TRIANGULAR POSITIONS OF PERTURBED GENERALIZED PHOTO-GRAVITATIONAL RESTRICTED THREE BODY PROBLEM WITH POYNTING-ROBERTSON DRAG EFFECT

by

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Abstract

In this work we investigated the effects of small perturbations in the coriolis (ε) and centrifugal (ε') on the position of the triangular libration points of oblate, radiating Restricted Three-Body Problem (RTBP) under the influence of the Poynting-Robertson (PR) drag force. The coordinates of the triangular libration points are obtained and are seen to be affected by the small perturbation in the centrifugal force due to the presence of its parameter in the equations. Our results were verified by applying it to the Kruger-60 and RXJ 0450,1−5856 binary system using the MATLAB Mathematical software.

Keywords: *Restricted Three Body Problem, Libration points, Perturbation, PR-Drag.*

Introduction

The Restricted Three Body Problem (RTBP) is one of the major problems in Astrophysics which helps to predict the existence, evolution and future motion of solar and extrasolar objects. Precisely, the RTBP describes the motion of an infinitesimal mass moving in the plane of two massive bodies, called the primaries, such that its motion does not influence their motions. The spacecraft moving in the vicinity of planets or the satellites orbiting the planets is a typical model of the RTBP.

Szebehely (1967a) discovered that the RTBP possesses three unstable collinear and two stable non-collinear points for the mass ratio, μ, $0 < \mu \le 1/2$.

Based on the fact that planetary bodies such as planets, natural and artificial satellites, asteroids, comets and meteorite exhibit properties (such as radiation, varying masses, atmospheric drag, solar wind drag, oblateness, triaxiality, coriolis and centrifugal forces etc) which affects the motion of the Classical system, thereby leading to a change in the general solution, various generalisations have been made by researchers on the classical RTBP.

Kunitsyn (2000, 2001) obtained the equation of the relative equilibrium positions (collinear libration points) of the circular photo-gravitational RTBP, and observed that the triangular points form isosceles triangles due to the radiation effect from either or both of the primaries on the RTBP.

The effect of the degree of flattening known as oblateness have also investigated. This unique condition occurs due to the rotation of planetary bodies. Sharma and Subbarao (1976, 1979) established that the triangular shape formed by the libration point were affected by the presence of the oblateness property. Abouelmagd (2012) observed that there

exist five equilibrium points for which due to oblateness, the triangular points deviate from its positions but does not influence the motion of the system in the x-y plane, in the linear sense.
Sharma (1982) studied triangular libration points of the RTBP when the bigger primary is an oblate spheroid as well as a source of radiation. Sharma (1987) further generalized this study by considering an oblate primary and radiating secondary. The shape of the triangular libration points were distorted more, but the motion of this system was not affected.
Poynting (1903) was the first to study the process by which solar radiation causing meteors and dust grain orbiting the stars to lose angular momentum. Robertson (1937) later modified it by using the relativistic approach of the first order in the ratio of the velocity of the particle to the speed of light to establish the expression for the net drag force given by:

$$\vec{F} = F_P\left[\frac{\vec{r}}{r} - \frac{\vec{v}}{c}\frac{\vec{r}}{r}\frac{\vec{r}}{r} - \frac{\vec{v}}{c}\right] \qquad (1)$$

where,
$F_P = \dfrac{3Lm}{16\pi r^2 \rho s c}$ denotes the measure of radiation pressure, $\bar{r}$ the position vector of a particle with respect to the radiation source, $\bar{v}$ is the corresponding velocity, c is the speed of light, L is the luminosity of the radiating body, m is the mass of the particle, ρ is the density of the particle, s is the cross section of the particle.
The first term expresses the radiation pressure effect, the second represents the Doppler shifts owing to the motion of the particle and the third is due to the absorption and subsequent re-emission of part of the radiation. The last two terms of Eq. (1) constitute the PR-drag effect. Colombo *et. al* (1996) studied the effect of radiation pressure and PR-drag on the RTBP. Chernikov (1970) and Schuerman (1980) established the existence of six libration points in which one lie out of the orbital plane. Murray (1994) explained the dynamical effect of drag in general on the planar circular RTBP. Liou *et. al* (1995) studied the effect of radiation, PR and solar wind drag in the RTBP. Ragos and Zafiropoulos (1995) established the equations of motion for when the primaries are radiating with PR-drag effect from the expression of the net force acting on the system. He studied this problem numerically and discovered that the collinear points deviate from the axis while the triangular points are no longer symmetrical. Raj and Ishwar (2017) obtained the diagonalizable Hamiltonian for the photogravitational RTBP with the PR-drag.
Kushvah and Ishwar (2004) and Ishwar and Kushvah (2006) studied the triangular equilibrium points of the generalized photo-gravitational RTBP when the smaller primary is considered to be oblate spheroid and the bigger one radiating with PR-drag. Das *et. al* (2009) determined the out of plane equilibrium points of a passive micron size particle and their stability in the field of radiating binary stars. Lhotka and Celletti (2014) examined the effect of PR-drag on the triangular libration points in the framework of elliptic RTBP. This is an extension of Murray (1994) work. Singh and Amuda (2014) studied the photo-gravitational RTBP when the bigger primary is considered to be oblate and the smaller one a

source of radiation with PR-drag. Singh *et. al* (2014) using analytical and numerical methods, obtained the triangular libration points which were found to move towards the line joining the primaries in the presence of any of perturbations (such as oblateness up to J_4 of the less massive primary, electromagnetic radiation of the more massive primary and potential from the belt), except in the presence of oblateness up to J_4 where the points move away from the line joining the primaries and examined their linear stability. A practical application of their model is the study of the motion of a dust particle near a radiating star and an oblate body surrounded by a belt. Jaiyeola *et. al* (2016) extended their works to understand the effects of various perturbing factors on the dynamics of a particle orbiting the primaries. They concluded that the P-R drag renders unstable those libration points that are conditionally stable in the classical case. The study of the effects of small perturbation in the coriolis and centrifugal forces on the stability of libration points of the RTBP cannot be over-emphasized because of their peculiar nature.

Amongst the many forces affecting the motion of the RTBP are the fictitious or pseudo forces known as the Coriolis and Centrifugal forces. Perturbations in these forces must be introduced particularly, since our study is centered on objects in the inertial frame considered from a rotating frame of reference. These forces are weak compared to most typical forces in everyday life.

In the case of a distant star observed from a rotating spacecraft in the reference frame co-rotating with the spacecraft, the star appears to move along a circular trajectory around the spacecraft, the resultant force F_R of centrifugal and Coriolis force must be taken into account. The vector sum of the centrifugal and the Coriolis force is the total fictitious force given by

$$\bar{F}_R = -m\bar{\omega}\times(\bar{\omega}\times\bar{r}) - 2m\bar{w}\times\bar{v} \qquad (2)$$

where m is the mass of the object, $\bar{\omega}$ is its angular velocity of the rotating frame, $\bar{r}$ the position vector and $\bar{v}$ the corresponding velocity as seen in the rotating frame.

Generalizing the classical case, Witner (1941) showed that the triangular libration points still forms an equilateral shape and this was due to the presence of the Coriolis parameter in the equation of motion. Szebehely $(1967b)$ considered similar problem keeeping the Centrifugal force constant and established for the triangular points a relation between the critical value of the mass parameter μ_c and the change ε in the coriolis (ε) force and concluded that the coriolis force has a corrective effect. Subbarao and Sharma (1975) showed that with oblateness, the Coriolis force is not always a stabilizing force. Bhatagar and Hallan (1979) extended their work to include the centrifugal (ε') force and showed that the coordinate of the triangular and non triangular points obtained were affected by the force. Also, it was seen that an increase or decrease in the range of stability depend upon the points $(\varepsilon, \varepsilon')$. Abdulraheem and Singh (2006), Singh (2009,2011), Singh and Aminu (2014), Singh and Omale (2015) and many other researchers have introduced and studied the effects of the coriolis and centrifugal forces, radiation pressure force, oblateness, on the stability of the RTBP. They observed that small perturbation in the Coriolis and centrifugal forces, radiation pressure force and oblateness all have significant influence on the position of the triangular libration points in one way or the other, but the effect of small perturbations in the Coriolis and centrifugal forces on the RTBP under the combined influence of oblateness, radiation pressure force with PR-drag has not yet been taken into consideration.

In this present study, we build upon the work of Jaiyeola *et. al* (2016) to study the effects of the coriolis and centrifugal forces on the location of the triangular libration points of the Photogravitational RTBP under the influence of the P-R drag force, in the linear sense. The primaries were taken to oblate and radiating with P-R drag effect. The motion of the kruger-60 and RXJ0450, 1-5856 binary systems were our model system. The results obtained here would serve as a form of reference to achieving more interesting and vital results in Space Dynamics and also an added value to designers of space crafts and aerospace agencies.

Equations of Motion

With reference to an inertial or fixed coordinates OXYZ, let $m(x,y,z)$, $m_1(-x_1,0,0)$ and $m_2(x_2,0,0)$ be the coordinates of the infinitesimal body, massive, and less massive, primary respectively. let r_1, r_2 be the distances between each of the primary and the infinitesimal while r is the distance between the primaries. Introducing a rotating coordinate system Oxyz with the origin O at the barycenter of the primaries in which the axis rotate relative to the inertial space with an angular velocity $\omega = nk$, the net force on the infinitesimal body due to the oblateness, radiation pressure effects PR-drag effect from the primaries is given by the coriolis theorem as,

$$\bar{F} = -m\bar{\omega}\times(\bar{\omega}\times\bar{r}) - 2m\bar{\omega}\times\bar{v} + \gamma m\left[\frac{m_i\bar{r}_i}{r_i^3} + \frac{3m_iA_i\bar{r}_i}{2r_i^5}\right] + F_{p_i}\left\{\frac{\bar{r}_i}{r_i} - \frac{\bar{v}.\bar{r}_i.\bar{r}_i}{c.r_i^2} - \frac{\bar{v}_i}{c}\right\}$$

where,

$F_p = F_g(1-q),\ 0<(1-q)<<1,\ q = 1-\frac{F_p}{F_g}$ is the mass reduction factor and $F_g = -\frac{\gamma m_1 m_2}{r^2}$ is the force due to gravity.

Now,

$$\bar{a} = -\bar{\omega}\times\left(\bar{\omega}\times\bar{r}\right) - 2\bar{\omega}\times\bar{v}$$
$$+\gamma\left[\frac{m_iq_i}{r_i^3}\bar{r}_i + \frac{3m_iA_i}{2r_i^5}\bar{r}_i + \frac{m_i(1-q_i)}{r_i^2}\left(\frac{(\dot{\bar{r}}_i + \bar{\omega}\times\bar{r}_i)\bar{r}_i\bar{r}_i}{cr_i^2} + \frac{(\dot{\bar{r}}_i + \bar{\omega}\times\bar{r}_i)}{c}\right)\right] \tag{3}$$

where,

$$i = 1,2,\ \bar{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \bar{r}_1 = (x+x_1)\hat{i} + y\hat{j} + z\hat{i},\ \bar{r}_2 = (x-x_2)\hat{i} + y\hat{j} + z\hat{k}$$

$$\dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k},\ \bar{r}_1^2 = (x+x_1)^2 + y^2 + z^2,\ \bar{r}_2^2 = (x-x_2)^2 + y^2 + z^2,\ v = \dot{\bar{r}} + \omega\times\bar{r}$$

we assume the distance between the primaries along the x- axis to be equal to one. The sum of the masses of the primary is also taken as 1 so that if $m_2 = \mu$ then $m_1 = 1-\mu$ and the origin the barycenter of the masses $x_1 = \mu$ and $x_2 = 1-\mu$ where $\mu = \frac{m_2}{m_1+m_2} < \frac{1}{2}$ is the mass ratio parameter. The unit of time is so chosen so as to make the gravitational constant γ to be equal to unity. The speed of light c is given as $c = c_d$. Assuming that the mass reduction factor, $q_i (i=1,2)$ are constant (neglecting fluctuations in the beam of solar radiation and the effect of the planet shadow). The $n^2 = 1+\frac{3}{2}A_i$, (i=1,2), where $0 < A_i = \frac{a_{e_i}^2 - a_{p_i}^s}{5r^2} \ll 1$ (McCuskey (1963)) are the oblateness coefficient. Comparing and equating the coefficients of **i** and **j** in Eq. (3), gives the dimensionless equation of the particle in the $x-y$ orbital plane, without perturbations in the coriolis and centrifugal forces as,

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= U_x \\ \ddot{y} + 2n\dot{x} &= U_y \end{aligned} \tag{4}$$

where

$$\begin{aligned} U_x &= n^2 x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2}{2r_2^5} \\ &\quad - \frac{\vartheta_1}{r_1^2}\left\{\frac{(x+\mu)}{r_1^2}[\dot{x}(x+\mu)+\dot{y}y] + \dot{x} - ny\right\} \\ &\quad - \frac{\vartheta_2}{r_2^2}\left\{\frac{(x+\mu-1)}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y] + \dot{x} - ny\right\} \end{aligned} \tag{5}$$

$$\begin{aligned} U_y &= n^2 y + \left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu q_2}{r_2^3} + \frac{3(1-\mu)A_1}{2r_1^5} + \frac{3\mu A_2}{2r_2^5}\right\} y \\ &\quad - \frac{\vartheta_1}{r_1^2}\left\{\frac{y}{r_1^2}[(x+\mu)\dot{x}+\dot{y}y] + \dot{y} + n(x+\mu)\right\} \\ &\quad - \frac{\vartheta_2}{r_2^2}\left\{\frac{y}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y] + \dot{y} + n(x+\mu-1)\right\} \end{aligned} \tag{6}$$

$$\vartheta_1 = \frac{(1-\mu)(1-q_1)}{c_d}, \vartheta_2 = \frac{\mu(1-q_2)}{c_d}, \tag{7}$$

$$r_1^2 = (x+\mu)^2 + y^2 \quad and \quad r_2^2 = (x+\mu-1)^2 + y^2 \tag{8}$$

The mean motion n is given by

$$n^2 = 1 + \frac{3A_1}{2} + \frac{3A_2}{2} \tag{9}$$

Introducing ε and ε' to represent small perturbations in the Coriolis and the centrifugal forces, using the parameter φ and ψ respectively such that

$$\varphi = 1+\varepsilon, \psi = 1+\varepsilon' \quad |\varepsilon|, \ |\varepsilon'| \ll 1 \tag{10}$$

The equations of motion in Eq. (4) now become

$$\begin{aligned} \ddot{x} - 2n\varphi\dot{y} &= \Omega_x = \Omega^*{}_x + \Omega_{PR_x} \\ \ddot{y} + 2n\varphi\dot{x} &= \Omega_y = \Omega^*{}_y + \Omega_{PR_y} \end{aligned} \tag{11}$$

where

$$\Omega^* = \frac{n^2\psi}{2}(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1}{2r_1^3} + \frac{\mu A_2}{2r_2^3} \tag{12}$$

$$\begin{aligned} \Omega_{PR} &= \vartheta_1\left[\frac{(x+\mu)\dot{x}+y\dot{y}}{2r_1^2} - n\arctan\left(\frac{y}{x+\mu}\right)\right] \\ &+\vartheta_2\left[\frac{(x+\mu-1)\dot{x}+y\dot{y}}{2r_2^2} - n\arctan\left(\frac{y}{x+\mu-1}\right)\right] \end{aligned} \tag{13}$$

Ω^* and Ω_{PR} are the negative of the gravitational potential due to attraction on the infinitesimal body under the influence of radiation, oblateness and P-R drag from the primaries. These are functions of position, velocity and dependent on the small perturbation in the centrifugal forces due to the presence of their parameters.

The equations of motion of the problem obtained in Eq. (11) admits the Jacobi integral given by

$$\dot{x}^2 + \dot{y}^2 = 2\Omega^* + 2W_{PR} - C$$

where $W_{PR} = \int(\dot{x}\Omega_{PR_x} + \dot{y}\Omega_{PR_Y})dt$ and C is the Jacobi constant.

The equation of the Zero Velocity Curves (ZVC) are given by

$$C = 2\Omega^*(x,y) \tag{14}$$

and the curve C represent various regions of possible motion.

The Triangular Libration Points

The triangular libration points are the solutions of Eq. (11) when the velocity and acceleration is zero (i.e $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$) and $y \neq 0$

$$\begin{aligned} \Omega_x &= n^2\psi x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} \\ &- \frac{3(1-\mu)(x+\mu)A_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2}{2r_2^5} + \frac{n\vartheta_1 y}{r_1^2} + \frac{n\vartheta_2 y}{r_2^2} = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} \Omega_y &= [n^2\psi - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3(1-\mu)A_1}{2r_1^5} - \frac{3\mu A_2}{2r_2^3}]y \\ &- \frac{n\vartheta_1(x+\mu)}{r_1^2} - \frac{n\vartheta_2(x+\mu-1)}{r_2^2} = 0 \end{aligned} \tag{16}$$

where ϑ_1, ϑ_2, r_1, r_2 and n are given in Eqs. (7) - (9).

In the absence of the radiation, oblateness and PR-drag (i.e. $n = q_1 = q_2 = 1,\ \vartheta_1 = \vartheta_2 = A_1 = A_2 = 0$) Eqs. (15) - (16) give $r_1 = r_2 = \frac{1}{\psi^{1/3}}$. Now, we let $\alpha,\ |\alpha| \ll 1\, and\ \beta,\ |\beta| \ll 1$ to represent the presence small perturbations in the coriolis and centrifugal forces, radiation pressure, oblateness and PR-drag so that

$$r_1 = \frac{1}{\psi^{1/3}}(1+\alpha)\ and\ r_2 = \frac{1}{\psi^{1/3}}(1+\beta) \qquad (17)$$

Substituting Eq. (17) in Eq. (8) and solving simultaneously for x and y , considering only linear terms of small quantities, gives

$$x = \frac{1}{2} - \mu + \frac{1}{\psi^{\frac{1}{3}}}(\alpha - \beta) \qquad (18)$$

$$y = \frac{\sqrt{4-\psi^{\frac{2}{3}}}}{2\psi^{\frac{1}{3}}}\left[1 + \frac{2\psi^{\frac{1}{3}}}{4-\psi^{\frac{2}{3}}}(\alpha+\beta)\right] \qquad (19)$$

Where,

$$\alpha = -\frac{w_1}{3\psi^{\frac{1}{3}}} - \frac{(1-\psi^{\frac{2}{3}})A_1}{2\psi^{\frac{1}{3}}} - \frac{A_2}{2\psi^{\frac{1}{3}}} - \frac{[\vartheta_1(2-\psi^{\frac{2}{3}}) + 2\vartheta_2]}{3(1-\mu)\psi\sqrt{4-\psi^{\frac{2}{3}}}} \qquad (20)$$

$$\beta = -\frac{w_2}{3\psi^{\frac{1}{3}}} - \frac{A_1}{2\psi^{\frac{1}{3}}} - \frac{(1-\psi^{\frac{2}{3}})A_2}{2\psi^{\frac{1}{3}}} + \frac{[2\vartheta_1 + \vartheta_2(2-\psi^{\frac{2}{3}})]}{3\mu\psi\sqrt{4-\psi^{\frac{2}{3}}}}$$

Which is obtained by solving Eqs. (15) - (16) using elimination method. Taking $q_1 = (1-w_1)$, $q_2 = (1-w_2)$, $|w_i| = 1 (i = 1,2)$ and substituting the values for $r_1,\ r_2,\ n,\ x\ and\ y$ obtained from Eqs. (8), (9), (18) and (19) into the equations above and considering only linear terms of $w_1, w_2, A_1, A_2, \vartheta_1, \vartheta_2$.

Therefore using Eq. (20) in Eqs. (18) - (19), produces the coordinates of the triangular libration points, $L_4(x,+y)$ and $L_5(x,-y)$ as:

$$x = \frac{1}{2} - \mu - \frac{w_1}{3\psi^{\frac{2}{3}}} + \frac{w_2}{3\psi^{\frac{2}{3}}} + \frac{A_1}{2} - \frac{A_2}{2} - \frac{[\vartheta_1(2-\mu\psi^{\frac{2}{3}}) + \vartheta_2(2-\psi^{\frac{2}{3}}+\mu\psi^{\frac{2}{3}})]}{3\mu(1-\mu)\psi^{\frac{4}{3}}\sqrt{4-\psi^{\frac{2}{3}}}} \qquad (21)$$

$$y = \pm\frac{\sqrt{4-\psi^{\frac{2}{3}}}}{2\psi^{\frac{1}{3}}}\left\{1 - \frac{2}{4-\psi^{\frac{2}{3}}}\left[\frac{w_1}{3} + \frac{w_2}{3} + \frac{(2-\psi^{\frac{2}{3}})A_1}{2} + \frac{(2-\psi^{\frac{2}{3}})A_2}{2}\right.\right. \qquad (22)$$
$$\left.\left. - \left[\frac{\vartheta_1(2-\mu(4-\psi^{\frac{2}{3}})) + \vartheta_2(2-\psi^{\frac{2}{3}}-\mu(4-\psi^{\frac{2}{3}}))}{3\mu(1-\mu)\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right]\right]\right\}.$$

The presence of the parameters ψ in these equations shows that the positions of the primaries are affected by the small perturbation in the centrifugal force.

Putting Eq. (10) in Eqs. (21) - (22) neglecting product of ε' with other small quantities $(|w_1|,|w_2|,|A_1|,|A_2|,|\vartheta_1|,|\vartheta_2|)$ the coordinates become

$$x_p \quad = \frac{1}{2} - \mu - \frac{w_1}{3} + \frac{w_2}{3} + \frac{A_1}{2} - \frac{A_2}{2} - \frac{[\vartheta_1(2-\mu) + \vartheta_2(1+\mu)]}{3\mu(1-\mu)\sqrt{3}} \tag{23}$$

$$y_p \quad = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{4\varepsilon'}{9} - \frac{2w_1}{9} - \frac{2w_2}{9} - \frac{A_1}{3} - \frac{A_2}{3} + \frac{\left(\vartheta_1(2-3\mu) + \vartheta_2(1-3\mu)\right)}{9\mu(1-\mu)\sqrt{3}}\right] \tag{24}$$

In order to appreciate the impact of the centrifugal force on the location of the libration points, we obtain further, the product of ε' with the small quantity parameters, taking only the first order terms in ε'. The coordinate were obtained as

$$\begin{aligned} x_{p^*} \quad &= \frac{1}{2} - \mu - \frac{w_1(3-2\varepsilon')}{9} + \frac{w_2(3-2\varepsilon')}{9} + \frac{A_1}{2} - \frac{A_2}{2} - \frac{\vartheta_1\left(18 - 22\varepsilon' - \mu(9-5\varepsilon')\right)}{27\mu(1-\mu)\sqrt{3}} \\ &- \frac{\vartheta_2\left(9 - 17\varepsilon' + \mu(9-5\varepsilon')\right)}{27\mu(1-\mu)\sqrt{3}} \end{aligned} \tag{25}$$

$$\begin{aligned} y_{p^*} = \pm\frac{\sqrt{3}}{2}\Bigg[&1 - \frac{4\varepsilon'}{9} - \frac{2(9-2\varepsilon')w_1}{81} - \frac{2(9-2\varepsilon')w_2}{81} - \frac{(9-8\varepsilon')A_1}{27} - \frac{(9-8\varepsilon')A_2}{27}. \\ &+ \frac{\vartheta_1\left(18 - 14\varepsilon' - \mu(27-54\varepsilon')\right)}{81\mu(1-\mu)\sqrt{3}} + \frac{\vartheta_2\left(9 - 11\varepsilon' - \mu(27-23\varepsilon')\right)}{81\mu(1-\mu)\sqrt{3}}\Bigg] \end{aligned} \tag{26}$$

In line with the work of Narayan and Shrivasta (2013), Singh and Umar (2014), we vary the values for the parameters ε' in studying the effect of small perturbation in the centrfugal force on the location around the triangular libration points. Specifically for the binary system Kruger–60 $(\mu = 0.3937, c_d = 48002.33, q_1 = 0.99992, q_2 = 0.99996)$ and $RXJ\,0450{,}1-5856(\mu = 0.0967, c_d = 2997924.58, q_1 = 0.9963, q_1 = 0.9965)$ with the aid of microsoft Excel and Maple 18 Mathematical Software. The values obtained are given in the table below.

Table 1

Centrifugal effect on the location of the triangular libration points for Kruger-60

ε'	x	x_p	y	y_p	C_p
-0.45	0.101286665	0.101282665	1.030547137	1.027080725	1.719567305
-0.40	0.101286665	0.101283110	1.011302128	1.008220873	1.763213802
-0.35	0.101286665	0.101283554	0.992057119	0.989361021	1.803923940
-0.30	0.101286665	0.101283999	0.972812110	0.970501169	1.841823436
-0.25	0.101286665	0.101284443	0.953567101	0.951641317	1.877038427
-0.20	0.101286665	0.101284888	0.934322092	0.932781465	1.909695458
-0.15	0.101286665	0.101285332	0.915077083	0.913921612	1.939921448
-0.10	0.101286665	0.101285777	0.895832074	0.895061760	1.967843663
-0.05	0.101286665	0.101286221	0.876587065	0.876201908	1.993589669
0	0.101286665	0.101286666	0.857342056	0.857342056	2.017287280
0.05	0.101286665	0.101287110	0.838097047	0.838482204	2.039064490
0.10	0.101286665	0.101287555	0.818852038	0.819622351	2.059049388

0.15	0.101286665	0.101287999	0.799607029	0.800762499	2.077370061
0.20	0.101286665	0.101288444	0.780362020	0.781902647	2.094154466
0.25	0.101286665	0.101288888	0.761117011	0.763042795	2.109530294
0.30	0.101286665	0.101289333	0.741872002	0.744182943	2.123624788
0.35	0.101286665	0.101289777	0.722626993	0.725323090	2.136564542
0.40	0.101286665	0.101290222	0.703381984	0.706463238	2.148475262
0.45	0.101286665	0.101290666	0.684136975	0.687603386	2.159481482

$x_c = 0.1063$ $y_c = \pm 0.866025404$ $C_c = 0.986949845$ (subscript c indicate evaluation for the classical case) and the values for the Jacobi Constant, C_p associated with the ZVCs that contain those point for Kruger – 60 is on the table.

Table 2

Centrifugal on the location of the triangular libration points for $RXJ\,0450{,}1-5856$.

ε'	x	x_p	y	y_p	C_p
-0.45	0.397183333	0.396848333	1.029790808	1.034331668	2.317252484
-0.40	0.397183333	0.396885555	1.010545799	1.015223619	2.376375280
-0.35	0.397183333	0.396922778	0.991300790	0.996115570	2.432777094
-0.30	0.397183333	0.396960000	0.972055781	0.977007522	2.486591281
-0.25	0.397183333	0.396997222	0.952810772	0.957899473	2.537951831
-0.20	0.397183333	0.397034444	0.933565763	0.938791424	2.586993342
-0.15	0.397183333	0.397071667	0.914320754	0.919683376	2.633850980
-0.10	0.397183333	0.397108889	0.895075745	0.900575327	2.678660433
-0.05	0.397183333	0.397146111	0.875830736	0.875437175	2.721557847
0	0.397183333	0.397183333	0.856585727	0.856585727	2.762679742
0.05	0.397183333	0.397220556	0.837340718	0.837734278	2.802162914
0.10	0.397183333	0.397257778	0.818095709	0.818882830	2.840144313
0.15	0.397183333	0.397295000	0.798850700	0.805035084	2.876760888
0.20	0.397183333	0.397332222	0.779605691	0.785927035	2.912149413
0.25	0.397183333	0.397369444	0.760360682	0.766818986	2.946446262
0.30	0.397183333	0.397406667	0.741115673	0.747710938	2.979787160
0.35	0.397183333	0.397443889	0.721870664	0.728602889	3.012306869
0.40	0.397183333	0.397481111	0.702625655	0.709494840	3.044138833
0.45	0.397183333	0.397518333	0.683380646	0.690386792	3.075414756

$x_c = 0.4033$ $y = \pm 0.866025404$ $C_c = 1.359625445$ for the cassical RTBP and the values of the Jacobi Constant, C_p associated with the ZVCs that contain those points for $RXJ\,0450{,}1-5856$.

The Table 1 and 2 above shows that there is a significant change in the values of the coordinates of the triangular libration points $(x_c,\ y_c)$ (classical case of the system) due to the presence of all the perturbing factors. For the two model systems (Kruger-60 and RXJ0450,1-5856), it can be seen that as the value of the small perturbation in the centrifugal force, ε', $\left(|\varepsilon'| << 1\right)$ is increasing, the values of x coordinate is not affected by the change.

But when the equation of the coordinate, x_p is extended up to the first order product of ε' with other small quantities, the values of x coordinate is increasing.

However, the values of the y decreases with increase in ε' thereby, affecting the isosceles triangle made by the coordinate in the classical case and other generalizations. This can be observed in the figures below.

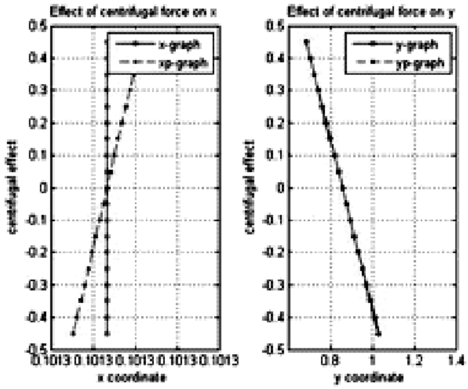

Figure 1: Coordinate points for Kruger-60

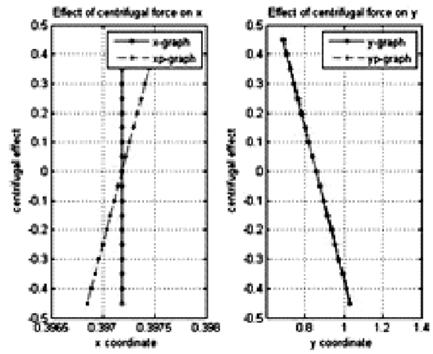

Figure 2: Coordinate point for RXJ 0450, 1-5856

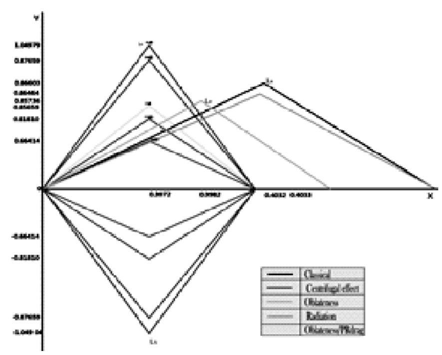

Figure 3: $L_{4,5}$ for Kruger-60

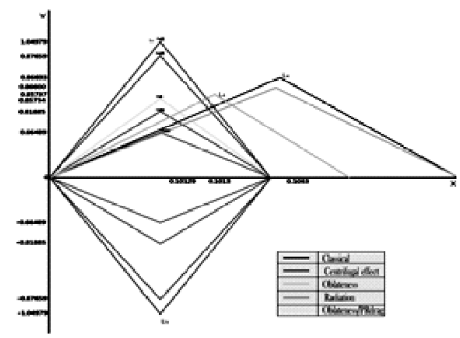

Figure 3: $L_{4,5}$ for RXJ 0450, 1-5856

Disscusion

We have studied the effects of small perturbations in the Coriolis and Centrifugal forces on the position of the and the other radiating with PR-drag without perturbations in the Coriolis and centrifugal forces reduces to the results of Kushvah and Ishwar, (2004) and Singh and Amuda, (2014).

Extending Eq. (24) to consider the product of ε' with small quantities up to the first order linear terms, as seen in Eqs. (25) - (26) shows that both the x and y coordinate of the triangular libration points are dependent on the parameter ε'.

Varying the values of ε', $|\varepsilon'| \ll 1$, and using the astronomical data's for the Kruger-60 and RXJ0450, 1-5856 binary system, also taking the oblateness coefficients for the primaries, A_1 and A_2 to be 0.01 and 0.02 respectively, Tables 1 - 2 shows that, as ε' is increasing, the values of the x coordinate is not changing in the linear sense, but is seen to be increasing significantly when its first order product with other small quantities is considered. On the other-hand, the values of the y coordinates is seen to be decreasing as ε' is increasing at the same rate. This can be veiwed in Figs. 1 and 2 , showing the graphs of ε' plotted against the x and y coordinates of Kruger-60 and RXJ0450, 1-5856 respectively.

Figs. 3 and 4 shows the sketch of the triangles formed by L_4 *and* L_5 showing the small pertubations in the centrifugal force, oblateness, radiation and the PR-drag force libration points on RTBP.

Conclusion

A satellite (natural or artificial) is expected to navigate in the neighbourhood of the planets in our solar system around the libration points under the influence of perturbing forces as Astrophysical evidence has revealed that these forces are natural activities in our solar, extrasolar and stellar systems.

In line with existing research results of various generalizations involving small perturbations in the centrifugal force, radiation pressure forces, oblateness of primaries, Poynting-Robertson drag, we have been able to obtain the coordinates of the triangular points and seen that the small perturbations in the centrifugal force have significant influence on the coordinate , therefore, it study should not be overlooked. Hence, our result provides information for Space/Astronomical Engineers to take into consideration, the effects all these perturbing forces particularly, perturbations in the centrifugal force when designing spacecraft that will navigate in the vicinity of the planets and binary stars.

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