Abacus (Mathematics Science Series) Vol. 44, No 1, Aug. 2019

GENERALIZED EFFICIENCY OF KERNEL DENSITY DERIVATIVE ESTIMATION (KDDE): A UNIVARIATE PERSPECTIVE

by

Ejakpovi, S. U, Ozobokeme, J. K. and Uveruveh, F. O

Department of Mathematics, College of Education, Warri, Delta State, Nigeria.

[1]*Email: simeonejakpovi@yahoo.com*

Abstract

The focus of this paper is on obtaining the generalized expression of efficiency of Kernel Density Derivative Estimation (KDDE) from the univariate perspective. The expressions were derived for the higher order Kernel functions and subsequently were implemented using some selected higher order Kernel functions. The results showed the efficiency for the classical functions for derivative order one and highly depreciative efficiency value of higher order Kernel functions as the derivative order, r and Kernel order, j increase.

Keywords and Phrases: *Kernel Density Derivative Estimation (**KDDE**); Asymptotic Mean Integrated Squared Error ($AMISE^r$); mean integrated squared error ($\boldsymbol{MISE^r}$) and efficiency (Eff^r).*

1.0 Introduction

Kernel Density Derivative Estimation (KDDE) is a novel form of nonparametric estimation after decades of the giant strides of Kernel Density Estimation (KDE). It exists in both univariate and multivariate classes with few and recent applications in distributional shape form (Mayorov, 2017); bump hunting (Good and Gaskins, 1980) and mean shift clustering analysis (Fukunaga and Hosteler, 1975). In both forms of its existence, it is an off shoot of the KDE which has been extensively studied. The focus of this paper shall be on the univariate class of the KDDE where $\{X_i\}_{i=1}^{\infty}$ be a random sample which is independent and identically distributed (i.i.d) of a continuous random variable X with density function $f(x)$. An estimator for the derivative is obtained by taking the derivative of the univariate KDE and the Kernel function K is differentiable r times, then the r^{th} derivative of the density function $f(x)$ is:

$$\hat{\boldsymbol{f}}^{r}(\boldsymbol{x}) = \frac{\boldsymbol{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{K}_{\boldsymbol{h}}^{\boldsymbol{r}}(\boldsymbol{x}-\boldsymbol{X}_{\boldsymbol{i}}) \qquad (\mathbf{1})$$

where K^r is the r^{th} derivative of the Kernel function K, which is symmetric probability density with at least r non-zero derivative when estimating $\hat{\boldsymbol{f}}^{\boldsymbol{r}}(\boldsymbol{x})$ and h is the smoothing parameter that controls the degree of smoothing applied to the data (Bhattacharya, 1967) and (Schuster, 1969)). The choice of the smoothing parameter enables one to obtain expressions for the asymptotic mean integrated squared error with nomenclature $AMISE^r\{\hat{\boldsymbol{f}}^{\boldsymbol{r}}(\boldsymbol{x})\}$ from the estimator in Equation (1). However, the Kernel function K that is differentiated r^{th} times to obtain K^r which is a non-negative function must satisfy the following axioms:

$$\int_{-\infty}^{\infty} \boldsymbol{tK(t)dt = 0}, \qquad \int_{-\infty}^{\infty} \boldsymbol{t^2\,K(t)dt = m_2(k) \neq 0}\,, \qquad \int_{-\infty}^{\infty} \boldsymbol{K(t)dt = 1} \qquad (\mathbf{2})$$

And if the regular axioms in Equation (2) are relaxed for higher order Kernels function K, which is symmetric function, the axioms in equation (2) becomes:

$$\boldsymbol{m_j(k)} = \int_{-\infty}^{\infty} \boldsymbol{t^j K(t) dt} = \begin{cases} \mathbf{1}, & \boldsymbol{j = 0} \\ \mathbf{0}, \boldsymbol{for\ j = 1, 2, \ldots, p-1} & \\ \neq \mathbf{0} & \boldsymbol{j = p} \end{cases} \qquad (\mathbf{3})$$

The estimator in Equation (1) must also satisfy the consistency properties of probability density function (Baired, 2014). Then, after moment of algebraic simplifications, the global error reduction technique of the estimator (1) is the mean integrated squared error with its nomenclature $MISE^r\{\hat{\boldsymbol{f}}^{\boldsymbol{r}}(\boldsymbol{x})\}$ involving the squared $Bias^r$ and $Variance^r$ is:

$$\boldsymbol{MISE^r\{\hat{f}^r(x)\} = \frac{1}{4} h^4 m_2(K)^2 \nabla(x) + \frac{R(K^r)}{nh^{1+2r}} + O\left(h^4 + \frac{1}{nh^{1+2r}}\right)} \qquad (\mathbf{4})$$

where $R(K^r) = \int K^r(t)^2 dt$ and $\nabla(x)$ is the unknown density distribution derivative upon the $(r+2)^{th}$ derivative. After ignoring the higher order term of Equation (4), we obtained the asymptotic mean integrated squared error with nomenclature $AMISE^r\{\hat{\boldsymbol{f}}^{\boldsymbol{r}}(\boldsymbol{x})\}$ whose minimization with respect to the smoothing parameter is h^r_{AMISE}. Then, substituting the h^r_{AMISE} into the $AMISE^r\{\hat{\boldsymbol{f}}^{\boldsymbol{r}}(\boldsymbol{x})\}$ we have the minimum $AMISE^r\{\hat{\boldsymbol{f}}^{\boldsymbol{r}}(\boldsymbol{x})\}$ as:

$$\boldsymbol{AMISE^r\{\hat{f}^r(x)\} = C \left\{\frac{R(K^r)}{n}\right\}^{\frac{4}{5+2r}} \left\{\frac{m_2(K)^2}{\varphi}\right\}^{\frac{\varphi}{5+2r}} \{R(\nabla(x)\}^{\frac{\varphi}{5+2r}}} \qquad (\mathbf{5})$$

Where C is a scalar and φ depends on the derivative order (r) of the kernel function K. Thus, the remaining part of the paper is organized as follows: In Section II, we shall construct the proposed classical kernel functions and their respective higher order polynomial functions; Section III, we defined the optimal kernel function for the first derivative and prove its efficiency expression, Section IV, we derived the generalized efficiency expression of KDDE; Section V, we implement the efficiency expressions in Section III and IV at the first and second order derivatives with selected kernel functions from section II and conclude the paper in Section VI.

2.0 The Proposed Higher Order Kernel Polynomials Expression

There are various techniques of constructing classical Kernels functions with their respective higher order kernels. The Izenman (2005) construction format is for the classical kernels functions for $(s, r) \rightarrow \{(2,3), (2,4), (2,5), \ldots\}$ which generates kernel functions that are compactly supported at $|t| \leq 1$. In an attempt to generate their respective higher order forms, we adopted the Legendre duplication constant of Muller (1984) and Grandvskey and Muller (1991) of the format: $\left(\frac{1}{2}\right)_m = \frac{(2m)!}{m! 2^{2m}}$ with $m = \alpha + 1$ and α as the order of the kernel function which leads to a modification of the Jones and Foster (1993) higher order kernel function format as:

$$\boldsymbol{K_{\alpha+2}(t)} = \begin{cases} \dfrac{\left(\frac{\mathbf{1}}{\mathbf{2}}\right)_{\boldsymbol{\alpha+1}}}{\mathbf{2}\boldsymbol{\alpha}!} (\mathbf{1} - \boldsymbol{t}^{\mathbf{2}})^{\boldsymbol{\alpha-1}} (\mathbf{3} - \{\mathbf{3} + \mathbf{2}\boldsymbol{\alpha}\}\boldsymbol{t}^{\mathbf{2}}), |\boldsymbol{t}| \leq \mathbf{1} \\ \mathbf{0}, \ \boldsymbol{elsewhere} \end{cases} \qquad (\mathbf{6})$$

where $\left(\frac{1}{2}\right)_{\boldsymbol{\alpha+1}}$ is the normalized constant. This was adopted to obtain the higher order kernel functions from their respective univariate forms.

3.0 The KDDE Efficiency Expression

The issue of optimal Kernel function is also important in KDDE. The research work of Muller (1984) showed that for the first derivative order, the optimal member in KDDE is the Biweight Kernel function and follows in that manner as the derivative order increases. Then, efficiency of derivative Kernels of the estimator given in Equation (1) above is the ratio of the $AMISE^r\{\hat{f}^r(x)\}$ using the optimal derivative Kernel to the $AMISE^r\{\hat{f}_o^r(x)\}$ of any other derivative Kernel function. It is denoted as $Eff^r(K)$ as

$$\boldsymbol{Eff^r(K) = (R(K^r))(m_2(K)^2)^{\frac{3}{4}}(R(K_o^r))^{-1}(m_2(K_o)^2)^{\frac{-3}{4}}} \qquad (7)$$

where K_o^r is the derivative Kernel function of any other function in the density estimator. Then, incorporating the $(R(K^1))$ value of the optimal derivative Kernel into Equation (7), we have:

$$\boldsymbol{Eff^r(K) = \Phi\{(R(K_o^r))\}^{-1}(m_2(K_o)^2)^{\frac{-3}{4}}} \qquad (8)$$

where Φ is a scalar constant achieved from the evaluation of $AMISE^r\{\hat{f}^r(x)\}$ for the optimal derivative Kernel function. The efficiency expression in Equation (8) can be used to generate the efficiency of the univariate kernel functions in the given estimator.

4.0 The Generalized Univariate KDDE Efficiency Expression

The above classical efficiency expression of KDDE in Equation (8), can be extended to order 4, 6, 8, …, up to the j^{th} order. In achieving this, we followed the efficiency meaning via Equation (5) to obtain the generalized efficiency expression for any higher order symmetric Kernel at j^{th} order as:

$$\boldsymbol{Eff_{2j}^r(K_h) = \left\{\frac{\beta^{2j}(K_{hb})}{\beta_o^{2j}(K_{ho})}\right\}^{\frac{4j+7}{4j+4}}} \qquad (9)$$

where the numerator is the normalized Biweight higher order $AMISE^r\{\hat{f}^r(x)\}$ constant and the denominator is the $AMISE^r\{\hat{f}_o^r(x)\}$ for any other higher order derivative Kernel function. Then, simplifying Equation (9) with the appropriate higher order kernel functions, we achieved the generalized univariate KDDE expression for efficiency as:

$$\boldsymbol{Eff_{2j}^r(K_h) = \emptyset\{(2j+3)(2j+5)(2j+7)\}^{\frac{-7}{2j+2}}\{m_{2j+2}(K_O)\}^{\frac{-7}{2j+2}}\{(R(K_o^r)\}^{-1}} \qquad (10)$$

where α and Λ are integers related with $\emptyset$ in $\emptyset = \boldsymbol{\alpha^{-1}(\Lambda j)^{\frac{7}{2j+2}}}$.

5.0 Implementing the KDDE Efficiency Expressions

In this Section, we shall examine the efficiency expressions for the univariate KDDE for both the classical kernel functions and their respective higher order kernel functions emanating from Section II, using the univariate KDDE efficiency expressions in Equations (8) and (10) at first and second order derivatives.

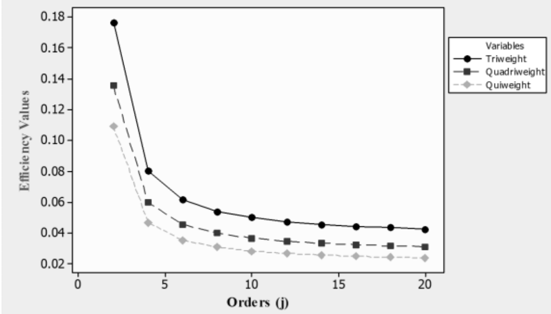

Fig. 1: Graph of Efficiency for classical Kernels

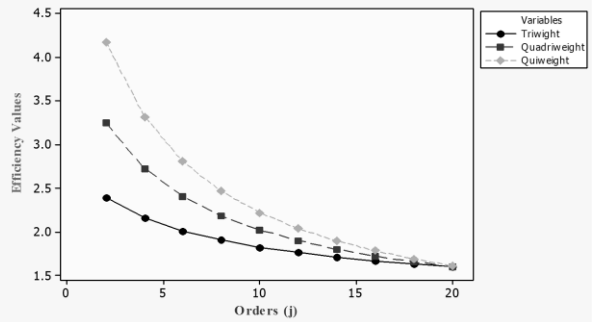

Fig. 2: Graph of Efficiency for Higher Order Kernels at first derivative

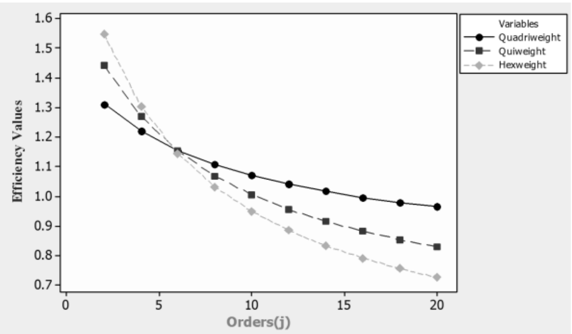

Fig.3: Graph of Efficiency for Higher order Kernels at second derivatives

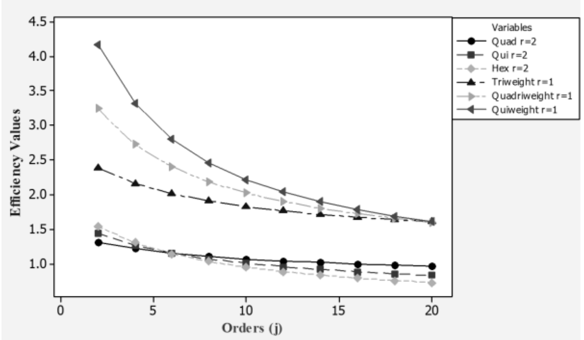

Fig. 4: Graph of Comparison of Efficiency for Higher Order Kernel at first and second derivative

The graphs in Figure 1, Figure 2, Figure 3 and Figure 4 showed the Efficiency values of the classical Kernel functions and their respective higher-order Triweight, Quadriweight, Quiweight and Hexweight Kernels. In Figure 1, the efficiencies of the classical Kernels exhibit faster loss efficiency values as the order increases. Also, Figures 2 and 3 showed the highly efficiency values for the various higher order Kernels for first and second derivatives, but their values begin to depreciate, as the order increases showing their inefficiencies. Then, Figure 4, gives the comparison of the efficiency values for both first and second order derivative for the various Kernels. The Figure 4, shows that the Kernels at derivative order one is having a minimal value of efficiency than the second order derivative Kernels.

6.0 Conclusions.

This paper examines the efficiencies of some classical Kernel functions and their respective higher order Kernel functions. It is obvious that the classical Kernel functions exhibit depreciative efficiency values at derivative order one while their higher order Kernel functions at derivative order one showed a point of coincidence with minimal efficiency values when compared to the higher order Kernels functions at derivative order two. This evidence unveiled the depreciative efficiency in KDDE for some selected higher order Kernel functions, which has also becomes a major contribution of this work to nonparametric estimation. The need to extend this work to the multivariate KDDE is a grey area for further research study.

References

Baird, M (2014): Cross Validation Bandwidth Selection for Derivatives Multidimensional Densities. The RAND Corporation

Bhattacharya, P.K (1967): Estimation of a probability density function and its derivatives. Sankhy, the Indian Journal of Statistics, Series A,29(4): 373-382.

Fukunaga, K and Hosteller, L. D (1975): The estimation of the gradient of a density functions with application in pattern Recognition. IEEE Transaction information Theory 21, 32-40

Izenman, A, J (1991): Recent Development in Nonparametric Density Estimation. Journal of the American Statistical Association, Volume 86, No 413, 205-224.

Jones, M. C. and Foster, P. J. (1993): Generalized Jackknifing and higher-order kernels'. Journal of Nonparametric Statistics**3**, 81 – 94.

Schuster, E.F (1969): Estimation of a probability density function and its derivatives. The Annals of Mathematical Statistics, 40(4):1187-1195.

THE EFFECT OF FRACTIONAL-ORDERALPHA, α,ON THE BEHAVIOR OF DYNAMICAL SYSTEMS FOR EPIDEMIC MODEL

by

A. E. Owoyemi, S. E. Olowo, D. Y. Idris, E. A. Olujosun, M. N. Umunagbu, I. Abdulmalik, W. O. Okedokun ,M. O, Aigbavboa,U. Dahiru And S. K. Gbaden

Federal College of Agricultural Produce Technology, Plot 54-56 Central Bank Road, Hotoro G.R.A. Extension, P.M.B. 3013, Kano, Nigeria.

Email: abiodunowoyemi9@gmail.com

Abstract

Recently, many deterministic mathematical model had been extended to fractional model with some fractional differential equations. It was believed that these fractional models are more realistic to represent the daily life phenomena than its integral–order counterpart. The main focus of this paper is to extend the model of asusceptible-infected-refractory(SIR) epidemic model to fractional model. More specifically, the fractional SIR epidemic model with sub-optimal immunity, nonlinear incidence and saturated recovery rate was discussed. The fractional ordinary differential equations were defined in the sense of the Caputo derivative. Existence, equilibrium points and the stability analysis for the fractional models were analyzed. We applied Adams-type predictor-corrector method to the numerical solutions of the models. Maple 18 is used as the software platform. The result also confirmed that choosing appropriate values of the fractional $\alpha \in [0, 1]$ increase the stability region of the equilibrium points.

Keywords:

Sub-optimal immunity, nonlinear incidence, saturated recovery rate, fractional-order, stability analysis.

INTRODUCTION

Over the years, there have been various types of epidemiological models but the basiccompartmental model was created by Kermack in 1927. The model describes space of communicable diseases. This compartmental model is often calledthe Kermack-McKendrick epidemic model. SIR model is a symbolic model,where *S* denotes the number of individuals who are susceptible to the disease, *I*, denotes the number of infected individuals and *R* denotes the number of individuals who have been recovered.

Some simple examples of epidemic for this basic compartmental model are measles, mumps and rubella. But, several of these SIR models have not been established based on fractional order equation (FODs). The basic compartmental model is usedto describe a disease which confers immunity against re-infection. TheSevere acute respiratory syndrome(SARS) outbreak in 2003 and Ebola outbreak in 2014 had speeding the research in this area.

In this research direction, mathematicians and physicists are among the academic workers who contributed to the knowledge of mathematical epidemiology. They keep on working in modeling the epidemics outbreak. The models are able to integrate realistic aspects of disease spreading.

One of the main parts of epidemiological research is focused on rate-based differential-equation models, i.e. compartmental models on completely mixing population. This epidemiology modeling has been used in planning, implementing and evaluating various prevention therapy and control programs. Before the era of research in complex networks, the theoretical approach to epidemic spreading is based on compartmental models in term of system of ordinary differential equations (ODEs). There are many other models, such as the

SI (susceptible-infected) model and the SIRS (susceptible-infected-refractory-susceptible) model, and the sub-optimal immunity model which lies in between the SIS (susceptible-infected-susceptible) and the SIR models.

In recent years, the research in this ODEs epidemic modeling had been shifted to fractional differential equations model. Various fractional epidemic models had been studied[1]–[5].In a wider context, the compartmental models such as epidemic models, pharmacokinetics, and in-host virus dynamicswere discussed in a SIAM(Society for Applied and Industrial Mathematics)paper by Angstmann, C. Net. al. in 2017 [5].

Fractional calculus has been investigated to be one of the best tools to characterize long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviours, power laws, algometric scaling laws etc[6]. On top of this, the area of fractional order derivative has actually received numerous attentions in the recent years most especially in the area of mathematical biology. Several researchers have worked and made several conclusions in modelling the outbreak of disease with the extension of fractional order derivative to different area of fields. It is an advantageous approach, which has been used to study the behaviour of disease (like pertussis and influenza) because the fractional order is a generalization of integer order differential equations [7].

On the other hand, the well-known Caputo fractional derivative (defined by Michele Caputo in 1967) and famous Riemann-Liouville fractional integral are the main subject of many study in fractional calculus[6], [8], [9]. The research works in this area is under a huge development such as study of theory of fractional calculus[10], [11], develop efficient numerical scheme[3], [12], [13], and application on physical problem[14]. In addition to that, the fractional derivative is used to increase the stability region of the system, which is more suitable than integer in an epidemic model [7], [15].However, in this research work, the focus is on the analysis in a fractional-order epidemic model with sub-optimal immunity, nonlinear incidence and saturated recovery rate.

[16] proposed a fractional order calculus based on the model for the simulation of an outbreak of dengue fever with the aim of a well befitting model of the real dynamic order for the transmission among heterogeneous population. The result obtain shows that fractional-order differential equations are very much stable than the integer-order.

[17] proposed a mathematical model of fractional order for the simulation of the dynamic of the outbreak of dengue fever. In the paper, a better conclusion was reached that the behaviour of the human population follows a model of a different order than the mosquito population. Still on the dengue disease, another conclusion was reached by a group of researchers when they proposed a work on dengue disease, basic reproduction number and control, where their model was based on two populations, mosquitoes and humans with insecticide control [18]. The research showed the possibility of reducing the number of infected human and mosquitoes with a steady insecticide, which can prevent an outbreak that could transform an epidemiological episode to an endemic disease.

Definition 1

Several definitions of fractional-order derivatives have been given by several authors and the commonly used is the Riemann - Liouville definition and the Caputo definitions. It has been proved that Caputo derivative only requires initial conditions by means of integer-order derivative, representing well-understood features of physical situation, which, however, making it more applicable to real world problems [19].

In this paper work, the Caputo definition is addressed.The fractional integral with fractional order $\beta \in \Re^{+}$ of function $x(t), t > 0$ is defined as:

$$I^{\beta}x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} x(s)ds \qquad (1)$$ where

$t = t_0$ is the initial time and $\Gamma(.)$ is the Euler's gamma function.

Definition 2 The Caputo fractional derivative with order $\alpha \in (n-1), n$ of function $x(t), t > 0$ is defined as:

$$_cD_*^{\alpha}x(t) = I^{n-\alpha}D^n x(t), D_* = \frac{d}{dt}. \qquad (2)$$

Comparing these two fractional derivatives, one easily arrives at the fact that Caputo's derivative of a constant is equal to zero, which is not the case for the Riemann-Liouville derivative[7].The main concern of the paper thus focuses on the Caputo derivative of order $\alpha \in [0,1]$,which is rather applicable in real application.

Stability Analysis of Fractional Order System

This section investigates the local stability analysis of fractional order system, which is based on the stability theory of fractional-order system. Fractional-order system has the same point of equilibrium with the corresponding integer-order system but stability conditions ofboth orders are quite different, the point of equilibrium of fractional-order may still be stable, even if the real part of the eigenvalue carries positive values.

Theorem 1 (condition for Caputo fractional derivative).

The necessary and sufficient condition for Caputo fractional derivative to be locally asymptotically stable, with system (3) where $\alpha\epsilon[0,1]$is if and only if eigenvalues,λ_i,of the Jacobian matrix, $\frac{\delta}{\delta x}f(t,y)$(t, y), evaluated at the equilibrium points is satisfied by $|\arg \lambda_i| > \frac{\alpha\pi}{2}, i = 1,2$,[7].

Proof. Consider the following commensurate fraction-order system:

$$cD_t^{\alpha}y_i(t) = f\big(t, y_i(t)\big), y_i(t_0) = y_o \qquad (3)$$

where $cD_t^{\alpha}$is the Caputo fractional derivative with order $\alpha \in (0,1]$. In order to evaluate the equilibrium points, let us put

$$cD_t^{\alpha}y_i(t) = 0 \Longrightarrow f_i\big(f_1^{eqn}, f_2^{eqn}, f_3^{eqn}\big) = 0. \qquad (4)$$

for which we can get the equilibrium points $f_1^{eqn}, f_2^{eqn}, f_3^{eqn}n$.

Now to evaluate the asymptotic stability, let us consider the system $cD_t^{\alpha}y_i(t) = (x,y)$ in the sense of Caputo and to find the asymptotic stability, let $y_i(t) = f_i^{eqn} \in_i (t)$. The equilibrium point $\big(f_1^{eqn}, f_2^{eqn}, f_3^{eqn}\big)$is locally asymptotically stable if the eigenvalues of the

Jacobian(=J), $J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} \end{bmatrix}$, matrix A evaluated at the equilibrium point is

satisfied by $\left|\arg(\lambda_{1,2,3})\right| > \alpha\frac{\pi}{2}$ [15], [20]–[22]■

Fig.1 displays the stability region of the fractional-order system, which explains that the stability region of the fractional order case is greater than the stability region of the integer-order case for $0 < \alpha \leq 1$. Thus, a and bi refer to the real and imaginary parts of the eigenvalues respectively where $i = \sqrt{-1}$.

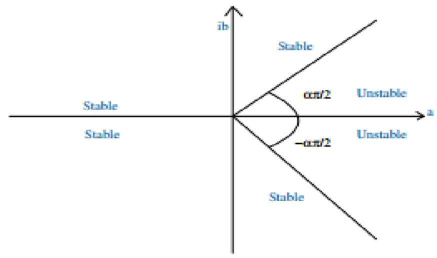

Fig. 1. Stability region of the Fractional-order system[15]

The Model

Many attempt have been made to develop realistic mathematical models in order to make prediction about the living organisms behaviour and at the end we help to simulate the dynamics, among such is a model presented in a seminar by Ruan and Wang in[23]. The paper was on SIR epidemic model with the specific nonlinear incidence rate andwas later developed in[24], which is an ODE Model. The main focus of this paper is to extend the model of [24]to fractional model. The model is given in the following system (5a) - (5b).

$$\frac{dI}{dt} = \beta\left(\frac{A}{\mu} - I - R\right)I^2 - \upsilon I - \frac{cL}{1+aI} - NI, \quad (5a)$$

$$\frac{dR}{dt} = K\left(\upsilon I + \frac{cI}{1+aI}\right) - \mu R \quad (5b)$$

The parameters interpretations are as follows: A is the recruitment rate of susceptible population, β is the disease transmission rate, μ is the natural death rate and v is the recovery rate. I and R still remain as infectious and recovered/removed, respectively. a and c are possible constants, where with the initial values

$$I(0) = I_0, R(0) = R_0 \quad (6)$$

The above integer-order derivatives of the system (5a) - (5b) are replaced with fractional derivatives of order $0 < \alpha \leq 1$ in the sense of Caputo as follows:

$$cD_t^\alpha = \beta\left(\frac{A}{\mu} - I - R\right)I^2 - \upsilon I - \frac{cL}{1+aI} - NI, \quad (7a)$$

$$cD_t^\alpha = K\left(\upsilon I + \frac{cI}{1+aI}\right) - \mu R \quad (7b)$$

where $0 < \alpha \leq 1$. All the parameters are positive constants. R_0 is used to establish the existence and stability conditions of both disease-free and endemic for the equilibrium points, which is the number of people that one sick person will infect (on average). There are two equilibriums in the system (7a) - (7b), when equating them to zero, namely, the existence of disease-free equilibrium, E^0, point, and the endemic equilibrium, E^e, point.

Existence and stability of disease-free equilibrium point

The asymptotic stability of the disease-free, E^0, is demonstrated in this section for when $R_0 < 1$. The basic reproduction number, R_0, of the system (7a) – (7b) is defined as:

$$R_0 = \frac{\beta A}{\mu(c-\mu+\upsilon)}, \quad (8)$$

The equilibrium of the disease-free is solved to be $E_{E^0} = (I = 0, R = 0)$

At E_{E^0}, system (7a) - (7b) is said to be asymptomatically stable if after obtaining the

Jacobian matrix and it's both eigenvalues are satisfied by using;

$$|arg\ \lambda_1| > \frac{\alpha\pi}{2}, \quad (9)$$

$$|arg\ \lambda_2| > \frac{\alpha\pi}{2}. \quad (10)$$

The establishment for the disease-free equilibrium, E^0,is given in the below theorem 2

Theorem 2

A sufficient condition for the system (7a) - (7b) to be locally asymptotically stable is if and only if

$$R_0 = \frac{\beta A}{\mu(c-\mu+v)} < 1 \quad (11)$$

Proof

To prove **theorem 2**, it is enough to show that all eigenvalues of Jacobian matrix of system (5a) – (5b) at E^0 have a negative real part. Hence, the Jacobian matrix is

$$\begin{bmatrix} -I^2\beta + 2\beta\left(\frac{A}{\mu} - I - R\right)I - v - \frac{c}{Ia+1} + \frac{cIa}{(Ia+1)^2} - \mu & -I^2\beta \\ K\left(v + \frac{c}{Ia+1} - \frac{cIa}{(Ia+1)^2}\right) & -\mu \end{bmatrix} \quad (12)$$

Then for $I_{eqn}, R_{eqn} = (I = 0, R = 0)$we calculated that

$$A = \begin{bmatrix} -v - c - \mu & 0 \\ K(v + c) & -\mu \end{bmatrix} \quad (13)$$

The eigenvalues are solved as

$\lambda_1 = -\mu$, and $\lambda_2 = -v - c - \mu.$ (14)

Hence, since the eigenvalues of the system (7a) - (7b)< 1, then the disease-free equilibrium E^0 is locally asymptotically stable, which implies that $R_0 < 1$and satisfy the condition in Equation 9 and 10.Conversely, it becomes unstable when $R_0 = \frac{\beta A}{\mu(c-\mu+v)} > 1$ ■

Existence and stability conditions of endemic equilibrium points, E^e

For the system (7a) - (7b), we evaluate the endemic equilibrium points by solving the quadratic equation$P(I) = AI^2 + BI + C = 0$, where

$$A = -2\beta I\left(\frac{A}{\mu} - I - R\right) + \beta I^2 + v + \frac{c}{(Ia+1)^2} + 2\mu, \quad (15)$$

$$B = \left(-2\beta I\left(\frac{A}{\mu} - I - R\right) + \beta I^2 + v + \frac{c}{(Ia+1)^2} + \mu\right)\mu + \beta I^2 k\left(v + \frac{c}{(Ia+1)^2}\right). \quad (16)$$

If$B < 0$, then the eigenvalue becomes $\frac{1}{2}\left(B \pm \sqrt{B^2 - 4AC}\right)$. However, It could be obtained that if $I^*$is a positive real root of the above quadratic equation, then the $E^e = (I^*, R^*)$ is the endemic equilibrium point of our system (7a) - (7b). However, if$\beta \leq \mu + v$, then system (7a) - (7b) posse no endemic equilibrium point.

Experimental simulation and numerical method

Here, we apply the Adams-type predictor-corrector method [4], [25], which is an implicit formula method. The method gives the error-free means of solving a problem with a sensible and logical choice of the time step[4].

To illustrate the stability of the fractional epidemic model as in system (7a) - (7b), the following parameters are chosen $\beta = \frac{1}{2}, v = 1.27, c = 2, \mu = 1, k = \frac{1}{2}, A = 6$with the following initial

values $(I, R) = (2,\ 1)$. By direct solving, and using Maple 18 software, it can be shown that the fractional-order model (7a) - (7b)have the following equilibrium points,
$E_1(I_1, R_1) = (1.615353698, 1.242243888)$, and
$E_2(I_2, R_2) = (2.046474276, 1.522295468)$
Hence, the Jacobian matrix for the corresponding equilibrium point (I_1, R_1) is given as

$$J = \begin{bmatrix} -\frac{1}{2}I^2 + (6 - I - R)I - 2.27 - 2(4I+1)^{-1} + 8\frac{I}{(4I+1)^2} & -\frac{1}{2}I^2 \\ 0.6350000000(4I+1)^{-1} & -1 \end{bmatrix} \quad (17)$$

and its eigenvalues for disease-free, E^0 are
$\lambda_1 = -1 + 0.I,$ (18)
$\lambda_2 = -4.27000000000000 + 0.I.$
(19)
Where that of the endemic,E^e , are
$\lambda_1 - 0.206140119950000\ +\ 0.851064558561176\,I,$ (20)
$\lambda_2 = -0.206140119950000 -\ 0.851064558561176\,I.$ (21)

And while the characteristic equation of the fractional epidemic model as in Equation (7a) - (7b) is: $P(\lambda) = -0.4106172840\ -\ 0.7053086420\lambda + \lambda^2$
Therefore, the argument $|arg\ \lambda_1| > \frac{\alpha\pi}{2}$ of matrix J at $\alpha = 0.8$ fall with the range of values, 3.141592654. The values of $|arg\ \lambda_1|$ of the $E_1(I_1, R_1)$ points is said to be stable and the system gives the asymptotically stable because all the eigenvalues fulfill $|arg\ \lambda_1| > \frac{\alpha\pi}{2}$ That is, $|arg\ \lambda_1| = 3.141592654 > 1.256800000 = \frac{\alpha\pi}{2}$.Also, by direct calculation, it is easy to show that $R_0 = \frac{\beta A}{\mu(c-\mu+v)} = 0.7025761124$, which obtained result is in agreement and compatible with Theorem 2, where $R_0 <\ 0.7025761124$. It means that the conditions for existence and asymptotically stable as discussed above are satisfied. It indicates that the spread of a disease depends on the contact rates with infected individual within the population. Basic reproduction number, R_0, which is the number of people that one sick person will infect (on average) also affects the model behaviour. We used R_0 to establish the existence and stability conditions at the equilibrium points. This parameter determines a threshold whenever $R_0 > 1$, a typical infective gives rise, on average, to more than one secondary infection, leading to an epidemic. In contrast, when $R_0 < 1$, infective typically give rise (on average) to less than one secondary infection, and the prevalence of infection cannot increase.[26]stated that the condition $R_0 < 1$ is a necessary and sufficient condition for the eradication when disease for the forward bifurcation occurs. However, it is no longer a sufficient condition for the occurrence of backward bifurcation.

However, a lot of numerical simulations were observed to see the effects that α's parameter has on dynamics behaviour of the fractional-order model after applying the Adams-type predictor-corrector method with $0 < \alpha \leq 1$
Fig. 2-3 shows the phase portrait plot of the infected and recovered individual in a particular time, t, in a stable endemic equilibrium when parameters are taken as $\beta = \frac{1}{2}, v = 1.27, c = 2, a = 4, \mu = 1, k = \frac{1}{2}, A = 6$. It indicated a stable infected situation.

Fig.4 displays the stable endemic equilibrium for system (7a-7b) when parameters are taken as $\beta = \frac{1}{2}, v = 1.27, c = 2, a = 4, \mu = 1, k = \frac{1}{2}, A = 6.$ The values of equilibrium are(2.046474276, 1.522295468). It indicated a stable recovered situation.

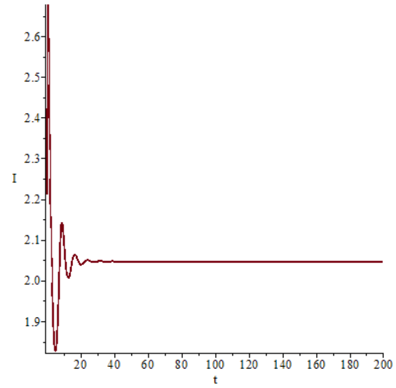

Fig. 2: Infected stable waveform plot of the fractional-order model for the system (7a-7b) with α = 0.95

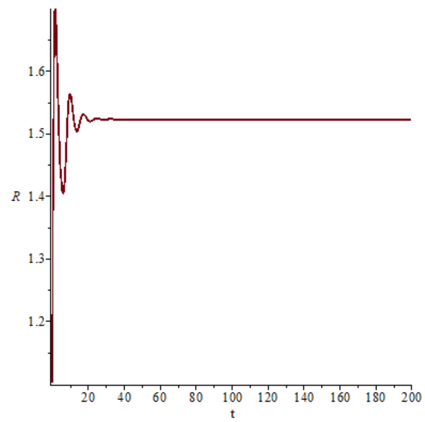

Fig. 3: Recovered stable waveform plot of the fractional-order model for the system (7a-7b) with α = 0.95

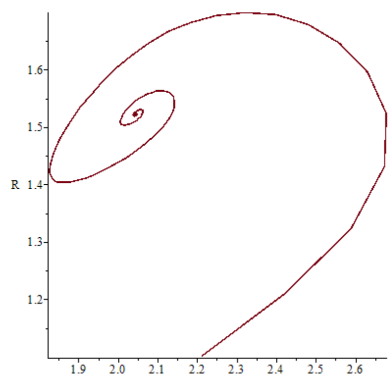

Fig.4: Recovered versus infected stable orbit plot of the fractional-order model for the system (7a-7b) with α = 0.95

Some effects of the fractional-order α on the behavior of dynamical systems of the epidemic model

In this section, we show that the stability region of the equilibrium points of the system (7a-7b) can be affected by choosing an appropriate value of fractional order α. However, the fractional-order system is achieved in the steady state when parameters which affect the value of alphaα, are controlled well [7].

Theorem 3

SupposeI^* is a positive real root of the quadratic equation, then $E^e$is the endemic equilibrium point of system (7a-7b)

a) The endemic equilibrium point E^e is unstable.
b) If $\alpha \leq \frac{2}{3}$,the endemic equilibrium point $E^e$are locally asymptotically α stable.
c) If $\alpha > \frac{2}{3}$ and $v \leq k$, the endemic equilibrium point $E^e$are locally asymptotically.

Proof

To prove Theorem 3, it is sufficient to show that all eigenvalues of Jacobian matrix of system (7a-7b) at E^e satisfied the condition (9)-(10). Hence, the Jacobian matrix is

$$\begin{bmatrix} -I^2\beta + 2\beta\left(\frac{A}{\mu} - I - R\right)I - v - \frac{c}{Ia+1} + \frac{cLa}{(Ia+1)^2} - \mu & -I^2\beta \\ K\left(v + \frac{c}{Ia+1} - \frac{cLa}{(Ia+1)^2}\right) & -\mu \end{bmatrix}$$

From the second equation (7a-7b), the characteristic of $P(I) = KI^2 + MI + N = 0$where

$$k = \frac{\beta(kv+kc+\mu-Aa)}{\beta\alpha(kv+\mu)}, \tag{22}$$

$$M = \frac{-\beta A+v\mu a+\mu^2 a}{\beta\alpha(kv+\mu)}, \tag{23}$$

$$N = \frac{c\mu+\mu^2+v\mu}{\beta\alpha(kv+\mu)}. \tag{24}$$

We basically follow the fundamental fractional order Routh-Hurwitz conditions in [20]. For every endemic equilibrium point, E^e, it is obvious that the condition for (9-10) is $0 \leq K, 0 \leq M, 0 \leq N$ and $\alpha \leq \frac{2}{3}$. As earlier stated, by Descartes' rule of sign, we will have at most one positive root (real or imaginary root). Since all the parameters are real and positive and the quadratic equation has real coefficients, the complex value roots will be in conjugate pair. However, we now assume that $P(I) = 0$possess one non-positive real root says, $\lambda_1 = -t$ and a pair of complex value roots says $\lambda_{2,3} = 0 \pm bi$ as thus;

$$P(I) = I^3 + I^2(-2a + t) + I(a^2 + b^2 - 2at) + t(a^2 + b^2) \tag{25}$$

It implies that

$$K = -2a + t, M = a^2 + b^2 - 2a, N = t(a^2 + b^2) \tag{26}$$

We know that $0 \leq N,\ 0 \leq M$. It implies that $2at \leq a^2 + b^2$and $2a = t$, from which we obtain

$$4a^2 \leq 2at \leq a^2\left(1 + \frac{a^2}{b^2}\right) \tag{27}$$

It then show from equation (27) that $4 \leq sec^2\left(Arg\lambda_{2,3}\right)$ and $\frac{\pi}{3} \leq \left(Arg\lambda_{2,3}\right) \leq \frac{2\pi}{3}$. Therefore, if $\alpha \leq \frac{\pi}{3}$, then condition (9-10) is satisfied and E^e are considered stable. Likewise, $KM - N > 0$, then $0 < -2a(a-t)^2 + b^2$. In the same manner, if $KM - N > 0$, then $\lambda_{2,3}$

must have non-positive real parts. It can be shown that if $v \leq k$, then $KM - N > 0$ and the root of the equation $P(I) = 0$ have a non-positive real parts■

In order to make comparison with the uncontrolled fractional-order for the system (7a-7b), we discuss the control model with different value. Fig. 5 - 7 shows that at $\alpha = 27$and for lowering the parameter of α namely, $\alpha = 1, 0.95, 0.90, 0.85$ it has effect on the stability, and as a result can stabilize the stable fixed point. From the epidemiological point of view, this feature is very important because the interpretation shows a longer periodic which infected persons can affect the health system [7]. It appeared in the figs that the fractional figure, $\alpha = 0.95, 0.90, 0.85$, entered stable point than it integer-order counterpart i.e at $\alpha = 1$.

The combined (infected and recovered class over time) numerical simulations for the system (7a-7b) are shown in Fig. 5-7 for α = 1 (colour blue), α = 0.99 (colour green) andα = 0.89 (colour red). The profiles indicated that system (7a-7b) speedy approaches the steady state based on the different values of α. This indicated that a change of the value α affects the dynamics of the epidemic.The fractional-order differential equations are very much stable than the integer-order. However, the value of α actually dictates when the model approaches the steady state. That is, the dynamics of the epidemic is affected by the change in the value of α. It shows how long a person can be affected with disease such as Pertussis and Influenza. The result indicated that fractional models are more realistic to represent the daily life phenomena than its integral–order counterpart.

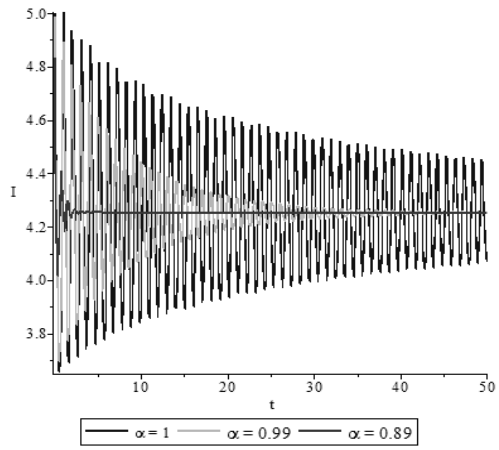

Fig. 5: Infected stable waveform plot of the fractional-order model for the system (7a-7b)

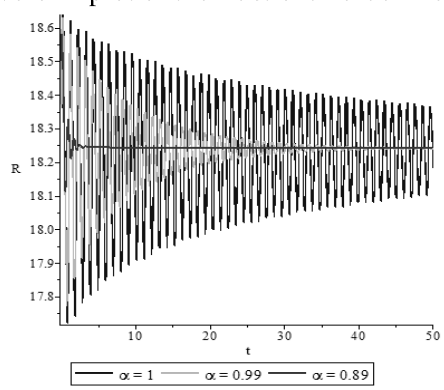

Fig. 6: Recovered stable waveform plot of the fractional-order model for the system (7a-7b)

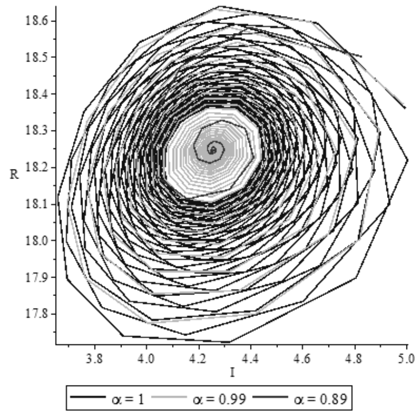

Fig.7: Recovered versus infected stable orbit plot of the fractional-order model

CONCLUSION

In this paper, we worked on a proposed epidemic model with sub-optimal immunity, nonlinear incidence rate and saturated treatment/recovery rate. The model is extended in the sense of Caputo derivative of order $\alpha \in [0, 1]$. The model indicated that the spread of a disease depends on the contact rates with infected individual within the population. Basic reproduction number, R_0, affects the model behaviour. We used R_0 to establish the existence and stability conditions at the equilibrium points. For simple epidemic processes, this parameter determines a threshold whenever $R_0 > 1$, a typical infective gave rise, on average, to more than one secondary infection, leading to an epidemic. Theorem 3 indicated that the stability of endemic equilibrium points can be managed and controlled by modifying the value of α. In fact, the fractional-order model can be achieved in the steady state by controlling the parameters which affect the value of α.We see that the value of α controls the stability region of the equilibrium points. We arrived at a conclusion that fractional-order is a very interesting order in the area of mathematical biology. The fractional modeling is an advantageous approach which has been used to study the behavior of diseases because the fractional derivative is a generalization of the integer-order derivative[7].

References:

A. Atangana and J. F. Botha, "A generalized groundwater flow equation using the concept of variable-order derivative," *Bound. Value Probl.*, 2013.

E. Ahmed and A. S. Elgazzar, "On fractional order differential equations model for nonlocal epidemics," *Phys. A Stat. Mech. its Appl.*, 2007.

K. Al-Khaled, "Numerical solution of time-fractional partial differential equations using sumudu decomposition method," *Rom. J. Phys.*, 2015.

I. Ameen and P. Novati, "The solution of fractional order epidemic model by implicit Adams methods," *Appl. Math. Model.*, 2017.

C. N. Angstmann, A. M. Erickson, B. I. Henry, A. V. Mcgann, J. M. Murray, and J. A. Nichols, "Fractional order compartment models," *SIAM J. Appl. Math.*, 2017.

V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Anal. Theory, Methods Appl.*, 2008.

D. Rostamy and E. Mottaghi, "Stability analysis of a fractional-order epidemics model with multiple equilibriums," *Adv. Differ. Equations*, 2016.

I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*. 1999.

J. Rogel-Salazar, "Fractional Calculus: An Introduction for Physicists (2nd Edition) by Richard Herrmann," *Contemp. Phys.*, 2014.

K. Diethelm and N. J. Ford, "Analysis of fractional differential equations," *J. Math. Anal. Appl.*, 2002.

D. Baleanu, S. Rezapour, H. Mohammadi, D. Baleanu, and S. Rezapour, "Some existence results on nonlinear fractional differential equations Some existence results on nonlinear fractional differential equations," no. April, 2013.

A. H. Bhrawy, T. M. Taha, and J. A. T. Machado, "A review of operational matrices and spectral techniques for fractional calculus," *Nonlinear Dynamics*. 2015.

P. Rahimkhani, Y. Ordokhani, and E. Babolian, "A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations," *Numer. Algorithms*, 2017.

M. Caputo and M. Fabrizio, "A new Definition of Fractional Derivative without Singular Kernel," *Progr. Fract. Differ. Appl*, 2015.

H. A. A. El-Saka, "The Fractional-order SIR and SIRS Epidemic Models with Variable Population Size," *Math. Sci. Lett. Math. Sci. Lett. An Int. J.*, vol. 2, no. 3, pp. 195–200, 2013.

G. González-Parra, A. J. Arenas, and B. M. Chen-Charpentier, "A fractional order epidemicmodel for the simulation of outbreaks of influenza A(H1N1)," *Math. Methods Appl. Sci.*, 2014.

K. Diethelm, "A fractional calculus based model for the simulation of an outbreak of dengue fever," *Nonlinear Dyn.*, 2013.

H. S. Rodrigues, M. T. T. Monteiro, D. F. M. Torres, and A. Zinober, "Dengue disease, basic reproduction number and control," in *International Journal of Computer Mathematics*, 2012.

C. Huang, J. Cao, M. Xiao, A. Alsaedi, and F. E. Alsaadi, "Controlling bifurcation in a delayed fractional predator–prey system with incommensurate orders," *Appl. Math. Comput.*, 2017.

E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, "Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models," *J. Math. Anal. Appl.*, 2007.

D. Matignon, "Stability results for fractional differential equations with applications to control processing," *Comput. Eng. Syst. Appl.*, 1996.

J. Singh, D. Kumar, M. Al Qurashi, and D. Baleanu, "A new fractional model for giving up smoking dynamics," *Adv. Differ. Equations*, 2017.

S. Ruan and W. Wang, "Dynamical behavior of an epidemic model with a nonlinear incidence rate," *Journal of Differential Equations*. 2003.

C. Phang, Y. Wu, and B. Wiwatanapataphee, "Computation of the Domain of Attraction for Suboptimal Immunity Epidemic Models Using the Maximal Lyapunov Function Method," *Abstr. Appl. Anal.*, 2013.

K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dyn.*, 2002.

A. M. Yousef and S. M. Salman, "Backward Bifurcation in a Fractional-Order SIRS Epidemic Model with a Nonlinear Incidence Rate," vol. 17, pp. 401–412, 2016.