using Jacobian matrix approach. It is shown that the introduced quarantine parameter helps in controlling and eradication of the Lassa fever virus in the population with respect to time. The analysis further reveals that the disease can be controlled if the basic reproduction number$R_0$is less than one regardless of the initial population.

Keywords: *SIQR model, quarantine parameter, disease free equilibrium, numerical simulation.*

Introduction
The greatest threat to human is infection diseases, the outbreak of infection diseaseshas caused the loss of millions of lives, great pain to families and also involves expenditure of huge amount of money in controlling the disease. The whole world has devoted efforts to control the spread of diseases. Mathematical models which describe the dynamics of infectious diseases have recently become important tools in analyzing the spread and control ofinfectious diseases[1,2,3,4]. Many mathematical models have already been proposed and studied to investigate the transmission and control of the dynamics of infectious diseases, these models provides the theoretical and quantitative bases for the prevention and control of infectious diseases [1].

Lassa fever isa form of such infectious diseases, it is an acute viral hemorrhagic fever (VHF) caused by the Lassa viruswhich is endemic in the belt of West Africa (Nigeria, Guinea Liberia, Sierra Leone)affecting about 2 – 3 million persons with 5,000 - 10,000 fatalities annually[5,6,7]. Transmission to Man occurs from exposure to excreta and blood of the rat, eating of contaminated food and water, or eating the rat as food.There may also be transmissions due to seasonal variations [6,8]. Infections also occur through contact with the fluid from an infected person [7,8,10]. Since its initial discovery in Lassa-Nigeria, outbreaks of Lassa fever have occurred repeatedly in other parts of Nigeria [9].

Lassa fever outbreaks in endemic areas are increased by factors that promote activities of man to rodents which include poor sanitation, crowding, deforestation, bush burning, rodent hunting and some other Agricultural activities [11].

In this article we study and formulate susceptible-infectious-quarantine-recovered (SIQR) model for the transmission and control dynamics of Lassa fever.

The schematic description of our model is given in the figure below

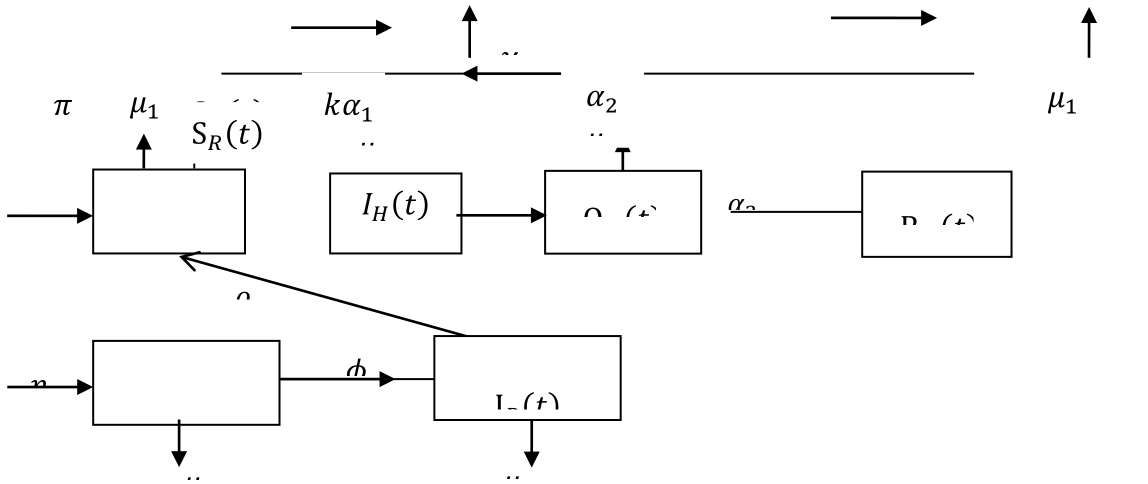

Fig. 1: Flow diagram of the dynamics Lassa Fever with Quarantine

2.0 Model Formulation

Lassa fevermodels usually encompassed individuals who have not come into contact with the virus known as susceptible humans (S_H (t)). The susceptible rodents (S_R (t)) become infected at the rate ϕ and infectious rodent infects human at the rate ρ, the infected human are treated at the rate δ, while some moved to the quarantine human class (Q_H (t)) at the rate α2. Those who are not aware of the treatment will be removed from the population through death at the rate α3,While the quarantine human class return to the susceptible human class at the rate γ1,The existence of region where the model is epidemiologically feasible is established. Stability analysis of the disease free equilibrium is investigated through the reproduction number obtained using the next generation operator approach.

In this model, individuals are recruited into the susceptible population of human at the rate π, susceptible population of rodent at the rate η, The infection spread at the rate k, where k is the probability of getting Lassa fever, c is the contact rate, both human and rodent die naturally at the rate μ1 and μ2 respectively.

The total population of human and Rodent are given by

N_H (t)=S_H (t)+I_H (t)+Q_R(t)+R_H (t) and N_R(t)=S_R (t)+I_R (t) respectively.

N(t)=N_R(t)+N_R(t)= Total population size at

$$\frac{dS_H}{dt} = \pi + \rho I_R + \gamma_1 R_H - k\alpha_1 S_H - \mu_1 S_H \quad (1)$$

$$\frac{dI_H}{dt} = k\alpha_1 S_H - (\mu_1 + \alpha_2 + \omega) I_H \quad (2)$$

$$\frac{dQ_H}{dt} = \alpha_2 I_H - (\mu_1 + \alpha_3) Q_H \quad (3)$$

$$\frac{dR_H}{dt} = \alpha_3 Q_H - (\mu_1 + \gamma_1) R_H \quad (4)$$

For the Rodent Populations:

$$\frac{dS_R}{dt} = \eta - (\mu_2 + \phi) S_H \quad (5)$$

$$\frac{dI_R}{dt} = \phi S_R - (\mu_2 + \rho) I_H \quad (6)$$

With initial conditions

$S_H(t) > 0, I_H(t) > 0, Q_R(t) > 0, R_H(t) = 0, S_R(t) > 0, I_R(t) > 0.$

The force of the infection $k = \frac{\beta c\, I_H}{N}$

$$\left.\begin{aligned} N_H(t) = S_H(t) + I_H(t) + Q_R(t) + R_H(t) \\ N_R(t) = S_R(t) + I_R(t) \end{aligned}\right\} \quad (7)$$

$$N(t) = N_H(t) + N_R(t) \quad (8)$$

Existence of Disease Free Equilibrium (DFE) E_f

In the absence of the disease, it implies that$I_H(t) = 0, Q_R(t) = 0, R_H(t) = 0,\ I_R(t) = 0.$ Therefore the above system of equations is reduced to

$$\frac{dS_H}{dt} = \pi - k\alpha_1 S_H - \mu_1 S_H \quad (9)$$

$$\frac{dS_R}{dt} = \eta - (\mu_2 + \phi) S_R \quad (10)$$

Hence letting equation (9) and (10) to zero and solving them simultaneously, we get

$$S_H = \frac{\pi}{\mu_1}, S_R = \frac{\eta}{\mu_2 + \phi},$$

Hence,

$$E_f = (S_H, I_H, Q_H, R_H, S_R, I_R) = \left(\frac{\pi}{\mu_1}, 0, 0, 0, \frac{\eta}{\mu_2 + \phi}, 0\right) \quad (11)$$

Computation of the Basic Reproductive Number $(\boldsymbol{R_0})$ of the Model

The basic reproductive number (R_0) is define as the number of secondary infections that one infectious individual would create over the duration of the infectious period, provided that everyone else is susceptible. R_0=1 is a threshold below which the generation of secondary cases is not sufficient to maintain the infection in human community. If R_0<1, the number of infected individuals will decrease from generation to next and the disease dies out and if R_0>1 the number of infected individuals will increase from generation to the next and the disease will persist.

We first rearranged the model Eqs (1) – (7) beginning with the infective classes to obtain the following equations below:

$$\frac{dI_H}{dt} = k\alpha_1 S_H - (\mu_1 + \alpha_2 + \omega) I_H \quad (12)$$

$$\frac{dI_R}{dt} = \phi S_R - (\mu_2 + \rho) I_H \quad (13)$$

$$\frac{dQ_H}{dt} = \alpha_2 I_H - (\mu_1 + \alpha_3) Q_H \quad (14)$$

$$\frac{dR_H}{dt} = \alpha_3 Q_H - (\mu_1 + \gamma_1) R_H \quad (15)$$

$$\frac{dS_R}{dt} = \eta - (\mu_2 + \phi) S_H \quad (16)$$

$$\frac{dS_H}{dt} = \pi + \rho I_R + \gamma_1 R_H - k\alpha_1 S_H - \mu_1 S_H \quad (17)$$

To compute the basic reproductive number (R_0) of the model Eqs (1) – (7), we employ the next generation method as applied in [3]. Using the approach in [3] we have

$$\mathcal{F}_i = \begin{pmatrix} \frac{\beta c \alpha_1}{N} I_H S_H \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (18)$$

$$\mathcal{F}_i = \begin{pmatrix} \frac{\beta c \alpha_1}{N} I_H S_H \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (19)$$

Where$\mathcal{F}_i$ and $\mathcal{V}_i$ are the rate of appearances of new infections in compartment i and the transfer of individuals into and out of compartment i by all other means respectively. Using the linearization method, the associated matrices at disease-free equilibrium (E_0) and after taking partial derivatives as defined by

$\mathcal{D}\mathcal{F}_i(E_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}and\mathcal{D}\mathcal{V}_i(E_0) = \begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix}$

Where$F$ is non-negative and V is a non-singular matrix, in which both are the $m \times m$ matrices defined by

$F = \left[\frac{\partial \mathcal{F}_i}{\partial x_j}(E_f)\right]$and $V = \left[\frac{\partial \mathcal{V}_i}{\partial x_j}(E_f)\right]$, with $1 \leq i, j \leq m$ and m is the number of infected classes. In particular $m = 3$, we have

$$f_i = \begin{pmatrix} k\alpha_1 S_H \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

$$v_i = \begin{pmatrix} (\mu_1 + \alpha_2 + \omega) I_H \\ -\phi S_R + (\mu_2 + \rho) I_H \\ -\alpha_2 I_H + (\mu_1 + \alpha_3) Q_H \end{pmatrix} \quad (21)$$

$$F = \begin{pmatrix} \frac{\beta c \alpha_1 \pi}{\mu_1 N} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (22)$$

$$V = \begin{pmatrix} (\mu_1 + \alpha_2 + \omega) & 0 & 0 \\ 0 & (\mu_2 + \rho) & 0 \\ -\alpha_2 & 0 & (\mu_1 + \alpha_3) \end{pmatrix} \quad (23)$$

and inverse of V is given such that

$$|FV^{-1} - \lambda| = \begin{vmatrix} \frac{\beta c \alpha_1 \pi}{\mu_1 N(\mu_1 + \alpha_2 + \omega)} - \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \quad (24)$$

And characteristics polynomial of Eq. (24) is given as

$\lambda^3 + \frac{\beta c\alpha_1\pi}{\mu_1 N(\mu_1+\alpha_2+\omega)}\lambda^2 = 0$(25)

and the eigenvalues is given by

$$\lambda_1 = 0,\ \lambda_2 = 0, \lambda_3 = \frac{\beta c\alpha_1\pi}{\mu_1 N(\mu_1+\alpha_2+\omega)} \qquad (26)$$

The most positive eigenvalues being the λ_2 is the Basic Reproduction Number (R_0)

Hence, we have

$$\boldsymbol{R_0} = \frac{\beta c\alpha_1\pi}{\mu_1 N(\mu_1+\alpha_2+\omega)} \qquad (27)$$

Stability Analysis of Disease Free Equilibrium State($\boldsymbol{E_0}$)

To study the behavior of the system Eqs. (1) – (7) around the disease-free equilibrium state $E_f = \left(\frac{\pi}{\mu_1}, 0, 0, 0, \frac{\pi}{\mu_2+\phi}, 0\right)$ we resort to the linearized stability approach.

Let

$$f_1 = \pi + \rho I_R + \gamma_1 R_H - k\alpha_1 S_H - \mu_1 S_H \qquad (28)$$

$$f_2 = k\alpha_1 S_H - (\mu_1+\alpha_2+\omega)I_H \qquad (29)$$

$$f_3 = \alpha_2 I_H - (\mu_1+\alpha_3)Q_H \qquad (30)$$

$$f_4 = \alpha_3 Q_H - (\mu_1+\gamma_1)R_H \qquad (31)$$

$$f_5 = \eta - (\mu_2+\phi)S_R \qquad (32)$$

$$f_6 = \phi S_R - (\mu_2+\rho)I_H \qquad (33)$$

The Jacobian (J_{E_f}) is given by

$$(J_{E_f}) = \begin{pmatrix} -\mu_1 & \frac{\beta c\alpha_1\pi}{\mu_1 N} & 0 & \gamma_1 & 0 & \rho \\ 0 & -(\mu_1+\alpha_2+\omega) + \frac{\beta c\alpha_1\pi}{\mu_1 N} & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & -(\mu_1+\alpha_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\mu_2+\phi) & 0 & 0 \\ 0 & 0 & 0 & \phi & -(\mu_2+\rho) & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -(\mu_1+\gamma_1) \end{pmatrix} \qquad (34)$$

Rewriting the matrix in Eq.(34), we get

$$(J_{E_f}) = \begin{pmatrix} -\mu_1 & \frac{\beta c\alpha_1\pi}{\mu_1 N} & 0 & \gamma_1 & 0 & \rho \\ 0 & -A + \frac{\beta c\alpha_1\pi}{\mu_1 N} & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & -B & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & 0 & 0 \\ 0 & 0 & 0 & \phi & -E & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -F \end{pmatrix} \qquad (35)$$

The determinant and the trace of matrix $\left(\boldsymbol{J_{E_f}}\right)$represented by Eq. (35) above is given

$$Det\left(J_{E_f}\right) = \frac{(AN - \beta c\pi\alpha_1)DF\mu_1 BE}{N} \quad (36)$$

$$Trace\left(\boldsymbol{J}_{\boldsymbol{E}_f}\right) = -\left(\mu_1 + A - \frac{\beta c\alpha_1\pi}{\mu_1 N} + B + D + E + F\right) \quad (37)$$

where,

$$\begin{cases} A = (\mu_1 + \alpha_2 + \omega), B = (\mu_1 + \alpha_3) \\ D = (\mu_2 + \phi), E = (\mu_2 + \rho), F = (\mu_1 + \gamma_1) \end{cases} \quad (38)$$

3.0 Numerical Simulations of the Experiments Model

In order to verify the theoretical predictions of the model, the numerical simulation of the Lassa fever dynamics control model incorporating quarantine class Eqs. (1)–(6) was solved numerically using Runge-Kutta-Fehllberg 4-5th order method and implemented using Maple 17 Software.

The parameters used in the implementation of the model are given by [13,14] as

Variables: $S_H(t)$= 0.017, $S_R(t)$= 0.0087, $I_H(t)$ = 0.000014,$I_R(t)$ = 0.007, $R_H(t)$ = 0.00002,$Q_H(t)$ = 0.000001.

Parameters: π = 0.0000215, μ_1 = 0.00000548, $\mu_2 = 0.00000213$, α_1 = 0.03, α_2 = 0.08, α_3 = 0.77, ω = 0.01, ρ = 0.00005, ϕ = 0.06, η = 0.05, c = 0.00018, γ_1 = 0.52.

List of Numerical Experiments

(1) The effect of treatment on the infected population when the quarantine rate is constant
(2) The effect of quarantine rate on the infected population when contact rate is constant.
(3) The effect of quarantine rate on the infected population with treatment rate when contact rate is constant
(4) The effect of quarantine rate on the recovered population contact rate is constant.

Experiment 1: The effect of treatment on the infected population when the qurantine rate is constant

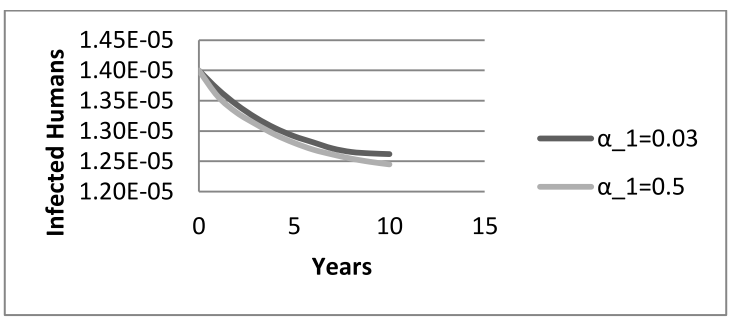

Fig.2 Graph showing the effecttreatment on the infected population at low and high ($\boldsymbol{\alpha_1 = 0.03}$, $\boldsymbol{\alpha_1 = 0.0, c = 0.00018}$)when the contact rate is constant.

Experiment 2: The effect of quarantine rate on the infected population when contact rate is constant

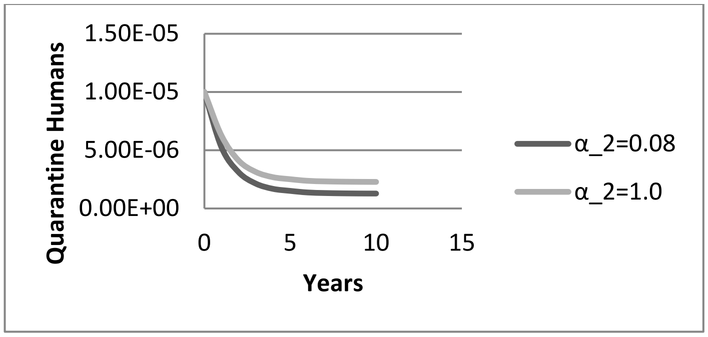

Fig.3 Graph showing the effectof quarantine on the infected population, when the quarantine rate is constant ($\boldsymbol{\alpha_2 = 0.1.0}$, $\boldsymbol{\alpha_2 = 0.008}$, $\boldsymbol{c = 0.00018}$)

Experiment 3: The effect of quarantine rate and treatment rate on the infected population when contact rate is constant

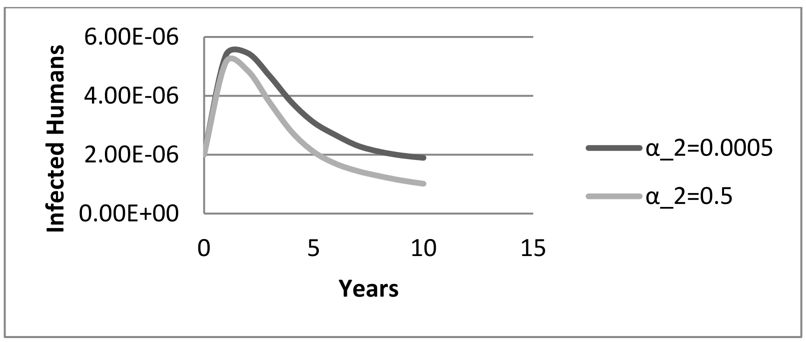

Fig.3 Graph showing the effect ofquarantine and treatment rate, when the quarantine rate is constant ($\boldsymbol{\alpha_2 = 0.0005}$, $\boldsymbol{\alpha_2 = 0.5}$, $\boldsymbol{c = 0.00018}$)

Experiment 4: The effect of quarantine rate on the recovered population contact rate is constant

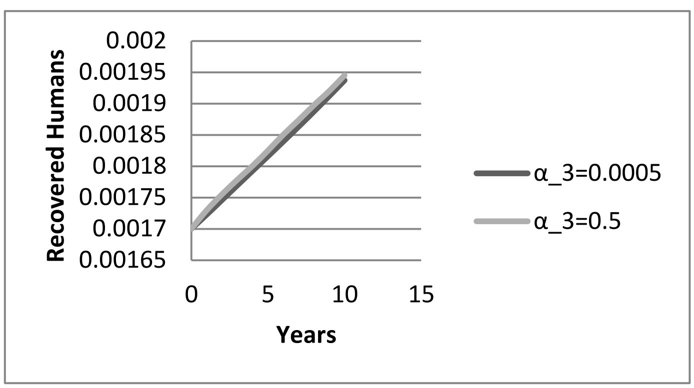

Fig.5: Graph showing the effect ofquarantine rate on recovered population when the quarantine rate is constant ($\boldsymbol{\alpha_3 = 0.0005}$, $\boldsymbol{\alpha_3 = 0.5}$, $\boldsymbol{c = 0.0001}$

Conclusion

In this article, a new mathematical model which incorporated some important factors that plays significant role in the control of Lassa fever was developed. These factors are disease induced death rate and the quarantine parameter. The introduced quarantine parameter helps in controlling and eradication of Lassa fever virus with respect to time. Furthermore, the basic reproduction numbers R_0 was calculated using the next generation approach. The analysis reveals that the disease can be control if the basic reproduction numberR_0,is less than one regardless of the initial population profile. Thus, every effort must be put in place by all concerned to prevent the virus infection by reducing R_0strictly to less than unity.

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ORDERING POLICY FOR AMELIORATING INVENTORY WITH LINEAR DEMAND RATE AND UNCONSTRAINED RETAILER'S CAPITAL

by

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Abstract

Items that incur a gradual increase in quality, quantity or both while in inventory are referred to as ameliorating items. Fruits, wine, high breed fishes in breeding yard (fish culture facility), fast growing animals like broiler, goose, rabbit etc. in farming yard provide good examples. When these items are stoked in the inventory or in the production centre, they undergo amelioration at some stages of their storage. This paper proposes a model that determines the optimal replenishment cycle time, such that the total variable cost is minimized.The amelioration rate and holding cost are constants, the demand rate throughout the cycle is linear time dependent, and shortages are not allowed. Numerical examples are provided to illustrate the application of the model developed.

Keywords:*Inventory, Amelioration, Ordering Policy, Linear demand rate, Unconstrained retailer's capital*

1.0 Introduction

In the classical inventory models, one of the assumptions was that the items preserved their physical characteristics while they were kept in the inventory or in the production centers. This assumption is not always true because some items are subject to risks of breakage, damage, spoilage, evaporation, obsolescence, etc. The decay that prevents items from being used for their original purpose is termed as deterioration.Although degradation (or loss) of value or utility or quantity of some physical goods is a commonexperience, Moon *et al. (*2003)observed that, there are some items whose value or utility increase over time by amelioration activation,e.g. wine. It is a common experience in wine manufacturing circle that utility or value of some kind of wine increases by time. Other examples can be seen with fruits (like orange, pineapple, mango etc.), high breed fishes in breeding yard (fish culture facility)or fast growing animals like broiler, goose, rabbit etc. in farming yard. These items at the initial stage of their storage or production environment undergo amelioration.

Two basic key factors are necessary in the development of models for ameliorating items these are: demand and amelioration rate. Demand acts as driving force of the entire inventory system and the amelioration rate stands for the characteristics of the ameliorating items. Li *et al*. (2010) classified demand into two types: the one that can be determined over certain period of time (Deterministic demand), for example, constant demand, time-dependent demand, inventory-level-dependent demand, price-dependent demand and among them Ramp-type demand are all deterministic demand, and Stochastic demand (the one that is characterized by a known distribution and the one that is characterized by arbitrary distribution). Hill (1995) was the first researcher to use Ramp type demand in his inventory model followed by Mandal and Pal (1998). Ameliorating rate and deteriorating rate are other key factors to be considered in developing model of ameliorating or deteriorating items. Earlier researchers like Ghare and Schrader (1963), Shah and Jaiswal (1977), Aggarwal (1978), Padmanabhana and Vrat (1995) and many others considered constant deteriorating rates in their models. However, recent researchers considered several scenarios of relationship between time and deteriorating rates. Some of these scenarios include: deteriorating rate as linear function of time, two-parameter Weibull distribution, three-parameter Weibull distribution and or deteriorating rate as other function of time.It is common experience in the market palace that the demand for inventory items increases with time in the growth phase, and decreases in the decline phase. So researchers commonly use a time-varying demand pattern to reflect sales in different phases of product circle. It all started with Silver and Meal in the early 1970's who came up with replenishment lot size model with deterministic time varying demand rate. Later, Donaldson (1977) came up with inventory replenishment policy for a linear trend in demand-an analytical solution. Recently, Ahmad and Musa (2016) developed an EOQ model with time dependent exponential declining demand.

The existing literature on inventory seems to ignore or give little attention to ameliorative inventories. Hwang (1997)was the first researcher to develop EOQ models for ameliorating items with the assumption that the ameliorating time follows the Weibull distribution. Again Hwang(1999) came up with other models for both ameliorating and deteriorating items separately considering LIFO and FIFO issuing policies. A partial selling inventory model for ameliorating items under profit maximization was developed by Mondal et al. (2003). Singh *et al.*(2011)used genetic algorithm to propose an optimal replenishment policy for ameliorating items under inflation and time value of money. Shortages were allowed and back-ordering was considered to be a decreasing function of waiting time. Panda *et al*. (2013) provides a note on inventory model for ameliorating items with time dependent second order demand rate. Harvest and sale decision problem of fresh agricultural products considering both amelioration of field items and deterioration of stored items was proposed by Chen (2011). This profit model of the farmer was developed via two situations: when the fresh agricultural products are harvested at maturing point and when they are harvested at critical ripeness point. Recent research on ameliorating items was carried out by Gwanda and Sani (2011). The model determined an optimum order quantity in which the demand rate, the amelioration rate and holding cost are all constants.

In the present article, an attempt has been made to propose an inventory model for ameliorating items in which the ameliorating rate and holding cost are constant. The demand rate is linear function of time throughout the cycle and retailer's capital is unconstrained, that is, the payment is affected on the receipt of the items in the inventory.

2.0 Assumptions and Notations

The model is developed based on the following assumptions and Notations:

i. Both the amelioration rate and holding cost are constants.

ii. The replenishment rate is instantaneous, lead time is zero.

iii. The inventory system involves only one single item and one stocking point.

iv. Shortages are not allowed.

v. Amelioration occurs when the items are effectively in stock.

vi. The demand rate λ is time dependent and linear i.e $\lambda(t) = \alpha + \beta t$.

vii.T is the length of the cycle and it is the time when the inventory level reaches zero.

$viii$ $I(t)$ is the inventory level at any time t.

ix Q is the optimal ordering quantity per cycle.

x. A_o is the fixed ordering cost per order.
xi. I_o is the initial inventory at $t = 0$.
xii. A_m is the ameliorated amount.
$xiii$. h is the inventory holding cost per unit per unit of time.
xiv c is the cost of each ameliorated item.
xv. $TVC(T)$ is the total (average) inventory cost per unit time.

3. 0 Mathematical formulation and solution

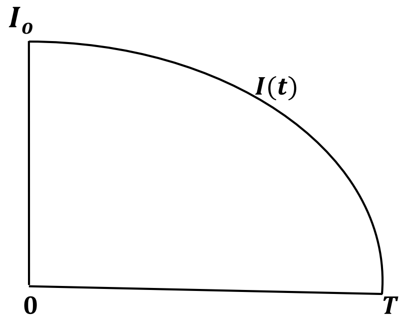

Fig 1: The inventory depletion in a constant amelioration system with no shortages

The depletion of the inventory during the interval $[0, T]$is a function of the ameliorating rate, demand rate and the remaining inventory level at the inventory system.Thus, the differential equation that describes the state of the inventory level $I(t)$during the time interval $(0 \leq t \leq T)$is given by: $\frac{dIt)}{dt} - AI(t) = -\lambda(t) 0 \leq t \leq T$Eq. (1)
Equation (1) is first order linear differential equation given by:

$$\frac{dI(t)}{dt} - AI(t) = -(\alpha + \beta t)$$

The integrating factor, $\rho = e^{-\int A dt} = e^{-At}$.
The solution is given by:

$$e^{-At} I(t) = -\int^{\gamma} e^{-At} (\alpha + \beta t) dt.$$

$\Longrightarrow I(t) = \frac{\alpha+\beta t}{A} + \frac{\beta}{A^2} + ke^{At}$, k is a constant, Eq. (2)

At $t = 0, I(0) = I_0$,

$$\therefore \quad I_0 = \frac{\alpha}{A} + \frac{\beta}{A^2} + k,$$

$\Longrightarrow \quad k = I_0 - \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right)$. Eq. (3)

Substituting (3) in (2),

$$I(t) = \frac{\alpha + \beta t}{A} + \frac{\beta}{A^2} + \left[I_0 - \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right)\right] e^{At}$$

$$= \frac{\alpha+\beta t}{A} + \frac{\beta}{A^2} + I_0 e^{At} - \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) e^{At} \quad \text{Eq. (4)}$$

At $t = T$, $I(t) = 0$,
Equation (4) becomes,

$$I_0 e^{AT} = \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) e^{AT} - \frac{\alpha+\beta T}{A} - \frac{\beta}{A^2}.$$

$\Longrightarrow I_0 = \frac{\alpha}{A} + \frac{\beta}{A^2} - \frac{\alpha+\beta T}{A} e^{-AT} - \frac{\beta}{A^2} e^{-AT}.$ Eq. (5)

Substituting (5) in (4),

$I(t) = \frac{\alpha+\beta t}{A} + \frac{\beta}{A^2} + \left(\frac{\alpha}{A} + \frac{\beta}{A^2} - \frac{\alpha+\beta T}{A} e^{-AT} - \frac{\beta}{A^2} e^{-AT}\right) e^{At} - \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) e^{At}.$

$= \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) - \left(\frac{\alpha+\beta T}{A} + \frac{\beta}{A^2}\right) e^{A(t-T)}.$ Eq. (6)

The ameliorated amount, A_m is given as Total demand in the cycle− The beginning inventory level.

$$\Longrightarrow A_m = \int_0^T \lambda(t)dt - \left[\left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) - \left(\frac{\alpha+\beta T}{A} + \frac{\beta}{A^2}\right) e^{-AT}\right].$$

$= \frac{\alpha+\beta T}{A} e^{-AT} - \frac{\beta}{A^2}(1 - e^{-AT}) + \alpha\left(T - \frac{1}{A}\right) - \frac{1}{2}\beta T^2.$ Eq. (7)

The total inventory carried in the cycle T, H_T is given as:

$$H_T = \int_0^T I(t)dt = \int_0^T \left[\left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) - \left(\frac{\alpha+\beta T}{A} + \frac{\beta}{A^2}\right) e^{A(t-T)}\right]$$

$= \left(\frac{\alpha+\beta T}{A^2} + \frac{\beta}{A^3}\right)(e^{-AT} - 1) + \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) T.$Eq.(8)

The total average cost per unit time $TVC(T)$ is given by:

$TVC(T) = \frac{1}{T}[ordering\ cost, A_0 + holding\ cost, H_c - cost\ of\ ameliorated\ items].$

$= \frac{1}{T}[A_0 + hH_T - cA_m].$

$$= \frac{A_0}{T} + \frac{h}{T}\left[\left(\frac{\alpha+\beta T}{A^2} + \frac{\beta}{A^3}\right)(e^{-AT} - 1) + \left(\frac{\alpha}{A} + \frac{\beta}{A^2}\right) T\right] - \frac{c}{T}\left[\frac{\alpha+\beta T}{A} e^{-AT} - \frac{\beta}{A^2}(1 - e^{-AT}) + \alpha\left(T - \frac{1}{A}\right) - \frac{1}{2}\beta T^2\right]$$

Eq. (9)

The necessary condition for $TVC(T)$ to be minimized is given by: $\frac{dTVC(T)}{dT} = 0.$

$$\Longrightarrow -\frac{A_0}{T^2} + \frac{1}{A^2 T^2}\left[\left((cA - h)AT + (c - h)\right)(\alpha + \beta T)e^{-AT} + \left(\frac{\beta h}{A} - c\beta\right)(1 - e^{-AT}) + \alpha(h - cA) + c\right]$$

$-\frac{c\beta}{2} = 0$ Eq.(10)

Multiplying equation (10) by T^2 we get:

$$-A_0 + \frac{1}{A^2}\left[\left((cA - h)AT + (c - h)\right)(\alpha + \beta T)e^{-AT} + \left(\frac{\beta h}{A} - c\beta\right)(1 - e^{-AT}) + \alpha(h - cA) + c\right]$$

$-\frac{c\beta}{2} T^2 = 0$ Eq. (11)

The Economic Order Quantity, EOQ, is given by:

The total demand in a cycle period – ameliorated mount.

$\Longrightarrow$ EOQ $= \int_0^T \lambda(t)dt - \left[\frac{\alpha+\beta T}{A} e^{-AT} - \frac{\beta}{A^2}(1 - e^{-AT}) + \alpha\left(T - \frac{1}{A}\right) - \frac{1}{2}\beta T^2\right]$

$$= \alpha T + \frac{\beta}{2} T^2 - \left[\frac{\alpha+\beta T}{A} e^{-AT} - \frac{\beta}{A^2}(1 - e^{-AT}) + \alpha\left(T - \frac{1}{A}\right) - \frac{1}{2}\beta T^2\right]$$

$= \frac{\beta}{A^2}(1 - e^{-AT}) + \frac{\alpha}{A}[(\alpha + \beta T) + 1] + \beta T^2$Eq. (12)

4.0 Numerical examples

For the purpose of numerical examples, eight parameter values in proper units are considered (as input) and the output of the model using Maple (2015) Mathematical Software gives the corresponding Optimal cycle length (T), the minimum total inventory cost(TVC) and Economic Order Quantity (EOQ) in the table below:

Table 1: EOQ, Total variable cost and optimal cycle length for ameliorating items with linear demand rate.

S/N	A_0	A	h	C	α	β	T	TVC	AOQ
1	1000	0.25	0.03	10	3	700	0.0423 (9 days)	21678	291
2	1000	0.35	0.03	10	3	700	0.0176 (6 days)	50396	175
3	1000	0.45	0.03	10	3	700	0.0037 (1 day)	229785	89
4	4000	0.33	0.75	30	10	500	0.0100 (4 days)	279537	367
5	4000	0.33	0.45	30	10	500	0.0198 (7 days)	285231	497
6	4000	0.33	0.25	30	10	500	0.0104 (4 days)	268802	507
7	4500	0.15	0.4	15	5	400	0.0016 (1 day)	245321	222
8	4500	0.15	0.2	15	5	400	0.0014 (1 day)	280365	225

5.0 Discussion of the results

From the above numerical examples we observe that amelioration rate and the holding cost affect the EOQ. It is clear that the higher the amelioration rate, the lower the EOQ. Thus, the stockiest is advised to purchase less in items with higher amelioration rate but purchase more in items with lower rate of amelioration. This is obvious for the stockiest to avoid over stocking which leads to the increase in total variable costs.Also we observe that the EOQ reduces with the rise in holding cost. Of course as the holding cost rises, it will be more economical to reduce the order quantity in order to maximize profit.

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