AN INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEM WITH TWO-PHASE DEMAND RATES AND EXPONENTIAL BACKLOGGING

by

Y. Bello[1 *] & Y. M. Baraya[2]

Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria

**Corresponding Author: Email address; yusufbello2831@gmail.com, 08139007975*

Abstract

This work investigates an inventory model for non-instantaneous deteriorating items with two - phase demand rates and exponential backlogging. Most of the time, the demand rate of newly produced products such as plasma, satellite, fridge, stabilizer, mobile phone, computer and fashionable garments increase with time. The demand rate for such products is constant for some period of time and after that, when the items become popular in the market, the demand for the items increase due to the quality of the products. Shortages are allowed and exponentially backlogged for there could be a situation where an economic advantage may be gained by allowing shortages. The objective of the model is to find the optimal cycle length and order quantity that minimizes the total average cost. Newton-Raphson method has been used to find the optimal cycle length and order quantity which minimizes the total average cost. The result illustrated with numerical examples. Sensitivity analysis is performed to show the effect of changes in the parameters on the optimum solution.

Keywords*: Two–phase demand rate, non-instantaneous deterioration and exponential backlogging.*

1. Introduction

Deterioration is usually referred to as damage, evaporation, spoilage, loss of potential or utility and so on, of a product through time. Some items like radioactive substances and highly volatile chemicals start deterioration process as soon as they are held in stock. These items are referred to as instantaneous deteriorating items. However, items like computer, mobile phone, television, plasma, and so on, do not start deterioration immediately they stocked until later, these items are referred to as non-instantaneous deteriorating items**.** So decay or deterioration of physical goods in stock is a very realistic feature. Therefore, there is need to consider deterioration when analyzing inventory models. Harris (1915) was the first researcher to develop an economic order quantity model. Wilson (1934) gave a formula to obtain economic order quantity. Whitin (1957) considered deterioration of the fashionable goods at the end of a prescribed shortage period. Ghare and Schrader (1963) developed a model for exponentially decaying inventory. An order-level inventory model for items deteriorating at a constant rate was presented by Shah and Jaiswal (1977). Aggarwal (1978) develop a note on an order level model for a system with constant deterioration. All these models discussed above are based on the constant deterioration rate, constant demand, infinite replenishment and no shortages. Other researches related to this area with shortage are Goswami and Chaudhuri (1995), Roy (2008), Baraya and Sani (2013), Datta and Kumar (2015), Janssen et.al (2016), Rangarajan and Karthikeyan (2017).

Inventory problems involving time variable demand pattern has received attention of researchers. It is observed that the demand rate of newly lunched products such as electronic goods, mobile phones, computers and fashionable garments increase with time. Silver and Meal (1973) established approximate solution techniques of deterministic inventory model with time dependent demand rate. Donalson (1977) introduced an inventory replenishment policy for a linear trend in demand having an analytical solution. Other papers related to this area are Ritchie (1980), Datta and Pal (1990), Dave and Patel (1994), Hariga and Goyal

(1995), Bhunia and Maiti (1998), Shinbsankar and Chaudhuri (2003), Ghosh and Chaudhuri (2006), Sunshil and Ravendra (2015), Amutha (2017) and so on. Sandeep *et al.* (2017) presented EOQ model for weibull distribution deteriorating items with exponential demand under linearly time dependent and shortage.

Singh *et al.* (2017) considered an optimal inventory policy for deteriorating items with time proportional deterioration rate and constant and time dependent demand without shortage. In real life situation, some items start to deteriorate as soon as they are placed on the shelf while others do not. The items that do not start deterioration instantly include mobile phone, computer, fridge, plasma, satellite, stabilizer, and fashionable garments. In the market place the goal for many businesses is to have their item/service available with high quality when the demand arises. Failing to do so can result in not only loss in profit that could have been gained by selling the items service, but may also damage the business's reputation which indeed takes a long time to build up. In this work we extend the work of Singh *et al.* (2017) by developing an inventory model for non- instantaneous deteriorating items with two-phase demand with exponential partial backlogging rate.

2. Model Description and Formulation

The inventory system is developed on basic of the following assumption and notation.

Notation and Assumptions.

Notation:

t_d	The time at which deterioration sets in as well as demand rate changes with time
t_1	The time at which inventory depletes to zero
C_O	The ordering cost per order
C	The unit cost of item
i	The inventory carrying charges
C_s	The shortage cost per unit of time
C_e	The lost sale per unit per time
T	The cycle time (decision variable)
N	The maximum inventory level
B	The backorder level during the shortage period
R	The order quantity during the cycle length
$z(t)$	The inventory level at any instant of time
$T - t_1$	The length of the waiting time

Assumptions:

This model is developed under the following assumptions

1. The demand rate is defined as
$$D(t) = \begin{cases} \alpha & 0 \le t \le t_d \\ \alpha + \beta(t - t_d) & t_d \le t \le t_1 \end{cases}$$
2. ω, 0 < ω < 1 is the deterioration rate. Deterioration sets in at t = t_d with constant deterioration rate

3. Shortages are allowed to occur and it is considered that only a fraction of demand is backlogged and it is denoted by B (t) = = $\alpha e^{-\delta(T-t)}$ where $T - t_1$ is the length of the waiting time.
4. Replenishment is non- instantaneous and planning horizon is infinite
5. The ordering cost, holding cost and unit cost remain constant.

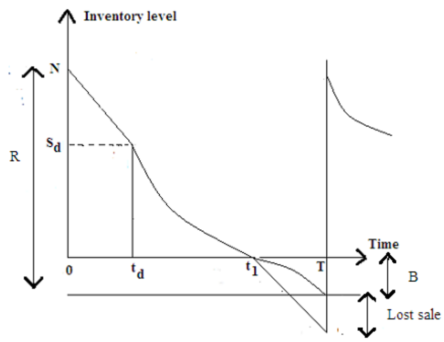

Figure 1: The graphical representation of the inventory level.

2.1 Mathematical Formulation of the Model

The inventory level as depicted in figure 1 depletes during period $[0, t_d]$ due to market demand only. At time $t = t_d$ deterioration sets in and depletion of inventory occurs due to demand and deterioration during the period t ε $[t_d, t_1]$. Shortages are allowed to occur during the period $[t_1, T]$.

The inventory level in the time interval $[0, t_d]$ is given by

$$\frac{dz(t)}{dt} = -\alpha \quad , \qquad 0 \le t \le t_d \tag{1}$$

with boundary conditions $z(t) = N$ at $t = 0$ and $z(t) = S_d$ at $t = t_d$.

The inventory level in the time interval $[t_{d,} t_1]$ is given by

$$\frac{dz(t)}{dt} + \omega z(t) = -[\alpha + \beta(t - t_d)] \quad , \qquad t_d \le t \le t_1 \tag{2}$$

with boundary conditions $z(t_d) = S_d$, and $z(t_1) = 0$

The inventory level in the time interval $[t_1, T]$ is given by

$$\frac{dz(t)}{dt} = -\alpha\, e^{-\delta(T-t)} \quad , \qquad t_1 \le t \le T \tag{3}$$

with boundary conditions $z(t_1) = 0$ and $z(T) = $ B

The solution of equation (1) during the period $[0, t_d]$ is as follows:

$$\int dz(t) = -\alpha \int dt$$

or

$$z(t) = -\alpha t + C_1 \tag{4}$$

where C_1 is the constant of integration.

Using the condition $z(t) = N$ at $t = 0$ in equation (4), we get

$$N = C_1$$

.and equation (4) become

$$z(t) = N - \alpha t \quad ,$$

$0 \le t \le t_d$ (5)

Using the boundary condition $z(t) = S_d$ at $t = t_d$ in (5) , we get

$$S_d = N - at_d \qquad (6)$$

This implies at $t = t_d$, the inventory level has been reduced by αt_d

Equation (2) being a first order linear differential equation whose integrating factor, $e^{\omega t}$ is solved as follows:

$$z(t)e^{\omega t} = -\int e^{\omega t}(\alpha + \beta t - \beta t_d)dt$$
$$= -\alpha\int e^{\omega t}dt - \beta\int t\, e^{\omega t}dt + \beta t_d\int e^{\omega t}dt$$
$$= -\frac{\alpha\, e^{\omega t}}{\omega} - \beta\left[\frac{te^{\omega t}}{\omega} - \frac{e^{\omega t}}{\omega^2}\right] + \frac{\beta t_d\, e^{\omega t}}{\omega} + C_2$$
$$= -\frac{\alpha\, e^{\omega t}}{\omega} - \frac{\beta\, t\, e^{\omega t}}{\omega} + \frac{\beta e^{\omega t}}{\omega^2} + \frac{\beta t_d\, e^{\omega t}}{\omega} + C_2 \qquad (7)$$

where C_2 is the constant of integration.

Using condition $z(t_1) = 0$ into (7), we get

$$0 = -\frac{\alpha\, e^{\omega t_1}}{\omega} - \frac{\beta t_1 e^{\omega t_1}}{\omega} + \frac{\beta\, e^{\omega t_1}}{\omega^2} + \frac{\beta t_d\, e^{\omega t_1}}{\omega} + C_2$$

or

$$C_2 = \frac{\alpha\, e^{\omega t_1}}{\omega} + \frac{\beta t_1 e^{\omega t_1}}{\omega} - \frac{\beta\, e^{\omega t_1}}{\omega^2} - \frac{\beta t_d\, e^{\omega t_1}}{\omega}$$

Substituting the value of C_2 into equation (7), we get

$$z(t)e^{\omega t} = \frac{\alpha\, e^{\omega t_1}}{\omega} + \frac{\beta t_1 e^{\omega t_1}}{\omega} - \frac{\omega\, e^{\omega t_1}}{\omega^2} - \frac{\beta t_d e^{\omega t_1}}{\omega} - \frac{\alpha\, e^{\omega t}}{\omega} - \frac{\beta t\, e^{\omega t}}{\omega} + \frac{\beta\, e^{\omega t}}{\omega^2} + \frac{\beta t_d\, e^{\omega t}}{\omega}$$

or

$$z(t) = \frac{\beta}{\omega^2}\left(1 - e^{\omega(t_1 - t)}\right) + \frac{\beta t_d}{\omega}\left(1 - e^{\omega(t_1 - t)}\right) - \frac{\alpha}{\omega}\left(1 - e^{\omega(t_1 - t)}\right) + \frac{\beta}{\omega}\left(t_1 e^{\omega(t_1 - t)} - t\right)$$

$$= \frac{\left(1 - e^{\omega(t_1 - t)}\right)}{\omega}\left[\frac{\beta}{\omega} + \beta t_d - \alpha\right] + \frac{\beta}{\omega}\left(t_1 e^{\omega(t_1 - t)} - t\right) , \quad t_d \le t \le t_1 \qquad (8)$$

Putting the condition $z(t_d) = S_d$ into equation (8) , we get

$$S_d = \frac{\left(1 - e^{\omega(t_1 - t_d)}\right)}{\omega}\left[\frac{\beta}{\omega} + \beta t_d - \alpha\right] + \frac{\beta}{\omega}\left(t_1 e^{\omega(t_1 - t_d)} - t_d\right) \qquad (9)$$

Combining (6) and (9), the maximum inventory level is given by

$$N = \alpha t_d + \frac{\left(1 - e^{\omega(t_1 - t_d)}\right)}{\omega}\left[\frac{\beta}{\omega} + \beta t_d - \alpha\right] + \frac{\beta}{\omega}\left(t_1 e^{\omega(t_1 - t_d)} - t_d\right) \qquad (10)$$

The solution of equation (3) can be obtained as follows:

$$\int dz(t) = -\alpha \int e^{-\delta(T-t)}$$

$$z(t) = -\frac{\alpha}{\delta} e^{-\delta(T-t)} + C_3 \qquad (11)$$

Where C_3 is constant of integration

Applying the boundary condition $z(t) = 0$ at $t = t_1$ in (11) yields

$$0 = -\frac{\alpha}{\delta} e^{-\delta(T-t_1)} + C_3$$

or

$$C_3 = \frac{\alpha}{\delta} e^{-\delta(T-t_1)}$$

Substituting C_3 in equation (11), we have

$$z(t) = -\frac{\alpha}{\delta} e^{-\delta(T-t)} + \frac{\alpha}{\delta} e^{-\delta(T-t_1)}$$

$$= \frac{\alpha}{\delta} e^{-\delta(T-t_1)} - \frac{\alpha}{\delta} e^{-\delta(T-t)} \qquad t_1 \le t \le T \qquad (12)$$

The order quantity per order is given by

$R =$ Maximum inventory level + Backorder

$$= \alpha t_d + \frac{\left(1 - e^{\omega(t_1-t_d)}\right)}{\omega}\left[\frac{\beta}{\omega} + \beta t_d - t_d\right] + \frac{\beta}{\omega}\left(t_1 e^{\omega(t_1-t_d)} - t_d\right) + \frac{\alpha}{\delta}\left[1 - e^{-\delta(T-t_1)}\right] \qquad (13)$$

The average total cost is composed of the following cost components:

(i) The ordering cost per order is given by C_0

(ii) The holding cost is given by

$$HC = iC\int_0^{t_1} z(t)dt \quad = iC\int_0^{t_d} z(t)\,dt + iC\int_{t_d}^{t_1} z(t)dt$$

Substituting equation (5) and equation (8) into above expression, we get

$$= iC\int_0^{t_d}(N - \alpha\, t)dt + iC\int_{t_d}^{t_1}\left(\frac{\left(1 - e^{\omega(t_1-t)}\right)}{\omega}\right)\left(\frac{\beta}{\omega} + \beta\, t_d - \alpha\right) + \frac{\beta iC}{\omega}\int_{t_d}^{t_1}\left(t_1 e^{\omega(t_1-t)} - t\right)dt$$

$$= iC\int_0^{t_d}\left[\alpha\, t_d + \frac{1}{\omega}\left(1 - e^{\omega(t_1-t_d)}\right)\left(\frac{\beta}{\omega} + \beta\, t_d - \alpha\right) + \frac{\beta}{\omega}\left(t_1 e^{\omega(t_1-t_d)} - t_d\right) - \alpha\, t\right]dt$$

$$+ \frac{iC}{\omega}\int_{t_d}^{t_1}\left(1 - e^{\omega(t_1-t)}\right)\left(\frac{\beta}{\omega} + \beta\, t_d - \alpha\right)dt + \frac{\beta iC}{\omega}\int_{t_d}^{t_1}\left(t_1 e^{\omega(t_1-t)} - t\right)dt$$

$$= iC\left[\alpha t_d\, t + \frac{1}{\omega}\left(\frac{\beta}{\omega} + \beta t_d - \alpha\right)\left(t - te^{\omega(t_1-t_d)}\right) + \frac{\beta}{\omega}\left(t_1 te^{\omega(t_1-t_d)} - t_d\, t\right) - \frac{\alpha t^2}{2}\right]_0^{t_d}$$

$$+ \left[\frac{iC}{\omega}\left(\frac{\beta}{\omega} + \beta t_d - \alpha\right)\left(t + \frac{e^{\omega(t_1-t)}}{\omega}\right)\right]_{t_d}^{t_1} + \frac{\beta iC}{\omega}\left[\frac{-t_1 e^{\omega(t_1-t)}}{\omega} - \frac{t^2}{2}\right]_{t_d}^{t_1}$$

$$= \alpha iC t_d^2 + \frac{iC}{\omega}\left(\frac{\beta}{\omega} + \beta t_d - \alpha\right)\left(t_d - t_d\, e^{(t_1-t_d)}\right) + \frac{\beta iC}{\omega}\left(t_d\, t_1 e^{\omega(t_1-t_d)} - t_d^2\right)$$

$$- \frac{\alpha t_d^2\, iC}{2} + \frac{iC}{\omega}\left(\frac{\beta}{\omega} + \beta t_d - \alpha\right)\left(t_1 - t_d + \frac{1}{\omega} - \frac{e^{\omega(t_1-t_d)}}{\omega}\right)$$

$$+ \frac{\beta iC}{\omega}\left(\frac{-t_1}{\omega} + \frac{t_1 e^{\omega(t_1-t_d)}}{\omega} - \frac{t_1^2}{2} + \frac{t_d^2}{2}\right)$$

$$= \alpha t_d^2 iC - \frac{\alpha t_d^2 iC}{2} + \frac{iC}{\omega}\left(\frac{\beta}{\omega} + \beta t_d - \alpha\right)\left(t_d - t_d e^{\omega(t_1-t_d)}\right)$$

$$+ \frac{\beta iC}{\omega}\left(t_d\, t_1 e^{\omega(t_1-t_d)} - t_d^2\right) + \frac{iC}{\omega}\left(\frac{\beta}{\omega} + \beta t_d - \alpha\right)\left(t_1 - t_d + \frac{1}{\omega} - \frac{e^{\omega(t_1-t_d)}}{\omega}\right)$$

$$+ \frac{\beta iC}{\omega}\left(\frac{-t_1}{\omega} + \frac{t_1 e^{\omega(t_1-t_d)}}{\omega} - \frac{t_1^2}{2} + \frac{t_d^2}{2}\right)$$

$$= \frac{\alpha t_d^2 iC}{2} + \frac{\beta t_d\, iC}{\omega^2} - \frac{\beta t_d\, iC}{\omega^2} e^{\omega(t_1-t_d)} + \frac{\beta t_d^2 iC}{\omega} - \frac{\beta t_d^2 iC}{\omega} e^{\omega(t_1-t_d)} - \frac{\alpha t_d\, iC}{\omega}$$

$$+ \frac{\alpha\, t_d\, iC}{\omega} e^{\omega(t_1-t_d)} + \frac{\beta\, t_d\, t_1 iC}{\omega} e^{\omega(t_1-t_d)} - \frac{\beta t_d^2 iC}{\omega} + \frac{\beta t_1 iC}{\omega^2} - \frac{\beta t_d\, iC}{\omega^2} + \frac{\beta iC}{\omega^3}$$

$$- \frac{\beta iC}{\omega^3} e^{\omega(t_1-t_d)} + \frac{\beta t_d\, t_1 iC}{\omega} - \frac{\beta \omega^2 iC}{\omega} + \frac{\beta t_d\, iC}{\omega^2} - \frac{\beta t_d\, iC}{\omega^2} e^{\omega(t_1-t_d)} - \frac{\alpha t_1 iC}{\omega}$$

$$+ \frac{\alpha t_d\, iC}{\omega} - \frac{\alpha iC}{\omega^2} + \frac{\alpha iC}{\omega^2} e^{\omega(t_1-t_d)} - \frac{\beta t_1 iC}{\omega^2} + \frac{\beta t_1 iC}{\omega^2} e^{\omega(t_1-t_d)} - \frac{\beta t_1^2 iC}{2\omega} + \frac{\beta t_d^2 iC}{2\omega}$$

$$= \frac{\alpha\, t_d^2 iC}{2} - \frac{2\beta t_d\, iC}{\omega^2} e^{\omega(t_1-t_d)} - \frac{\beta t_d^2 iC}{\omega} e^{\omega(t_1-t_d)} + \frac{\alpha t_d\, iC}{\omega} e^{\omega(t_1-t_d)}$$

$$+ \frac{\beta t_d\, iC\, t_1}{\omega} e^{\omega(t_1-t_d)} + \frac{\beta t_d\, iC\, t_1}{\omega} - \frac{\alpha\, iC t_1}{\omega} - \frac{\beta iC\, t_d^2}{\omega} + \frac{\beta iC}{\omega^3} + \frac{\beta t_d\, iC}{\omega^2} - \frac{\alpha\, iC}{\omega^2}$$

$$- \frac{\beta iC}{\omega^3} e^{\omega(t_1-t_d)} + \frac{\alpha\, iC}{\omega^2} e^{\omega(t_1-t_d)} + \frac{\beta iC\, t_1}{\omega^2} - \frac{\beta iC\, t_1^2}{2\omega} + \frac{\alpha\, t_d^2}{2\omega}$$

$$= \frac{\beta iC}{\omega^3}\left[1 - e^{\omega(t_1-t_d)}\right] + \frac{\alpha\, iC}{\omega^2}\left[e^{\omega(t_1-t_d)} - 1\right] + \frac{\alpha\, t_d\, iC}{\omega}\left[e^{\omega(t_1-t_d)} + \frac{t_d\, \omega}{2}\right]$$

$$- \frac{\beta t_d^2 iC}{\omega}\left[e^{\omega(t_1 - t_d^2)} + \frac{1}{2}\right] + \frac{\beta t_d\, iC}{\omega^2}\left[1 - 2e^{\omega(t_1-t_d)}\right] + \frac{\beta t_d\, t_1 iC}{\omega}\left[e^{\omega(t_1-t_d)} + 1\right] - \frac{\alpha\, t_1 iC}{\omega}$$

$$+ \frac{\beta t_1 iC}{\omega^2} e^{\omega(t_1-t_d)} - \frac{\beta iC\, t_1^2}{2\omega} + \frac{\beta\, t_d^2 iC}{2\omega} \qquad (14)$$

(iii) The inventory deterioration cost during the period [0, T] is given by

$$DC = C\omega\int_{t_d}^{t_1} z(t)dt$$

Putting equation (5) in the above expression, we have

$$= \omega C\left[\int_{t_d}^{t_1}\left(\frac{\left(1-e^{\omega(t_1-t)}\right)}{\omega}\right)\left(\frac{\beta}{\omega}+\beta t_d-\alpha\right)\,dt+\frac{\beta}{\omega}\int_{t_d}^{t_1}\left(t_1e^{\omega(t_1-t)}-t\right)\right]dt$$

$$= \omega C\left[\int_{t_d}^{t_1}\left(\frac{1}{\omega}\left(1-e^{\omega(t_1-t)}\right)\right)\left(\frac{\beta}{\omega}+\beta t_d-\alpha\right)\,dt+\frac{\beta}{\omega}\int_{t_d}^{t_1}\left(t_1e^{\omega(t_1-t)}-t\right)\right]dt$$

$$= \omega C\left[\frac{1}{\omega}\left(\frac{\beta}{\omega}+\beta t_d-\alpha\right)\left(t+\frac{e^{\omega(t_1-t)}}{\omega}\right)+\frac{\beta}{\omega}\left(\frac{-t_1e^{\omega(t_1-t)}}{\omega}-\frac{t^2}{2}\right)\right]_{t_d}^{t_1}$$

$$= \omega C\left[\frac{1}{\omega}\left(\frac{\beta}{\omega}+\beta t_d-\alpha\right)\left(t_1-t_d+\frac{1}{\omega}-\frac{e^{\omega(t_1-t_d)}}{\omega}\right)\right]$$

$$+\frac{\beta\omega C}{\omega}\left[\left(\frac{-t_1}{\omega}+\frac{t_1}{\omega}e^{\omega(t_1-t_d)}-\frac{t_1^2}{2}+\frac{t_d^2}{2}\right)\right]$$

$$= \omega C\left[\frac{\beta t_1}{\omega^2}-\frac{\beta t_d}{\omega^2}+\frac{\beta}{\omega^3}-\frac{\beta}{\omega^3}e^{\omega(t_1-t_d)}+\frac{\beta t_d t_1}{\omega}-\frac{\beta t_d^2}{\omega}+\frac{\beta t_d}{\omega^2}-\frac{\beta t_d}{\omega^2}e^{\omega(t_1-t_d)}-\frac{\alpha t_1}{\omega}+\frac{\alpha t_d}{\omega}-\frac{\alpha}{\omega^2}\right.$$
$$\left.+\frac{\alpha}{\omega^2}e^{\omega(t_1-t_d)}-\frac{\beta t_1}{\omega^2}+\frac{\beta t_1}{\omega^2}e^{\omega(t_1-t_d)}-\frac{\beta t_1^2}{2\omega}+\frac{\beta t_d^2}{2\omega}\right]$$

$$= C\omega\left[\frac{\beta}{\omega^3}\left(1-e^{\omega(t_1-t_d)}\right)+\frac{\alpha}{\omega^2}\left(e^{\omega(t_1-t_d)}-1\right)+\frac{\beta t_1}{\omega^2}\left(e^{\omega(t_1-t_d)}-\frac{\omega t_1}{2}\right)-\frac{\beta t_d\,e^{\omega(t_1-t_d)}}{\omega^2}\right.$$
$$\left.-\frac{\alpha(t_1-t_d)}{\omega}+\frac{\beta t_d t_1}{\omega}+\frac{\beta t_d^2}{2\omega}\right]$$

$$= C\left[\frac{\beta}{\omega^2}\left(1-e^{\omega(t_1-t_d)}\right)+\frac{\alpha}{\omega}\left(e^{\omega(t_1-t_d)}-1-\omega(t_1-t_d)\right)+\frac{\beta t_1}{\omega}\left(e^{\omega(t_1-t_d)}-\frac{\omega t_1}{2}\right)+\beta t_d t_1\right.$$
$$\left.+\frac{\beta t_d^2}{2}\right] \tag{15}$$

(iv) The shortage cost is given by

$$Sc = C_s\int_{t_1}^{T}-z(t)dt$$
$$= -\frac{\alpha C_s}{\delta}\int_{t_1}^{T}\left(e^{-\delta(T-t_1)}-e^{-\delta(T-t)}\right)dt$$
$$= -\frac{\alpha C_s}{\delta}\left[te^{-\delta(T-t_1)}-\frac{e^{-\delta(T-t)}}{\delta}\right]_{t_1}^{T}$$
$$= -\frac{\alpha C_s}{\delta}\left[(T-t_1)e^{-\delta(T-t_1)}-\frac{1}{\delta}+\frac{e^{-\delta(T-t_1)}}{\delta}\right]$$
$$= -\frac{\alpha C_s}{\delta}\left[(T-t_1)e^{-\delta(T-t_1)}+\frac{1}{\delta}\left(e^{-\delta(T-t_1)}-1\right)\right] \tag{16}$$

(v) The lost sale during cycle is given by

$$\mathrm{LS} = \int_{t_1}^{T}\alpha C_e\left(1-e^{-\delta(T-t)}\right)dt$$

$$= \alpha\, C_e \left[t - \frac{1}{\delta} e^{-\delta(T-t)}\right]_{t_1}^{T}$$
$$= \alpha\, C_e \left[(T - t_1) - \frac{1}{\delta} + \frac{1}{\delta} e^{-\delta(T-t_1)}\right] \quad (17)$$

(vi) The total amount of backorder at the end of cycle is

$$\mathrm{B} = \int_{t_1}^{T} \alpha\, e^{-\delta(T-t)} dt$$
$$= \left[\frac{\alpha}{\delta} e^{-\delta(T-t)}\right]_{t_1}^{T}$$
$$= \frac{\alpha}{\delta} e^{-\delta(T-T)} - \frac{\alpha}{\delta} e^{-\delta(T-t_1)}$$
$$= \frac{\alpha}{\delta}\left[1 - e^{-\delta(T-t_1)}\right] \quad (18)$$

The total average cost per unit time is a function of t_1 and T and is given by
ATC$(t_{1\,,}\, T)$ = $\frac{1}{T}$[holding cost + deterioration cost + shortage cost + backorder cost +lost sale]
$= \frac{1}{T}[C_O + HC + DC + SC + B + LS]$
Substituting the components of cost function, we have

$$\begin{aligned} ATC(t_{1\,,} T\,) = \frac{1}{T}\Big\{ & C_O + \frac{\beta iC}{\omega^3}\left[1 - e^{\omega(t_1 - t_d)}\right] + \frac{\alpha iC}{\omega^2}\left[e^{\omega(t_1-t_d)} - 1\right] + \frac{\beta t_d\, iC}{\omega}\left[e^{\omega(t_1-t_d)} + \frac{t_d\,\omega}{2}\right] \\ & - \frac{\beta\, t_d^2 iC}{\omega}\left[e^{\omega(t_1 - t_d)} + \frac{1}{2}\right] + \frac{\beta\, t_d\, iC}{\omega^2}\left[1 - 2e^{\omega(t_1 - t_d)}\right] + \frac{\beta\, t_d\, t_1 iC}{\omega}\left[e^{\omega(t_1 - t_d)} + 1\right] \\ & - \frac{\alpha\, t_1 iC}{\omega} + \frac{\beta t_1 iC}{\omega^2} e^{\omega(t_1 - t_d)} - \frac{\beta t_1 iC}{2\omega} + \frac{\beta t_d^2 iC}{2\omega} + \frac{C}{\omega^2}\left[1 - e^{\omega(t_1 - t_d)}\right] \\ & + \frac{C\alpha}{\omega}\left[e^{\omega(t_1-t_d)} - 1 - \omega(t_1 - t_d)\right] + \frac{\beta t_1 C}{\omega}\left[e^{\omega(t_1-t_d)} - \frac{\omega\, t_1^2}{2}\right] + \beta\, t_d\, t_1 C - \frac{C\beta\, t_d^2}{\omega} \\ & - \frac{\alpha\, C_s}{\delta}\left[(T - t_1)e^{-\delta(T-t_1)} + \frac{1}{\delta}(e^{-\delta(T-t_1)} - 1)\right] + \frac{\alpha}{\delta}\left[1 - e^{-\delta(T-t_1)}\right] \\ & + \alpha\, C_e\left[(T - t_1) - \frac{1}{\delta} + \frac{1}{\delta} e^{-\delta(T-t_1)}\right] \Big\} \end{aligned} \quad (19)$$

Since $t_1 < T$, then by letting $t_1 = r\,T$ with $0 < \mathrm{r} < 1$ equation (19) reduces to

$$\begin{aligned} ATC(T) = & \frac{C_O}{T} + \frac{\beta iC}{T\omega^3} - \frac{\beta iC}{T\omega^3} e^{\omega(rT-t_d)} + \frac{\alpha\, iC}{T\omega^2} e^{\omega(rT-t_d)} - \frac{\alpha\, iC}{T\omega^2} + \frac{\beta t_d\, iC}{T\omega} e^{\omega(rT-t_d)} \\ & + \frac{\beta t_d^2 iC}{2T} - \frac{\beta\, t_d^2 iC}{T\omega} e^{\omega(rT-t_d)} - \frac{\beta t_d^2 iC}{2T\omega} + \frac{\beta t_d\, iC}{T\omega^2} - \frac{2\,\beta t_d\, iC}{T\omega^2} e^{\omega(rT-t_d)} \\ & + \frac{\beta t_d\, iC\, r}{\omega} e^{\omega(rT-t_d)} - \frac{\beta t_d\, iC\, r}{\omega} - \frac{\alpha\, r\, iC}{\omega} + \frac{\beta r\, iC}{\omega^2} e^{\omega(rT-t_d)} - \frac{\beta r\, iC}{2\omega} + \frac{\beta t_d^2 iC}{2\omega T} + \frac{\beta C}{T\omega^2} \\ & - \frac{\beta C}{T\omega^2} e^{\omega(rT-t_d)} + \frac{\alpha\, C}{T\omega} e^{\omega(rT-t_d)} - \frac{\alpha\, C}{T\omega} - \frac{\alpha\, C}{T}(rT - t_d) + \frac{\beta r\, C}{\omega} e^{\omega(rT-t_d)} - \frac{\beta r^3 T^2 C}{2} \end{aligned}$$

$$+\beta t_d\, r\, C - \frac{\beta C\, t_d^2}{T\omega} - \frac{\alpha C_s}{\delta} e^{-\delta T\,(1-r)} + \frac{\alpha r\, C_s}{\delta} e^{-\delta T\,(1-r)} - \frac{\alpha C_s}{\delta^2 T} e^{-\delta T\,(1-r)} + \frac{\alpha C_s}{\delta^2 T} + \frac{\alpha}{\delta T} - \frac{\alpha}{\delta T} e^{-\delta T\,(1-r)} + C_e \alpha - C_e \alpha\, r - \frac{C_e \alpha}{\delta T} + \frac{C_e \alpha}{\delta T} e^{-\delta T\,(1-r)} \qquad (20)$$

which is a cost function of a single variable T.

3. Optimal Decision

The necessary condition for the existence of optimum values of ATC (T) is given by

$$\frac{dATC(T)}{dT} = 0 \qquad (21)$$

and sufficiency condition is

$$\frac{d^2ATC(T)}{dT} > 0 \qquad (22)$$

Differentiating (20) with respect to T, yield

$$\frac{dATC(T)}{dT} = -\frac{C_O}{T^2} - \frac{\beta iC}{\omega^3 T^2} + \frac{\beta iC}{\omega^3 T^2} e^{\omega\,(rT-t_d)} - \frac{\beta r\, iC}{\omega^2 T} e^{\omega\,(rT-t_d)} - \frac{\alpha\, iC}{\omega^2 T^2} e^{\omega\,(rT-t_d)}$$
$$+ \frac{\alpha\, iCr}{\omega T} e^{\omega\,(rT-t_d)} + \frac{\alpha\, iC}{\omega^2 T^2} - \frac{\omega\, t_d\, iC}{\omega T^2} e^{\omega(rT-t_d)} + \frac{\beta t_d\, r\;\, iC}{T} e^{\omega(rT-t_d)} - \frac{\beta t_d^2 iC}{2T^2}$$
$$+ \frac{\beta iC\, t_d^2}{\omega T^2} e^{\omega(rT-t_d)} - \frac{\beta iC\, t_d^2 r}{T} e^{\omega\,(rT-t_d)} + \frac{\beta\, t_d^2 iC}{2\omega T^2} - \frac{\beta t_d\, iC}{\omega^2 T^2} + \frac{2\beta t_d\, iC}{T^2 \omega^2} e^{\omega\,(rT-t_d)}$$
$$- \frac{2\beta t_d\, iC}{\omega T} e^{\omega\,(rT-t_d)} + \beta r^2 t_d\, iC e^{\omega\,(rT-t_d)} + \frac{\beta iC\omega r^2}{\omega} e^{\omega\,(rT-t_d)} - \frac{\beta iC\, t_d^2}{2\omega T^2}$$
$$- \frac{\beta C}{T^2 \omega^2} + \frac{\beta C}{T^2 \omega^2} e^{\omega\,(rT-t_d)} - \frac{\beta C\, r}{\omega T} e^{\omega(rT-t_d)} - \frac{\alpha C}{\omega T^2} e^{(rT-t_d)} + \frac{\alpha r\, C}{T} e^{\omega(rT-t_d)} + \frac{\alpha C}{\omega T^2}$$
$$+ \frac{\alpha C}{T^2} (rT - t_d) - \frac{\alpha r\, C}{T} + \beta\, r^2 C e^{\omega\,(rT-t_d)} - \beta r^3 C\, T + \frac{\beta C\, t_d^2}{\omega T^2}$$
$$+ \frac{\alpha C_s}{\delta} (1-r) e^{-\delta T\,(1-r)} - \frac{\alpha r\, C_s}{\delta} (1-r) e^{-\delta T\,(1-r)} + \frac{\alpha C_s}{\delta^2 T^2} e^{-\delta T\,(1-r)}$$
$$+ \frac{\alpha C_s}{\delta^2 T} (1-r) e^{-\delta T\,(1-r)} - \frac{\alpha C_s}{\delta^2 T^2} - \frac{\alpha}{\delta T^2} + \frac{\alpha}{\delta T^2} e^{-\delta T\,(1-r)} + \frac{\alpha}{\delta T}\, (1-r) e^{-\delta T\,(1-r)}$$
$$+ \frac{C_e \alpha}{\delta T^2} - \frac{C_e\, \alpha}{\delta T^2} e^{-\delta T\,(1-r)} - \frac{C_e\, \alpha}{\delta T} (1-r) e^{-\delta T\,(1-r)}$$

$$= -\frac{C_O}{T^2} - \frac{\beta iC}{\omega^3 T^2} + \frac{\beta iC}{\omega^3 T^2} e^{\omega\,(rT-t_d)} - \frac{\beta r\, iC}{\omega^2 T} e^{\omega\,(rT-t_d)} - \frac{\alpha\, iC}{\omega^2 T^2} e^{\omega\,(rT-t_d)}$$
$$+ \frac{\alpha\, iC\, r}{\omega T} e^{\omega(rT-t_d)} + \frac{\alpha\, iC}{\omega^2 T^2} - \frac{\beta r\, iC}{\omega T^2} e^{\omega(rT-t_d)} + \frac{\beta t_d\, r\, iC}{T} e^{\omega\,(rT-t_d)} - \frac{\beta t_d^2 iC}{2T^2}$$
$$+ \frac{\beta iC\, t_d^2}{\omega T^2} e^{\omega\,(rT-t_d)} - \frac{\beta t_d^2\, iCr}{T}\, e^{\omega\,(rT-t_d)} - \frac{\beta t_d\, iC}{\omega^2 T^2} + \frac{2\,\beta t_d\, iC}{T^2 \omega^2} e^{\omega\,(rT-t_d)}$$

$$-\frac{2\beta t_d\, iC}{\omega T}e^{\omega(rT-t_d)}+\beta r^2 t_d\, iCe^{\omega(T-t_d)}+\frac{\beta\,\omega\, iCr^2}{\omega}e^{\omega(rT-t_d)}-\frac{\alpha\, C}{\omega T^2}e^{\omega(rT-t_d)}$$
$$+\frac{\alpha\, r\, C}{T}e^{\omega(rT-t_d)}+\frac{\alpha\, C}{\omega T^2}+\frac{\alpha\, C\, r}{T}-\frac{\alpha\, C\, t_d}{T^2}-\frac{\alpha\, r\, C}{T}+\beta r^2 Ce^{\omega(rT-t_d)}-\beta r^3 C\, T+\frac{\beta C\, t_d^2}{\omega T^2}$$
$$+\frac{\alpha\, C_s}{\delta}(1-r)e^{-\delta T(1-r)}-\frac{\alpha\, r\, C_s}{\delta}(1-r)e^{-\delta T(1-r)}+\frac{\alpha\, C_s}{\delta^2 T^2}e^{-\delta T(1-r)}$$
$$+\frac{\alpha\, C_s}{\delta^2 T}(1-r)e^{-\delta T(1-r)}-\frac{\alpha\, C_s}{\delta^2 T^2}-\frac{\alpha}{\delta T^2}+\frac{\alpha}{\delta T^2}e^{-\delta T(1-r)}+\frac{\alpha}{\delta T}(1-r)e^{-\delta T(1-r)}$$
$$+\frac{C_e\alpha}{\delta T^2}-\frac{C_e\,\alpha}{\delta T^2}e^{-\delta T(1-r)}-\frac{C_e\,\alpha}{\delta T}(1-r)e^{-\delta T(1-r)}$$

Setting $\frac{dATC(T)}{dT}=0$, we get

$$-\frac{C_O}{T^2}-\frac{\beta\, iC}{\omega^3 T^2}+\frac{\beta\, iC}{\omega^3 T^2}e^{\omega(rT-t_d)}-\frac{\beta r\, iC}{\omega^2 T}e^{\omega(rT-t_d)}-\frac{\alpha\, iC}{\omega^2 T^2}e^{\omega(rT-t_d)}$$
$$+\frac{\alpha\, iC\, r}{\omega T}e^{\omega(rT-t_d)}+\frac{\alpha\, iC}{\omega^2 T^2}-\frac{\beta r\, iC}{\omega T^2}e^{\omega(rT-t_d)}+\frac{\beta t_d\, r\, iC}{T}e^{\omega(rT-t_d)}-\frac{\beta t_d^2 iC}{2T^2}$$
$$+\frac{\beta\, iC\, t_d^2}{\omega T^2}e^{\omega(rT-t_d)}-\frac{\beta t_d^2\, iCr}{T}e^{\omega(rT-t_d)}-\frac{\beta t_d\, iC}{\omega^2 T^2}+\frac{2\,\beta t_d\, iC}{T^2\omega^2}e^{\omega(rT-t_d)}$$
$$-\frac{2\beta t_d\, iC}{\omega T}e^{\omega(rT-t_d)}-\frac{\beta C}{T^2\omega^2}+\frac{\beta C}{T^2\omega^2}e^{\omega(rT-t_d)}-\frac{\beta C\, r}{\omega T}e^{\omega(rT-t_d)}+\beta r^2 t_d\, iCe^{\omega(rT-t_d)}$$
$$+\frac{\beta\,\omega\, iCr^2}{\omega}e^{\omega(rT-t_d)}-\frac{\alpha\, C}{\omega T^2}e^{\omega(rT-t_d)}+\frac{\alpha\, r\, C}{T}e^{\omega(rT-t_d)}+\frac{\alpha\, C}{\omega T^2}+\frac{\alpha\, C\, r}{T}$$
$$-\frac{\alpha\, C\, t_d}{T^2}-\frac{\alpha\, r\, C}{T}+\beta r^2 Ce^{\omega(rT-t_d)}-\beta r^3 C\, T+\frac{\beta C\, t_d^2}{\omega T^2}+\frac{\alpha\, C_s}{\delta}(1-r)e^{-\delta T(1-r)}$$
$$-\frac{\alpha\, r\, C_s}{\delta}(1-r)e^{-\delta T(1-r)}+\frac{\alpha\, C_s}{\delta^2 T^2}e^{-\delta T(1-r)}+\frac{\alpha\, C_s}{\delta^2 T}(1-r)e^{-\delta T(1-r)}-\frac{\alpha\, C_s}{\delta^2 T^2}-\frac{\alpha}{\delta T^2}$$
$$+\frac{\alpha}{\delta T^2}e^{-\delta T(1-r)}+\frac{\alpha}{\delta T}(1-r)e^{-\delta T(1-r)}+\frac{C_e\alpha}{\delta T^2}-\frac{C_e\,\alpha}{\delta T^2}e^{-\delta T(1-r)}$$
$$-\frac{C_e\,\alpha}{\delta T}(1-r)e^{-\delta T(1-r)}=0 \tag{23}$$

Differentiating (23) with respect to T gives

$$\frac{d^2 ATC(T)}{dT^2}=\frac{2C_O}{T^3}+\frac{2\beta\, iC}{T^3\omega^3}-\frac{2\beta\, iC}{\omega^3 T^3}e^{\omega(rT-t_d)}+\frac{\beta r\, iC}{T^2\omega^2}e^{\omega(rT-t_d)}+\frac{\beta r\, iC}{T^2\omega^2}e^{\omega(rT-t_d)}$$
$$-\frac{\beta r^2 iC}{\omega T}e^{\omega(rT-t_d)}+\frac{2\alpha\, iC}{T^3\omega^2}e^{\omega(rT-t_d)}-\frac{\alpha\, iC\, r}{\omega T^2}e^{\omega(rT-t_d)}-\frac{\alpha\, iC\, r}{\omega T^2}e^{\omega(rT-t_d)}$$
$$+\frac{\alpha\, iCr^2}{T}e^{\omega(rT-t_d)}+\frac{2\beta t_d\, iC}{\omega T^3}e^{\omega(rT-t_d)}-\frac{\beta t_d\, iC\, r}{T^2}-\frac{\beta t_d\, iC\, r}{T^2}$$
$$+\frac{\beta t_d\, iC\omega r^2}{T}e^{\omega(rT-t_d)}+\frac{\beta t_d\, iC}{T^3}-\frac{2\beta\, t_d^2 iC}{\omega T^3}e^{\omega(rT-t_d)}-\frac{2\beta\, t_d^2 iC}{\omega T^3}e^{\omega(rT-t_d)}$$
$$+\frac{\beta ri\, C}{T^2}e^{\omega(rT-t_d)}+\frac{\beta\, t_d^2 iC}{T^2}e^{\omega(rT-t_d)}-\frac{\beta\, t_d^2\, r\,\omega\, iC}{T}e^{\omega(rT-t_d)}+\frac{2\beta t_d\, iC}{T^3\omega^2}-\frac{4\beta t_d\, iC}{T^3\omega^2}$$
$$+\frac{2\beta t_d\, iCr}{\omega T^2}+\frac{2\beta t_d\, iC\, r}{\omega T^2}+\frac{2\beta t_d\, iC\, r^2}{T}e^{(rT-t_d)}+\beta t_d\, iC\, r^3\omega\, e^{\omega(rT-t_d)}$$

$$+\beta Cr^3e^{\omega(rT-t_d)}+\frac{2\beta C}{T^3\omega^2}-\frac{2\beta C}{T^3\omega^2}e^{\omega(rT-t_d)}+\frac{\beta Cr}{\omega T^2}e^{\omega(rT-t_d)}$$
$$+\frac{\beta Cr}{\omega T^2}e^{\omega(rT-t_d)}-\frac{\alpha Cr}{T}e^{\omega(rT-t_d)}+\frac{2rC}{T^3}e^{(rT-t_d)}-\frac{\alpha Cr}{T^2}e^{\omega(rT-t_d)}-\frac{\alpha Cr}{T^2}e^{\omega(rT-t_d)}$$
$$+\frac{\alpha C\omega r^2}{T}+\frac{2\alpha C}{\omega T^3}+\frac{2\alpha t_d C}{T^3}+\beta C\omega r^3e^{\omega(rT-t_d)}-\beta C\omega r^3-\frac{2\beta Ct_d^2}{\omega T^3}$$
$$+\frac{\alpha C_s}{\delta}(1-r)^2e^{-\delta T(1-r)}-\frac{\alpha rC_s}{\delta}(1-r)^2e^{-\delta T(1-r)}+\frac{2\alpha C_s}{\delta^2T^3}e^{-\delta T(1-r)}$$
$$+\frac{\alpha C_s}{\delta^2T^2}(1-r)e^{-\delta T(1-r)}+\frac{\alpha C_s}{\delta^2\square}(1-r)e^{-\delta T(1-r)}+\frac{\alpha rC_s}{\delta^2T}(1-r)^2e^{-\delta T(1-r)}$$
$$-\frac{2\alpha C_s}{\delta^2T^3}+\frac{2\alpha}{\delta T^3}-\frac{2\alpha}{\delta T^3}e^{-\delta T(1-r)}-\frac{\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)}-\frac{\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)}$$
$$-\frac{\alpha}{\delta T}(1-r)e^{-\delta T(1-r)}-\frac{2C_e\alpha}{\delta T^3}+\frac{2C_e\alpha}{\delta T^3}e^{-\delta T(1-r)}+\frac{C_e\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)}$$
$$+\frac{C_e\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)}-\frac{C_e\alpha}{\delta T}(1-r)^2e^{-\delta T(1-r)}$$

$$=\frac{2C_O}{T^3}+\frac{2\beta iC}{T^3\omega^3}-\frac{2\beta iC}{\omega^3T^3}e^{\omega(rT-t_d)}+\frac{2\beta riC}{T^2\omega^2}e^{\omega(rT-t_d)}-\frac{\beta r^2iC}{\omega T}e^{\omega(rT-t_d)}+\frac{2\alpha iC}{T^3\omega^2}e^{\omega(rT-t_d)}$$
$$-\frac{2\alpha iC}{\omega^2T^3}-\frac{2\alpha iCr}{\omega T^2}+\frac{\alpha iCr^2}{T}e^{\omega(rT-t_d)}+\frac{2\beta t_d iC}{\omega T^3}e^{\omega(rT-t_d)}-\frac{2\beta t_d iCr}{T^2}$$
$$+\frac{\beta t_d iC\omega r^2}{T}e^{\omega(rT-t_d)}+\frac{\beta t_d^2iC}{T^3}-\frac{2\beta t_d iC}{\omega T^3}e^{\omega(rT-t_d)}+\frac{\beta t_d iC(r+1)}{T^2}$$
$$-\frac{t_d^2r\,iC}{T}e^{\omega(rT-t_d)}+\frac{2\beta t_d iC}{T^3\omega^2}-\frac{4\beta t_d iC}{T^3\omega^2}e^{\omega(rT-t_d)}+\frac{4\beta t_d r\,iC}{T^2}-\frac{2\beta r\,iCr^2}{T}e^{\omega(rT-t_d)}$$
$$+\beta t_d iCr^3(1+\omega)e^{\omega(rT-t_d)}+\frac{2bC}{T^3\omega^2}-\frac{2bC}{T^3\omega^2}e^{\omega(rT-t_d)}+\frac{2\beta Cr}{\omega T^2}e^{\omega(rT-t_d)}$$
$$-\frac{\beta Cr^2}{T}e^{\omega(rT-t_d)}+\frac{2\alpha C}{\omega T^3}e^{\omega(rT-t_d)}-\frac{2\alpha Cr}{T^2}e^{\omega(rT-t_d)}+\frac{\alpha C\omega r^2}{T}e^{\omega(rT-t_d)}$$
$$+\frac{2\alpha C}{\omega T^3}+\frac{2\alpha Ct_d}{T^3}+\beta Cr^3e^{\omega(rT-t_d)}-\beta Cr^3-\frac{2\beta Ct_d^2}{\omega T^3}+\frac{\alpha C_s}{\delta}(1-r)^2e^{-\delta T(1-r)}$$
$$+\frac{\alpha C_s}{\delta^2T^2}(1-r)e^{-\delta T(1-r)}+\frac{\alpha C_s}{\delta^2T}(1-r)e^{-\delta T(1-r)}+\frac{\alpha rC_s}{\delta^2T}(1-r)^2e^{-\delta T(1-r)}$$
$$-\frac{\alpha rC_s}{\delta}(1-r)^2e^{-\delta T(1-r)}+\frac{2\alpha C_s}{\delta^2T^3}e^{-\delta T(1-r)}-\frac{2\alpha C_s}{\delta^2T^3}+\frac{2\alpha}{\delta T^3}-\frac{2\alpha}{\delta T^3}e^{-\delta T(1-r)}$$
$$-\frac{\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)}-\frac{\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)}-\frac{\alpha}{\delta T}(1-r)e^{-\delta T(1-r)}-\frac{2C_e\alpha}{\delta T^3}$$
$$+\frac{2C_e\alpha}{\delta T^3}e^{-\delta T(1-r)}+\frac{C_e\alpha}{\delta T^2}(1-r)e^{-\delta T(1-r)} \tag{24}$$

Since, it is difficult to find the analytical solution of equation (24). We numerically show that equation (24) > 0.

4. Solution procedure

Algorithm

In order to find the optimal solutions, we propose the following algorithm:

Step1: Input the values of all parameters.

Step 2: Using Newton – Raphson Method determine the value of T from equation (23).
Step 3: Compare T with t_d , If $T > t_d$, go to step 4, otherwise T is infeasible,
Step 4: Compute the corresponding average total cost ATC (T), maximum inventory level N , ordering quantity R , t_1 from (10), (13) and (20) respectively.

5. Numerical Example

The following data is used to illustrate the model numerically.

$C_O = 500$, $i = \$0.50$, $C = 18.0$ units $\alpha = 20$ units, $\beta = 0.2$ $t_d = 0.4$, $\omega = 0.2$,

$r = 0.6$, $\delta = 0.4/unit$, $C_s = 0.04$, $C_e = 1$

The optimal cycle time T*= 2.4717 days using expression (23)

Using the value of T* in $t_1 = rT$ we obtain $t = t_1^* = 1.4830$ days

Maximum inventory level $N^* = 17.7552$ units are determined using equation (10) . The optimum ordering quantity, R = R*= 26.1635 units using expression (13).

The corresponding average total cost is \$ 220.0690 using equation (20).

The sufficient condition is $\frac{d^2\ ATC(T)}{dT^2} = 406.3906 > 0$

6. Sensitivity Analysis

We now study the effect of change in the parameters and observe the change in decision variable. The sensitivity analysis is performed by changing each parameter by -3% to 3%, taking one parameter at a time and keeping remaining parameters unchanged.

Table 1: Percentage change in the decision variables with respect to the percentage in parameter values.

Variation Parameters	Percentage change in Parameters	Change in decision Variables from -3% to 3%				
α		t_1^*	T^*	N^*	R^*	ATC(T^*)
	-3	0.7143	0.7143	-1.8738	-0.6270	-0.7313
	-2	0.4748	0.4748	-1.2488	-1.4554	-0.5247
	-1	0.2367	0.2367	-0.6230	-0.7279	-0.3184
	1	-0.2352	-0.2352	0.6309	0.7285	0.0932
	2	-0.4692	-0.4691	1.2590	1.4575	0.2986
	3	-0.7017	-0.7017	1.8879	3.8070	0.5037
β	-3	-0.0524	-0.0524	-0.5968	1.1707	-0.0651
	-2	-0.0349	-0.0350	-0.3969	1.3087	-0.0808
	-1	-0.0175	-0.0174	-0.1968	1.4469	-0.0967
	1	0.0175	0.0175	0.2042	1.7238	-0.1282
	2	0.0349	0.0349	0.4050	1.8624	-0.1440
	3	0.0175	0.0175	0.2042	1.7238	-0.1282
ω	-3	-0.8803	-0.8803	0.0240	1.5879	0.6897
	-2	-0.5829	-0.5829	0.0190	1.5884	0.4171
	-1	-0.2895	-0.2895	0.0122	1.5874	0.1498
	1	0.2857	0.2857	-0.0067	1.5818	-0.3698
	2	0.5677	0.5677	-0.0188	1.5772	-0.6224

	3	0.8460	0.8461	-0.0321	1.5715	-0.8703
C	-1	0.8666	0.8666	0.5017	1.9399	-1.1433
	-2	0.5754	0.5754	0.3343	1.8207	-0.7991
	-3	0.2865	0.2865	0.1683	1.7025	-0.4554
	1	-0.2843	-0.2842	-0.1598	1.4690	0.2298
	2	-0.5663	-0.5662	-0.3219	1.3535	0.5715
	3	-0.8460	-0.8460	-0.4827	1.2391	0.9124
C_s	-3	0.4264	0.4264	0.2487	1.7597	-0.4962
	-2	0.2836	0.2836	0.1665	1.7013	-0.3681
	-1	0.1415	0.1415	0.0848	1.6431	-0.2401
	1	-0.1408	0.1415	-0.0774	1.5276	0.0149
	2	-0.2810	0.1415	-0.1579	1.4703	0.1422
	3	-0.4205	0.1415	-0.2381	1.4132	0.2691

Comment of sensitivity analysis

When value of parameter α (constant demand) increases there is corresponding increase in order quantity R*, maximum inventory level N^* average total cost ATC (T^*) and decrease in cycle length and depletion period. This shows that managers need to increase the order quantity when constant demand increases in order to minimize the cost.

When the value of the parameter β increases, there is corresponding increase in depletion period, cycle time T^*, maximum inventory level N^*, and order quantity R^*. This indicates that increases in the value of β, the inventory management need to increase the cycle length and order quantity.

When the value of the parameter ω increases, lead to decrease in depletion period t_1^*, cycle time T^* while maximum inventory level N^*, order quantity R^* and average total cost ATC (T^*) increases This show that, inventory manager should order small quantity when deterioration increases.

As the value of the parameter C increases, lead to decrease in depletion period t_1^*, cycle time (T*), the maximum inventory level N^*, order quantity R^* and average total cost increases. This also indicates that when price increases manager should order small quantity.

When a value of parameter C_s increases, the inventory depletion period t_1^*, cycle length (T*), the maximum inventory level N^*, the order quantity decrease, there are corresponding increases in the average total cost.

8. Conclusion

This paper presents an inventory model which has direct application to the business enterprises that consider constant demand rate in the first part of the cycle, linear time dependent demand rate in the second part of the cycle and during the shortage period demand is exponentially backlogged. This can be seen in situation when new products such as computers, fridge, plasma, satellite, televisions, android mobiles, and so on, are launched in the market. Demand for such items are constant in the early stage and after for some time, the demand increases due to its popularity. There could also be situations in which an economic advantage may be gained by allowing shortages to occur and so exponential backlogged rate was factored. The optimal cycle time and optimal order quantity have been derived by minimizing the total average cost. Then Newton-Raphson has been used to solve the minimization problem so determined. Finally, the result is illustrated with a numerical example which was followed by sensitivity analysis.

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