AN INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH TIME DEPENDENT QUADRATIC DEMAND AND COMPLETE BACKLOGGING UNDER TRADE CREDIT POLICY

by

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Abstract

In the classical economic order quantity model, it was assumed that the purchaser must pay for the items received immediately. In real practices, the supplier may provide the purchaser a permissible delay of payments so as to settle for the goods supplied. This motivates retailers to order more which in turns lead higher turnover by the supplier. In this paper, an inventory model for non-instantaneous deteriorating items with two-phase demand under trade credit policy and complete backlogging has been considered. The demand rate before deterioration sets in is assumed to be time dependent quadratic function after which it is considered as constant. Shortages are allowed and are completely backlogged. Optimal time with positive inventory, cycle length and order quantity are determined so as to minimise the total variable cost. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions are provided. Three numerical examples are provided to demonstrate the application of the model for each case. Finally, sensitivity analysis of some model parameters on optimal solutions have been carried out and the implications are discussed. In the discussions, suggestions toward minimizing the total variable cost of the inventory system are also given.

***Keywords**: Non-instantaneous deteriorating item, Quadratic demand, Trade credit policy, Complete backlogging.*

1. Introduction

Since the formulation of economic order quantity (EOQ) model by Harris (1913), several models were developed in the inventory literature by assuming a constant demand rate. But in the real marketing situation, the demand rate of any item may vary with time. Silver and Meal (1969) were the first to modify a simple classical square root formula developed by Harris (1913) for time-varying demand rate. Later, Silver and Meal (1973) developed a heuristic approach to determine EOQ in the general case of a time varying-demand rate. Many researchers such as Dave and Patel (1981), Goyal (1986), Goswami and Chaudhuri (1991), Chang and Dye (1999), Khanra and Chaudhuri (2003), Ghosh and Chaudhuri (2006), Khanra *et al.* (2011), Sarkar *et al*. (2012) and Mishra (2016) made their valuable contributions in this direction.

In the conventional EOQ models, it is consider that the retailers should pay for the items as soon as they are received. Butin real life practice, a supplier/wholesaler offers the retailer a delay period in paying for purchasing cost, known as trade credit period. Retailer can accumulate revenues by selling items and by earning interests. The concept of trade credit was first introduced by Haley and Higgins (1973). Goyal (1985) was the first to consider the EOQ model under conditions of permissible delay in payments. Several valuable contributions in this field were made in this direction. This include articles developed by

Aggarwal and Jaggi (1995), Chu *et al*. (1998), Chung (2000), Sana and Chaudhuri (2008), Geetha *et.* al (2010), Khanra *et al*. (2011),and Sarkar [(2012a), (2012b), (2013)], Jaggi *et al.* (2008), Min *et al.* (2010) and so on.

Deb and Chaudhuri (1987) were the first to incorporate shortages in their model by extending the model of Silver (1979). This extension and incorporation of shortages was studied by Dave (1989), Goyal *et al*. (1992), Goswami and Chaudhuri, (1991), Giri *et al.* (1996), Teng (1996) and so on. Choudhury *et al* (2013) developed an inventory model for non-instantaneous deteriorating item with stock dependent demand, time varying holding cost and shortages with complete backlogging.

In this present model, an effort has been made to extend the work of Babangida and Baraya (2018) by allowing shortages which are completely backlogged. The analytical solution of the model is obtained and the solution is illustrated with the help of numerical examples. Finally, sensitivity analysis is carried out to show the effect of changes in some model parameters on decision variables. This is followed by discussions and conclusion.

2. Model Description and Formulation

This section describes the proposed model notation, assumptions and formulation.

2.1 Notation and assumptions

The inventory system is developed based on the following notation and assumptions.

Notation:

A The ordering cost per order.
C The purchasing cost per unit per unit time (\$/unit/ year).
S The selling price per unit per unit time (\$/unit/ year).
C_b Shortage cost per unit per unit of time.
h The holding cost (excluding interest charges) per unit per unit time (\$/unit/year).
I_c The interest charged in stock by the supplier per Dollar per year (\$/unit/year) $(I_c \geq I_e)$.
I_e The interest earned per Dollar per year (\$/unit/year).
M The trade credit period (in year) for settling accounts.
θ The constant deterioration rates function $(0 < \theta < 1)$.
t_d The length of time in which the product exhibits no deterioration.
t_1 Length of time in which the inventory has no shortage.
T The length of the replenishment cycle time (time unit).
Q_m The maximum inventory level.
B_m The backorder level during the shortage period.
Q The order quantity during the cycle length where $Q = (Q_m + B_m)$.

Assumptions

This model is developed under the following assumptions.

(i) The replenishment rate is infinite.
(ii) The lead time is zero.
(iii) A single non-instantaneous deteriorating item is considered.
(iv) During the fixed period, t_d, there is no deterioration and at the end of this period, the inventory item deteriorates at the constant rate θ.
(v) There is no replacement or repair for deteriorated items during the period under consideration.
(vi) Demand before deterioration begins is quadratic function of time t and is given by

$\alpha + \beta t + \gamma t^2$ where $\alpha \geq 0, \beta \neq 0, \gamma \neq 0$.

(vii) Demand after deterioration sets in is assumed to be constant and is given by λ.

(viii) During the trade credit period M $(0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.

(ix) Shortages are allowed and completely backlogged.

2.2 Formulation of the model

The inventory system is developed as follows. There are Q_m units arrival of a single product from the manufacturer at the beginning of the cycle (i.e., at time $t = 0$) arrive. During the time interval $[0, t_d]$, the inventory level $V_1(t)$ is depleting gradually due to market demand only and it is assumed to be quadratic function of time t. At time interval $[t_d, t_1]$ the inventory level $V_2(t)$ is depleting due to combined effects of demand from the customers and deterioration and the demand at time is reduced to λ, a constant demand. At time $t = t_1$, the inventory level depletes to zero. Shortages occur at the time $t = t_1$ and is completely backlogged. The behaviour of the inventory system is described in figures below.

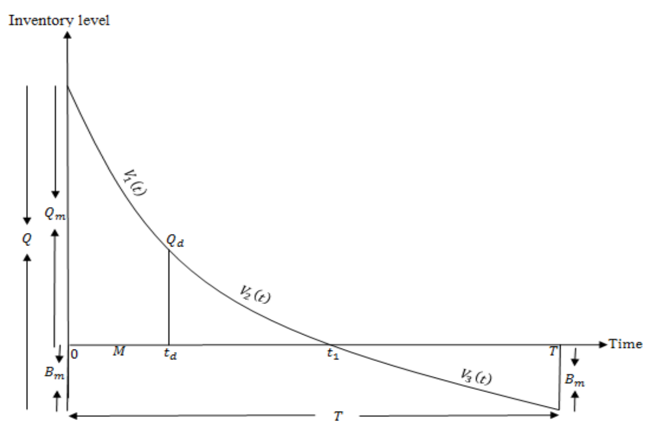

Figure 1: Inventory situation for case $(0 < M \leq t_d)$

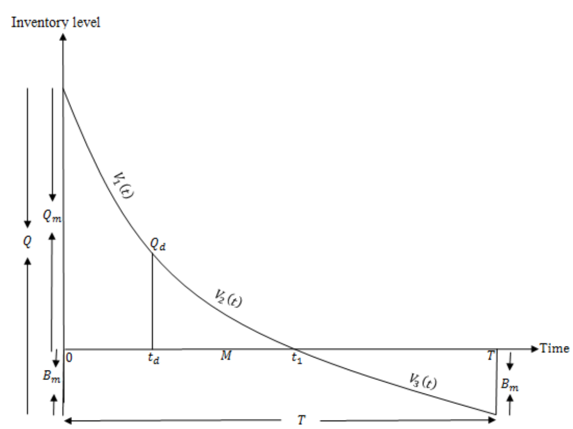

Figure 2: Inventory situation for case $(t_d < M \leq t_1)$

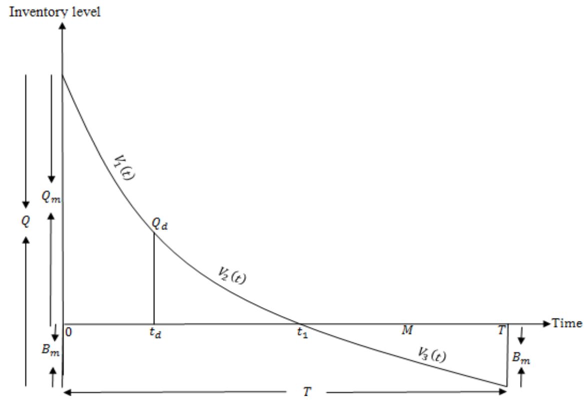

Figure 3: Inventory situation for case $(M > t_1)$

Based on the above description, during the time interval $[0, T]$, the change of inventory at any time t is represented by the following differential equations

$$\frac{dV_1(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad 0 \leq t \leq t_d \tag{1}$$

with boundary conditions $V_1(0) = Q_m$ and $V_1(t_d) = Q_d$.

$$\frac{dV_2(t)}{dt} + \theta V_2(t) = -\lambda, \qquad t_d \leq t \leq t_1 \tag{2}$$

with boundary conditions $V_2(t_1) = 0$ and $V_2(t_d) = Q_d$.

$$\frac{dV_3(t)}{dt} = -\lambda, \qquad t_1 \leq t \leq T \tag{3}$$

with condition $V_3(t_1) = 0$ at $t = t_1$.

The solution of equations (1), (2) and (3) are

$$V_1(t) = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \alpha(t_d - t) + \frac{\beta}{2}\left(t_d^2 - t^2\right) + \frac{\gamma}{3}\left(t_d^3 - t^3\right), \quad 0 \leq t \leq t_d \tag{4}$$

$$V_2(t) = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t)} - 1\right), \qquad t_d \leq t \leq t_1 \tag{5}$$

and

$$V_3(t) = \lambda(t_1 - t) \tag{6}$$

The maximum inventory level is obtained at $t = 0$, and then from equation (4), we have

$$Q_m = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(\alpha t_d + \beta \frac{t_d^2}{2} + \gamma \frac{t_d^3}{3}\right) \tag{7}$$

The value of Q_d is obtained at $t = t_d$, and then from equation (5), we have

$$Q_d = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) \tag{8}$$

The maximum backordered inventory B_m is obtained at $t = T$, and then from equation (6), we have

$$B_m = \lambda(T - t_1) \tag{9}$$

Thus the order size during total time interval $[0,\ T]$ is

$$Q = Q_m + B_m$$

$$= \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(\alpha t_d + \beta \frac{t_d^2}{2} + \gamma \frac{t_d^3}{3}\right) + \lambda(T - t_1) \tag{10}$$

(i) The total demand during the period $[t_d,\ t_1]$ is given by

$$D_M = \int_{t_d}^{t_1} \lambda\, dt$$

$$= \lambda(t_1 - t_d) \qquad (11)$$

(ii) The total number of deteriorated items per cycle is given by

$D_P = Q_d - D_M$

Substituting Q_d and D_M from equations (8) and (11) respectively into D_P, we obtain

$$D_P = \frac{\lambda}{\theta}\left[e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right] \qquad (12)$$

(iii) The deterioration cost is given by

$$D_C = C\frac{\lambda}{\theta}\left[e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right] \qquad (13)$$

(iv) The fixed ordering cost per order is given by A

(v) The inventory holding cost for the entire cycle is given by

$$C_H = h\left[\int_0^{t_d} V_1(t)dt + \int_{t_d}^{t_1} V_2(t)dt\right] \qquad (14)$$

Substituting equations (4) and(5) into equation (14), we obtain

$$C_H = h\left[\frac{\lambda t_d}{\theta} e^{\theta(t_1 - t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta}\right] \qquad (15)$$

(vi) The backordered cost per cycle is given by

$$C_B = C_b\int_{t_1}^{T} -V_3(t)dt$$

$$= \frac{C_b \lambda}{2}(T - t_1)^2 \qquad (16)$$

(vii) The average total cost per unit time for a replenishment cycle (denoted by $Z(T)$ is given by

$$Z(t_1, T) = \begin{cases} Z_1(t_1, T) & 0 < M \le t_d \\ Z_2(t_1, T) & t_d < M \le t_1 \\ Z_3(t_1, T) & M > t_1 \end{cases} \qquad (17)$$

where $Z_1(t_1, T)$, $Z_2(t_1, T)$, and $Z_3(t_1, T)$ are discussed for three different cases follows.

Case 1:$(\mathbf{0} < M \le \boldsymbol{t_d})$

The interest payable

This is the period before deterioration sets in, and payment for goods is settled with the capital opportunity cost rate I_c for the items in stock. Thus, the interest payable is given below.

$$I_{P1} = cI_c\left[\int_M^{t_d} V_1(t)dt + \int_{t_d}^{t_1} V_2(t)dt\right]$$

$$= cI_c\left[\frac{\lambda(t_d - M)}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{\lambda}{\theta^2}\left(e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right)\right] \qquad (18)$$

The interest earned

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although, the retailer has to settle the accounts at period M, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to t_d. The interest earned is

$$I_{E1} = sI_e\left[\int_0^M (\alpha + \beta t + \gamma t^2)t\,dt\right]$$

$$= sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right) \tag{19}$$

The average total variable cost per unit time ($0 < M \le t_d$) is

$Z_1(t_1, T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + interest payable during the permissible delay period – interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{A + h\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1 - \theta t_1\right)\right]$$
$$+ C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right] + \frac{C_b\lambda}{2}(T - t_1)^2$$
$$+ cI_c\left[\frac{\lambda(t_d - M)}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2\right.$$
$$\left. + \frac{\gamma}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2 + \frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right)\right]$$
$$- sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Bigg\} \tag{20}$$

Case 2: ($\boldsymbol{t_d} < M \le \boldsymbol{t_1}$)

The interest payable

This is when the end point of credit period is greater than the period with no deterioration but shorter than or equal to the length of period with positive inventory stock of the items. The interest payable is

$$I_{P2} = cI_c\left[\int_M^{t_1} V_2(t)\,dt\right]$$

$$= cI_c\left[\frac{\lambda}{\theta^2}\left(e^{\theta(t_1-M)} - 1 - \theta(t_1 - M)\right)\right] \tag{21}$$

The interest earned

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although, the retailer has to settle the accounts at period M, for that he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M to t_1. The interest earned is

$$I_{E2} = sI_e\left[\int_0^{t_d} (\alpha + \beta t + \gamma t^2)t\,dt + \int_{t_d}^M \lambda t\,dt\right]$$

$$= sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right] \tag{22}$$

The average total variable cost per unit time ($t_d < M \le t_1$) is

$Z_2(t_1,T)=\frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + interest payable during the permissible delay period – interest earned during the cycle}

$$=\frac{1}{T}\Bigg\{A+h\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)}+\frac{\alpha}{2}t_d^2+\frac{\beta}{3}t_d^3+\frac{\gamma}{4}t_d^4+\frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)}-1-\theta t_1\right)\right]$$
$$+C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)}-1-\theta(t_1-t_d)\right]+\frac{C_b\lambda}{2}(T-t_1)^2$$
$$+cI_c\left[\frac{\lambda}{\theta^2}\left(e^{\theta(\mathbf{t_1}-M)}-1-\theta(t_1-M)\right)\right]$$
$$-sI_e\left[\left(\alpha\frac{t_d^2}{2}+\beta\frac{t_d^3}{3}+\gamma\frac{t_d^4}{4}\right)+\frac{\lambda M^2}{2}-\frac{\lambda t_d^2}{2}\right]\Bigg\} \qquad (23)$$

Case 3: $(M>t_1)$

The interest payable

In this case, the period of delay in payment is greater than period with positive inventory. In this case the retailer pays no interest. Therefore, $I_{P3}=0$.

The interest earned

In this case, the period of delay in payment (M) is greater than period with positive inventory (t_1). In this case the retailer earns interest on the sales revenue up to the permissible delay period and no interest is payable during the period for the item kept in stock. Interest earned for the time period $[0,T]$

$$I_{E3}=sI_e\left[\int_0^{t_d}(\alpha+\beta t+\gamma t^2)tdt+(M-t_1)\int_0^{t_d}(\alpha+\beta t+\gamma t^2)dt+\int_{t_d}^{t_1}\lambda tdt+(M-t_1)\int_{t_d}^{t_1}\lambda dt\right]$$
$$=sI_e\left[\left(\alpha\frac{t_d^2}{2}+\beta\frac{t_d^3}{3}+\gamma\frac{t_d^4}{4}\right)+(M-t_1)\left(\alpha t_d+\beta\frac{t_d^2}{2}+\gamma\frac{t_d^3}{3}\right)-\frac{\lambda}{2}(t_1-t_d)^2\right.$$
$$\left.+M\lambda(t_1-t_d)\right] \qquad (24)$$

The average total variable cost per unit time $(M>t_1)$ is

$Z_3(t_1,T)=\frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost +backordered cost– interest earned during the cycle}

$$=\frac{1}{T}\Bigg\{A+h\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)}+\frac{\alpha}{2}t_d^2+\frac{\beta}{3}t_d^3+\frac{\gamma}{4}t_d^4+\frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)}-1-\theta t_1\right)\right]$$
$$+C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)}-1-\theta(t_1-t_d)\right]+\frac{C_b\lambda}{2}(T-t_1)^2$$
$$-sI_e\left[\left(\alpha\frac{t_d^2}{2}+\beta\frac{t_d^3}{3}+\gamma\frac{t_d^4}{4}\right)+(M-t_1)\left(\alpha t_d+\beta\frac{t_d^2}{2}+\gamma\frac{t_d^3}{3}\right)-\frac{\lambda}{2}(t_1-t_d)^2\right.$$
$$\left.+M\lambda(t_1-t_d)\right]\Bigg\} \qquad (25)$$

Since $0<\theta<1$, by utilizing a quadratic approximation for the exponential terms in equations (20), (23) and (25) we have

$$Z_1(t_1,T)=\frac{\lambda}{T}\left\{\frac{1}{2}A_1t_1^2-B_1t_1+C_1-C_bTt_1+\frac{C_bT^2}{2}\right\} \qquad (26)$$

where $A_1 = \left[h(t_d\theta + 1) + cI_c\big(1 + \theta(t_d - M)\big) + C\theta + C_b\right]$, $B_1 = [ht_d^2\theta + C\theta t_d + cI_c(M + \theta(t_d - M)t_d)]$ and $C_1 = \frac{1}{\lambda}\Big[A + h\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + \frac{C\theta\lambda t_d^2}{2} - sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right) + cI_c\Big(\frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2 - \frac{\lambda t_d^2}{2} + M\lambda t_d + \frac{\lambda\theta}{2}(t_d - M)t_d^2\Big)\Big]$.

Similarly

$$Z_2(t_1, T) = \frac{\lambda}{T}\left\{\frac{1}{2}A_2t_1^2 - B_2t_1 + C_2 - C_bTt_1 + \frac{C_bT^2}{2}\right\} \tag{27}$$

where $A_2 = [h(t_d\theta + 1) + cI_c + C\theta + C_b]$, $B_2 = [ht_d^2\theta + C\theta t_d + cI_cM]$ and $C_2 = \frac{1}{\lambda}\Big[A + h\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + \frac{C\theta\lambda t_d^2}{2} - sI_e\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right) + cI_c\frac{\lambda}{2}M^2\Big]$.

and

$$Z_3(t_1, T) = \frac{\lambda}{T}\left\{\frac{1}{2}A_3t_1^2 - B_3t_1 + C_3 - C_bTt_1 + \frac{C_bT^2}{2}\right\} \tag{28}$$

where $A_3 = [h(t_d\theta + 1) + sI_e + C\theta + C_b]$, $B_3 = \left[ht_d^2\theta + C\theta t_d - sI_e\left[(M + t_d) - \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda}\right]\right]$ and $C_3 = \frac{1}{\lambda}\left[A + h\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + \frac{C\theta\lambda t_d^2}{2} - sI_e\left(\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)M - \frac{\lambda}{2}(2M + t_d)t_d\right)\right]$.

3. Optimal Decision

In order to find the ordering policies that minimize the total variable cost per unit time, we established the necessary and sufficient conditions. The necessary condition for the total variable cost per unit time $Z_i(t_1, T)$ to be minimum are $\frac{\partial Z_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_i(t_1,T)}{\partial T} = 0$ for $i = 1, 2, 3$. The value of (t_1, T) obtained from $\frac{\partial Z_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_i(t_1,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 Z_i(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 Z_i(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 Z_i(t_1,T)}{\partial t_1\,\partial T}\right)^2\right\} > 0$ is satisfied gives a minimum for the total variable cost per unit time $Z_i(t_1, T)$.

Case 1: $(\mathbf{0} < M \leq \boldsymbol{t_d})$.

The necessary conditions for the average total variable cost in equation (26) to be the minimum are $\frac{\partial Z_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_1(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_1(t_1, T)}{\partial t_1} = \frac{\lambda}{T}\{A_1t_1 - B_1 - C_bT\} = 0 \tag{29}$$

which implies

$$T = \frac{1}{C_b}(A_1t_1 - B_1) \tag{30}$$

Note that $A_1t_1 - B_1 = \left[h(t_d\theta(t_1 - t_d) + t_1) + C\theta(t_1 - t_d) + C_bt_1 + cI_c\big((t_1 - M) + \theta(t_d - M)(t_1 - t_d)\big)\right] > 0$ since $(t_d - M) \geq 0, (t_1 - t_d), (t_1 - M) > 0$

Similarly

$$\frac{\partial Z_1(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}A_1t_1^2 - B_1t_1 + C_1 - \frac{C_bT^2}{2}\right\} = 0 \qquad (31)$$

Substituting T from equation (30) into equation (31), we obtain

$$A_1(A_1 - C_b)t_1^2 - 2B_1(A_1 - C_b)t_1 - (2C_bC_1 - B_1^2) = 0 \qquad (32)$$

Let $\Delta_1 = A_1(A_1 - C_b)t_d^2 - 2B_1(A_1 - C_b)t_d - (2C_bC_1 - B_1^2)$

Lemma 1. For $0 < M \le t_d$, we have

(i) If $\Delta_1 \le 0$, then the solution of $t_1 \in [t_d, \infty)$ (say t_{11}^*) which satisfies equation (32) not only exists but also is unique.

(ii) If $\Delta_1 > 0$, then the solution of $t_1 \in [t_d, \infty)$which satisfies equation (32) does not exist.

Proof of (i). From equation (32), we define a new function $F_1(t_1)$ as follows

$$F_1(t_1) = A_1(A_1 - C_b)t_1^2 - 2B_1(A_1 - C_b)t_1 - (2C_bC_1 - B_1^2),\ t_1 \in [t_d, \infty). \qquad (33)$$

Taking the first-order derivative of $F_1(t_1)$ with respect to $t_1 \in [t_d, \infty)$, we have

$$\frac{F_1(t_1)}{dt_1} = 2(A_1t_1 - B_1)(A_1 - C_b) > 0$$

Because $(A_1t_1 - B_1) > 0$ and $(A_1 - C_b) = \left[h(t_d\theta + 1) + cI_c\big(1 + \theta(t_d - M)\big) + C\theta\right] > 0$

Hence we obtain that $F_1(t_1)$ is increasing of t_1 in the interval $[t_d, \infty)$. Moreover, we have $\lim_{t_1 \to \infty} F_1(t_1) = \infty$ and $F_1(t_d) = \Delta_1 \le 0$

We have $F_1(t_d) \le 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{11}^* \in [t_d, \infty)$ such that $F_1(t_{11}^*) = 0$. Hence t_{11}^* is the unique solution of equation (32). Thus, the value of t_1 (denoted by t_{11}^*) can be found from equation (32) and is given by

$$t_{11}^* = \frac{B_1}{A_1} + \frac{1}{A_1}\sqrt{\frac{(2A_1C_1 - B_1^2)C_b}{(A_1 - C_b)}} \qquad (34)$$

Once we obtain t_{11}^*, then the value of T (denoted by T_1^*) can be found from equation (30) and is given by

$$T_1^* = \frac{1}{C_b}(A_1t_{11}^* - B_1) \qquad (35)$$

Equations (34) and (35) give the optimal values of t_{11}^* and T_1^* for the cost function in equation (26) only if B_1 satisfies the inequality given in equation (36)

$$B_1^2 < 2A_1C_1 \qquad (36)$$

Proof of (ii). If $\Delta_1 > 0$,then from equation (33), we have $F_1(t_1) > 0$. Since $F_1(t_1)$ is an increasing function of $t_1 \in [t_d, \infty)$, we have $F_1(t_1) > 0$ for all $t_1 \in [t_d, \infty)$. Thus, we cannot find a value of $t_1 \in [t_d, \infty)$ such that $F_1(t_1) = 0$. This completes the proof.

Theorem 1. When $0 < M \le t_d$, we have

(i) If $\Delta_1 \le 0$, then the total variable cost $Z_1(t_1, T)$ is convex and reaches its global minimum at the point (t_{11}^*, T_1^*), where (t_{11}^*, T_1^*) is the point which satisfies equations (32) and (29).

(ii) If $\Delta_1 > 0$, then the total variable cost $Z_1(t_1,\ T)$ has a minimum value at the point$(t_{11}^*,\ T_1^*)$ where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b}(A_1t_d - B_1)$

Proof of (i). When $\Delta_1 \le 0$, we see that t_{11}^* and T_1^* are the unique solutions of equations (32) and (29) from Lemma l(i). Taking the second derivative of $Z_1(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point $(t_{11}^*,\ T_1^*)$, we obtain

$$\left.\frac{\partial^2 Z_1(t_1,\ T)}{\partial t_1^2}\right|_{(t_{11}^*,\ T_1^*)} = \frac{\lambda}{T_1^*}A_1 > 0$$

$$\left.\frac{\partial^2 Z_1(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{11}^*,\ T_1^*)} = -\frac{\lambda}{T_1^*} C_b$$

$$\left.\frac{\partial^2 Z_1(t_1,\ T)}{\partial T^2}\right|_{(t_{11}^*,\ T_1^*)} = \frac{\lambda}{T_1^*} C_b > 0$$

and

$$\left(\left.\frac{\partial^2 Z_1(t_1,\ T)}{\partial t_1^2}\right|_{(t_{11}^*,\ T_1^*)}\right)\left(\left.\frac{\partial^2 Z_1(t_1,\ T)}{\partial T^2}\right|_{(t_{11}^*,\ T_1^*)}\right) - \left(\left.\frac{\partial^2 Z_1(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{11}^*,\ T_1^*)}\right)^2 = \frac{\lambda^2 C_b}{T_1^{*2}}(A_1 - C_b)$$

$$= \frac{\lambda^2 C_b}{T_1^{*2}}\left[h(t_d\theta + 1) + cI_c\big(1 + \theta(t_d - M)\big) + C\theta\right] > 0 \qquad (37)$$

We thus conclude from (37) and Lemma 1 that $Z_1(t_{11}^*,\ T_1^*)$ is convex and $(t_{11}^*,\ T_1^*)$ is the global minimum point of $Z_1(t_1,\ T)$. Hence the values of t_1 and T in equations (34) and (35) are optimal.

Proof of (ii). When $\Delta_1 > 0$, then we know that $F_1(t_1) > 0$ for all $t_1 \in [t_d,\ \infty)$. Thus, $\frac{\partial Z_1(t_1,\ T)}{\partial T} = \frac{F_1(t_1)}{T^2} > 0$ for all $t_1 \in [t_d, \infty)$ which implies $Z_1(t_1,\ T)$ is an increasing function of T. Thus $Z_1(t_1,\ T)$ has a minimum value when T is minimum. Therefore, $Z_1(t_1,\ T)$ has a minimum value at the point $(t_{11}^*,\ T_1^*)$ where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b}(A_1 t_d - B_1)$. This completes the proof.

Case 2: ($\boldsymbol{t_d} < M \leq \boldsymbol{t_1}$).
The necessary conditions for the average total variable cost in equation (27) to be the minimum are $\frac{\partial Z_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_2(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_2(t_1,T)}{\partial t_1} = \frac{\lambda}{T}\{A_2 t_1 - B_2 - C_b T\} = 0 \qquad (38)$$

which implies

$$T = \frac{1}{C_b}(A_2 t_1 - B_2) \qquad (39)$$

Note that $A_2 t_1 - B_2 = [h(t_d\theta(t_1 - t_d) + t_1) + C\theta(t_1 - t_d) + C_b t_1 + cI_c(t_1 - M)] > 0$ since $(t_1 - t_d) > 0, (t_1 - M) \geq 0$.
Similarly

$$\frac{\partial Z_2(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}A_2 t_1^2 - B_2 t_1 + C_2 - \frac{C_b T^2}{2}\right\} = 0 \qquad (40)$$

substituting T from equation (39) in equation (40), we obtain

$$A_2(A_2 - C_b)t_1^2 - 2B_2(A_2 - C_b)t_1 - (2C_b C_2 - B_2^2) = 0 \qquad (41)$$

Let $\Delta_2 = A_2(A_2 - C_b)M^2 - 2B_2(A_2 - C_b)M - (2C_b C_2 - B_2^2)$

Lemma 2. For $t_d < M \leq t_1$, we have

(i) If $\Delta_2 \leq 0$, then the solution of $t_1 \in [M, \infty)$ (say t_{12}^*) which satisfies equation (41) not only exists but also is unique.

(ii) If $\Delta_2 > 0$, then the solution of $t_1 \in [M, \infty)$ which satisfies equation (41) does not exist.

Proof of (i). From equation (41), we define a new function $F_2(t_1)$ as follows

$$F_2(t_1) = A_2(A_2 - C_b)t_1^2 - 2B_2(A_2 - C_b)t_1 - (2C_b C_2 - B_2^2), t_1 \in [M, \infty). \qquad (42)$$

Taking the first-order derivative of $F_2(t_1)$ with respect to $t_1 \in [M, \infty)$, we have

$$\frac{F_2(t_1)}{dt_1} = 2(A_2 t_1 - B_2)(A_2 - C_b) > 0$$

Because $(A_2 t_1 - B_2) > 0$ and $(A_2 - C_b) = [h(t_d\theta + 1) + cI_c + C\theta] > 0$

Hence we obtain that $F_2(t_1)$ is increasing of t_1 in the interval $[M, \infty)$. Moreover, we have $\lim_{t_1 \to \infty} F_2(t_1) = \infty$ and $F_2(M) = \Delta_2 \leq 0$

We have $F_2(M) \leq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{12}^* \in [M, \infty)$ such that $F_2(t_{12}^*) = 0$. Hence t_{12}^* isthe unique solution of equation (41). Thus, the value of t_1 (denoted by t_{12}^*) can be found from equation (41) and is given by

$$t_{12}^* = \frac{B_2}{A_2} + \frac{1}{A_2}\sqrt{\frac{(2A_2C_2 - B_2^2)C_b}{(A_2 - C_b)}} \tag{43}$$

Once we obtain t_{12}^*, then the value of T (denoted by T_2^*) can be found from equation (39) and is given by

$$T_2^* = \frac{1}{C_b}(A_2 t_{12}^* - B_2) \tag{44}$$

Equations (43) and (44) give the optimal values of t_{12}^* and T_2^* for the cost function in equation (27) only if B_2 satisfies the inequality given in equation (45)

$$B_2^2 < 2A_2C_2 \tag{45}$$

Proof of (ii). If $\Delta_2 > 0$, then from equation (42), we have $F_2(t_1) > 0$. Since $F_2(t_1)$ is an increasing function of $t_1 \in [M, \infty)$, we have $F_2(t_1) > 0$ for all $t_1 \in [M, \infty)$. Thus, we cannot find a value of $t_1 \in [M, \infty)$ such that $F_2(t_1) = 0$. This completes the proof.

Theorem 2. When $t_d < M \leq t_1$, we have

(i) If $\Delta_2 \leq 0$, then the total variable cost $Z_2(t_1, T)$ is convex and reaches its global minimum at the point (t_{12}^*, T_2^*), where (t_{12}^*, T_2^*) is the point which satisfies equations (41) and (38).

(ii) If $\Delta_2 > 0$, then the total variable cost $Z_2(t_1, T)$ has a minimum value at the point (t_{12}^*, T_2^*) where $t_{12}^* = M$ and $T_2^* = \frac{1}{C_b}(A_2 M - B_2)$

Proof of (i). When $\Delta_2 \leq 0$, we see that t_{12}^* and T_2^* are the unique solutions of equations (41) and (38) from Lemma 2(i). Taking the second derivative of $Z_2(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point $(t_{12}^*,\ T_2^*)$, we obtain

$$\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)} = \frac{\lambda}{T_2^*} A_2 > 0$$

$$\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{12}^*,\ T_2^*)} = -\frac{\lambda}{T_2^*} C_b$$

$$\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12}^*,\ T_2^*)} = \frac{\lambda}{T_2^*} C_b > 0$$

and

$$\left(\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)}\right)\left(\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12}^*,\ T_2^*)}\right) - \left(\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{12}^*,\ T_2^*)}\right)^2 = \frac{\lambda^2 C_b}{T_2^{*2}}(A_2 - C_b)$$

$$= \frac{\lambda^2 C_b}{T_2^{*2}} [h(t_d\theta + 1) + cI_c + C\theta] > 0 \qquad (46)$$

We thus conclude from equation (46) and Lemma 2 that $Z_2(t_{12}^*,\ T_2^*)$ is convex and $(t_{12}^*,\ T_2^*)$ is the global minimum point of $Z_2(t_1,\ T)$. Hence the values of t_1 and T in equations (43) and (44) are optimal.

Proof of (ii). When $\Delta_2 > 0$, then we know that $F_2(t_1) > 0$ for all $t_1 \in [M,\ \infty)$. Thus, $\frac{\partial Z_2(t_1,\ T)}{\partial T} = \frac{F_2(t_1)}{T^2} > 0$ for all $t_1 \in [M, \infty)$ which implies $Z_2(t_1,\ T)$ is an increasing function of T. Thus $Z_2(t_1,\ T)$ has a minimum value when T is minimum. Therefore, $Z_2(t_1,\ T)$ has a minimum value at the point$(t_{12}^*,\ T_2^*)$ where $t_{12}^* = M$ and $T_2^* = \frac{1}{C_b}(A_2 M - B_2)$. This completes the proof.

Case 3: $(\boldsymbol{M > t_1})$.

The necessary conditions for the average total variable cost in equation (28) to be the minimum are $\frac{\partial Z_3(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_3(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_3(t_1,T)}{\partial t_1} = \frac{\lambda}{T}\{A_3 t_1 - B_3 - C_b T\} = 0 \qquad (47)$$

and

$$T = \frac{1}{C_b}(A_3 t_1 - B_3) \qquad (48)$$

Note that

$$A_3 t_1 - B_3 = \left[h(t_d\theta(t_1 - t_d) + t_1) + C\theta(t_1 - t_d) + C_b t_1 + sI_e\left[(M + t_d + t_1) - \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda}\right]\right] > 0 \text{ since } (t_1 - t_d) > 0$$

Similarly

$$\frac{\partial Z_3(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}A_3 t_1^2 - B_3 t_1 + C_3 - \frac{C_b T^2}{2}\right\} = 0 \qquad (49)$$

Substituting T from equation (48) in equation (49), we obtain

$$A_3(A_3 - C_b)t_1^2 - 2B_3(A_3 - C_b)t_1 - (2C_b C_3 - B_3^2) = 0 \qquad (50)$$

Let $\Delta_{31} = A_3(A_3 - C_b)t_d^2 - 2B_3(A_3 - C_b)t_d - (2C_b C_3 - B_3^2)$ and $\Delta_{32} = A_3(A_3 - C_b)M^2 - 2B_3(A_3 - C_b)M - (2C_b C_3 - B_3^2)$

Lemma 3. For $M > t_1$, we have

(i) If $\Delta_{31} \le 0 \le \Delta_{32}$, then the solution of $t_1 \in [t_d, M]$ (say t_{13}^*) which satisfies equation (50) not only exists but also is unique.

(ii) If $\Delta_{32} < 0$, then the solution of $t_1 \in [t_d, M]$ which satisfies equation (50) does not exist.

Proof of (i). From equation (50), we define a new function $F_3(t_1)$ as follows

$$F_3(t_1) = A_3(A_3 - C_b)t_1^2 - 2B_3(A_3 - C_b)t_1 - (2C_b C_3 - B_3^2), t_1 \in [t_d, M]. \qquad (51)$$

Taking the first-order derivative of $F_3(t_1)$ with respect to $t_1 \in [t_d, M]$, we have

$$\frac{F_3(t_1)}{dt_1} = 2(A_3 - C_b)(A_3 t_1 - B_3) > 0$$

Because $(A_3 t_1 - B_3) > 0$ and $(A_3 - C_b) = [h(t_d\theta + 1) + sI_e + C\theta] > 0$

Hence we obtain that $F_3(t_1)$ is increasing of t_1 in the interval $[t_d, M]$. Moreover, we have $F_3(t_d) \leq 0$ and $F_3(M) \geq 0$. That is $F_3(t_d) \leq 0 \leq F_3(M)$. Thus, we can find a unique value t_1 say $t_{13}^* \in [t_d, M]$ such that $F_3(t_{13}^*) = 0$. Hence t_{13}^* is the unique solution of equation (50). Thus, the value of t_1 (denoted by t_{13}^*) can be found from equation (50) is given by

$$t_{13}^* = \frac{B_3}{A_3} + \frac{1}{A_3}\sqrt{\frac{(2A_3C_3 - B_3^2)C_b}{(A_3 - C_b)}} \tag{52}$$

Once we obtain t_{13}^*, then the value of T (denoted by T_3^*) can be found from equation (48) and is given by

$$T_3^* = \frac{1}{C_b}(A_3 t_{13}^* - B_3) \tag{53}$$

Equations (52) and (53) give the optimal values of t_{13}^* and T_3^* for the cost function in equation (28) only if B_3 satisfies the inequality given in equation (54)

$$B_3^2 < 2A_3C_3 \tag{54}$$

Proof of (ii). If $\Delta_{32} < 0$, $F_3(M) < 0$. Since $F_3(t_1)$ is increasing function of t_1 in the interval $[t_d, M]$ and $M > t_1$ we can get $F_3(t_1) < 0$ for all $t_1 \in [t_d,\ M]$. This implies that we cannot find a value of $t_1 \in [t_d, M]$ such that $F_3(t_1) = 0$. This completes the proof.

Theorem 3. When $M > t_1$, we have:

(i) If $\Delta_{31} \leq 0 \leq \Delta_{32}$, then the total variable cost $Z_3(t_1, T)$ is convex and reaches its global minimum at the point (t_{13}^*, T_3^*), where (t_{13}^*, T_3^*) is the point which satisfies equations (47) and (50).

(ii) If $\Delta_{32} < 0$, then the total variable cost $Z_3(t_1, T)$ has a minimum value at the point $(t_{13}^*,\ T_3^*)$ where $t_{13}^* = M$ and $T_3^* = \frac{1}{C_b}(A_3 M - B_3)$

(iii) If $\Delta_{31} > 0$, then the total variable cost $Z_3(t_1, T)$ has a minimum value at the point $(t_{13}^*,\ T_3^*)$ where $t_{13}^* = t_d$ and $T_3^* = \frac{1}{C_b}(A_3 t_d - B_3)$

Proof of (i). When $\Delta_{32} \leq 0 \leq \Delta_{32}$, we see that t_{13}^* and T_3^* are the unique solutions of equations (50) and (47) from Lemma 3(i). Taking the second derivative of $Z_3(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point $(t_{13}^*,\ T_3^*)$, we obtain

$$\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)} = \frac{\lambda}{T_3^*} A_3 > 0$$

$$\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)} = -\frac{\lambda}{T_3^*} C_b$$

$$\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)} = \frac{\lambda}{T_3^*} C_b > 0$$

and

$$\left(\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)}\right)\left(\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right) - \left(\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)}\right)^2 = \frac{\lambda^2 C_b}{T_3^{*2}}(A_3 - C_b)$$

$$= \frac{\lambda^2 C_b}{T_3^{*2}}\left[h(t_d\theta + 1) + sI_e + C\theta\right] > 0 \quad (55)$$

We thus conclude from equation (55) and Lemma 3 that $Z_3(t_{13}^*,\ T_3^*)$ is convex and $(t_{13}^*,\ T_3^*)$ is the global minimum point of $Z_3(t_1,\ T)$. Hence the values of t_1 and T in equations (52) and (53) are optimal.

Proof of (ii). When $\Delta_{32} < 0$, then we know that $F_3(M) < 0$. Since $F_3(t_1)$ is an increasing function of t_1 in the interval $[t_d, M]$, we can get $F_3(t_1) < 0$ for all $t_1 \in [t_d, M]$. This implies that $\frac{\partial Z_3(t_1,\ T)}{\partial T} = \frac{F_3(t_1)}{T^2}$, for all $t_1 \in [t_d, M]$. So, $Z_3(t_1,\ T)$ is a decreasing function of T in the interval $[t_d,\ M]$. Thus $Z_3(t_1,\ T)$ has a minimum value at $(t_{13}^*,\ T_3^*)$ where $t_{13}^* = M$ and the corresponding minimum value of T_3^* is $T_3^* = \frac{1}{C_b}(A_3 M - B_3)$.

Proof of (iii). When $\Delta_{31} > 0$, $F_3(t_d) > 0$, then we can get $F_3(t_1) > 0$ for all $t_1 \in [t_d, M]$, which implies $\frac{\partial Z_3(t_1,\ T)}{\partial T} = \frac{F_3(t_1)}{T^2} > 0$ for all $t_1 \in [t_d, M]$. So, $Z_3(t_1,\ T)$ is an increasing function of T in the interval $[t_d, M]$. Thus $Z_3(t_1,\ T)$ has a minimum value at $(t_{13}^*,\ T_3^*)$ where $t_{13}^* = t_d$ and the corresponding minimum value of T_3^* is $T_3^* = \frac{1}{C_b}(A_3 t_d - B_3)$.

Thus, the EOQ corresponding to the optimal cycle length T^* will be computed as follows:
EOQ^* =Total demand before deterioration set in+total demand after deterioration set in+total number of deteriorated items + total number of items backordered

$$= \int_0^{t_d}(\alpha + \beta t + \gamma t^2)dt + \int_{t_d}^{t_1^*}\lambda dt + \left[\frac{\lambda}{\theta}\left(e^{\theta(t_1^* - t_d)} - 1\right) - \lambda(t_1^* - t_d)\right] \quad + \lambda(T^* - t_1^*)$$

$$= \alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3} + \frac{\lambda}{\theta}\left(e^{\theta(t_1^* - t_d)} - 1\right) + \lambda(T^* - t_1^*) \quad (56)$$

4. Numerical Examples

This section will provide some numerical examples to illustrate the application of the proposed model by considering the following numerical examples.

Example 4.1 for (Case 1)

Consider an inventory system with the following input parameters: $A = \$300$/order, $C =$ \$50/unit/year, $S =$ \$60/unit/year, $h =$ \$10/unit/year, $C_b =$ \$30/unit/year, $\theta = 0.01$ units/year, $\alpha = 1000$ units, $\beta = 200$ units, $\gamma = 20$ units, $\lambda = 500$ units, $t_d = 0.2026$ year (74 days), $M = 0.0548$ year (20 days), $I_c = 0.12$, $I_e = 0.08$. Here we find that $M \leq t_d$. We first check the conditions $\Delta_1 = -24.6490 < 0$ and $B_1^2 = 0.1901$, $2A_1C_1 = 82.9575$ hence $B_1^2 < 2A_1C_1$. Substituting the above values in equations $(44), (45), (35)$ and (68), we obtain as follows the values of the optimal length of time with positive inventory $t_{11}^* = 0.2728$ year (100 days), the optimal cycle length $T_1^* = 0.4085$ year (149 days), the optimal average total cost $Z_1(T_1^*,\ t_{11}^*) =$ \$2036.4518 per year, and the economic order quantity $EOQ_1^* = 309.7469$ units per year respectively.

Example 4.2 for (Case 2)

The data are same as in Example 4.1 except that $M = 0.2333$ year (85 days). Here we find that $M > t_d$.We first check the conditions $\Delta_2 = -12.8615 < 0$and $B_2^2 = 2.2656$, $2A_2C_2 = 70.3302$ hence $B_2^2 < 2A_2C_2$. Substituting the above values in equations $(54), (55), (36)$ and (68), we obtain as follows the values of the optimal length of time with positive inventory $t_{12}^* = 0.2713$ year (99 days), the optimal cycle length $T_2^* = 0.3706$ year (135 days), the optimal average total cost $Z_2(T_2^*,\ t_{12}^*) = \$1488.7090$ per year, and the economic order quantity $EOQ_2^* = 290.7660$ units per year respectively.

Example 4.3 for (Case 3)

The data are same as in Example 4.1 except that $t_d = 0.1545$ year (56 days) $M = 0.2608$ year (95 days). Here we find that $M > t_d$.We first check the conditions that $\Delta_{31} = -9.6204 < 0$ and $\Delta_{32} = 22.3451 > 0$, hence $\Delta_{31} \le 0 \le \Delta_{32}$ and $B_3^2 = 0.1660$, $2A_3C_3 = 42.7192$ hence $B_3^2 < 2A_3C_3$. Substituting the above values in equations $(64), (65), (37)$ and (68), we obtain as follows the values of the optimal length of time with positive inventory $t_{13}^* = 0.1925$ year (70 days), the optimal cycle length $T_3^* = 0.3043$ year (111 days), the optimal average total cost $Z_3(T_3^*,\ t_{13}^*) = \$1677.6924$ per year, and the economic order quantity $EOQ_3^* = 231.8288$ units per year respectively.

5. Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-30\%, -20\%, -10\%$, $+10\%, +20\%$ to $+30\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these system parameter values on optimal length of time with positive inventory, cycle length, total variable cost and the order quantity per cycle are discussed.

Table 1: Percentage change in the decision variables with respect to the percentage change in parameters from $-30\%, -20\%, -10\%$, $+10\%, +20\%$ to $+30\%$ for examples 4.1, 4.2 & 4.3.

Parameter	% change in	Change in decision Variables from −30% to +30%											
		t^*_{11}	t^*_{12}	t^*_{13}	T^*_1	T^*_2	T^*3	EOQ_1	EOQ_2	EOQ_3	$Z_1(t^*_{11}, T^*_1)$	$Z_2(t^*_{12}, T^*_2)$	$Z_3(t^*_{13}, T^*_3)$
θ	−30	0.228	0.219	0.189	0.144	0.151	0.155	0.094	0.095	0.075	−0.024	−0.034	−0.011
	−20	0.151	0.146	0.125	0.096	0.101	0.077	0.062	0.063	0.050	−0.016	−0.022	−0.007
	−10	0.076	0.073	0.062	0.048	0.050	0.038	0.031	0.032	0.025	−0.008	−0.011	−0.004
	+10	−0.075	−0.072	−0.062	−0.048	−0.050	−0.038	−0.031	−0.031	−0.025	0.008	0.011	0.004
	+20	−0.150	−0.144	−0.124	−0.095	−0.099	−0.076	−0.062	−0.063	−0.049	0.016	0.022	0.007
	+30	−0.224	−0.215	−0.185	−0.142	−0.149	−0.113	−0.092	−0.094	−0.074	0.023	0.033	0.011
C	−30	6.047	1.867	0.183	2.710	1.322	0.112	1.790	0.843	0.073	−3.994	−0.167	−0.011
	−20	3.892	1.194	0.122	1.725	0.846	0.074	1.139	0.539	0.049	−2.628	−0.107	−0.007
	−10	1.882	0.574	0.061	0.825	0.406	0.037	0.545	0.259	0.024	−1.298	−0.052	−0.004
	+10	−1.765	−0.533	−0.060	−0.757	−0.377	−0.037	−0.500	−0.240	−0.024	1.268	0.048	0.004
	+20	−3.425	−1.028	−0.120	−1.453	−0.728	−0.073	−0.959	−0.464	−0.048	2.508	0.094	0.007
	+30	−4.988	−1.491	−0.180	−2.094	−1.055	−0.110	−1.382	−0.673	−0.072	3.722	0.136	0.011
S	−30	0.236	4.138	17.678	0.245	4.699	11.716	0.161	2.996	7.694	0.261	6.231	1.456
	−20	0.158	2.780	11.603	0.163	3.156	7.759	0.108	2.012	5.095	0.174	4.185	1.144
	−10	0.079	1.401	5.717	0.082	1.590	3.858	0.054	1.014	2.533	0.087	2.109	0.660
	+10	−0.079	−1.423	−5.567	−0.082	−1.616	−3.829	−0.054	−1.030	−2.514	−0.087	−2.143	−0.839
	+20	−0.158	−2.871	−11.002	−0.163	−3.259	−7.642	−0.108	−2.078	−5.017	−0.175	−4.322	−1.861
	+30	−0.237	−4.343	−16.323	−0.245	−4.931	−11.453	−0.162	−3.143	−7.519	−0.262	−6.538	−3.073
I_c	−30	5.756	1.617	0.000	2.527	1.149	0.000	1.669	0.733	0.000	−3.961	−0.131	0.000
	−20	3.718	1.038	0.000	1.615	0.737	0.000	1.066	0.470	0.000	−2.609	−0.084	0.000
	−10	1.803	0.500	0.000	0.775	0.355	0.000	0.512	0.227	0.000	−1.290	−0.041	0.000
	+10	−1.701	−0.467	0.000	−0.716	−0.332	0.000	−0.473	−0.21	0.000	1.261	0.038	0.000
	+20	−3.308	−0.904	0.000	−1.379	−0.642	0.000	−0.911	−0.409	0.000	2.497	0.074	0.000
	+30	−4.831	−1.313	0.000	−1.994	−0.933	0.000	−1.316	−0.595	0.000	3.707	0.107	0.000
I_e	−30	0.236	4.138	17.678	0.245	4.699	11.716	0.161	2.996	7.694	0.261	6.231	1.456
	−20	0.158	2.780	11.603	0.163	3.156	7.759	0.108	2.012	5.095	0.174	4.185	1.144
	−10	0.079	1.401	5.717	0.082	1.590	3.858	0.054	1.014	2.533	0.087	2.109	0.660
	+10	−0.079	−1.423	−5.567	−0.082	−1.616	−3.829	−0.054	−1.030	−2.514	−0.087	−2.143	−0.839
	+20	−0.158	−2.871	−11.002	−0.163	−3.259	−7.642	−0.108	−2.078	−5.017	−0.175	−4.322	−1.861
	+30	−0.237	−4.343	−16.323	−0.245	−4.931	−11.453	−0.162	−3.143	−7.519	−0.262	−6.538	−3.073
C_b	−30	−5.806	−3.491	−8.050	7.313	6.911	6.946	4.821	4.403	4.558	−6.428	−5.256	−7.073
	−20	−3.529	−2.110	−4.903	4.328	4.087	4.108	2.853	2.604	2.696	−3.907	−3.176	−4.308
	−10	−1.622	−0.965	−2.257	1.946	1.837	1.846	1.283	1.170	1.211	−1.796	−1.453	−1.983
	+10	1.397	0.825	1.949	−1.621	−1.528	−1.536	−1.069	−0.974	−1.008	1.546	1.242	1.712
	+20	2.613	1.538	3.649	−2.993	−2.820	−2.835	−1.973	−1.797	−1.860	2.893	2.316	3.206
	+30	3.681	2.162	5.147	−4.169	−3.927	−3.948	−2.748	−2.502	−2.590	4.076	3.255	4.522

6. Results and Discussion

(i) Based on the computational results shown in Table 1, the following managerial insights are obtained. As the rate of deterioration (θ) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and economic order quantity (EOQ^*) decrease while total variable cost ($Z(T^*, t_1^*)$) increase. Hence the retailer will order less quantity to avoid the items being deteriorating when the deterioration rate increases.

(ii) As the unit purchasing cost (C) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), and the economic order quantity (EOQ^*) decrease while the total variable cost ($Z(T^*, t_1^*)$) increase. In real market situation the higher the cost of an item, the higher the total variable cost. This result implies that the retailer will order a smaller quantity to enjoy the benefits of permissible delay in payments more frequently in the presence of an increased unit purchasing price and consequently shortening optimal time with positive inventory and cycle length.

(iii) As the unit selling price (S) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), the economic order quantity (EOQ^*) and the total variable cost $(Z(T^*, t_1^*))$ decrease. In real market situation the higher the selling price of an item, the lower the demand of that item and vice versa. This means that when the unit selling price is increasing, the retailer will order less quantity to take the benefits of the trade credit more frequently.

(iv) As the interest payable (I_c) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and the economic order quantity (EOQ^*) decrease while the total variable cost $(Z(T^*, t_1^*))$ increase when interest payable is high for both case 1&2. This means that when interest payable is high the retailer should order less amount of items. As for case 3,the increase/decrease in interest payable(I_c) do not affect the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $(Z(T^*, t_1^*))$. This is because the interest payable in this case is zero.

(v) As the interest earn (I_e) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $(Z(T^*, t_1^*))$ decrease. This implies that when the interest earned is high, the optimal time with positive inventory (t_1^*), cycle length, the economic order quantity and the total variable cost are low. Hence the retailer should order less items so as to effectively take the benefit of trade credit more frequently.

(vi) An increase in the shortage cost will lead to an increase in the optimal time with positive inventory (t_1^*) and total variable cost $(Z(T^*, t_1^*))$, decrease in cycle length (T^*), and the economic order quantity (EOQ^*).This means that when the shortages cost increase, the number of backordered items reduce drastically which in turn lead to the decrease of order quantity.

7. Conclusion

In this paper, we develop an economic order quantity model for non-instantaneous deteriorating items with time dependent quadratic demand function of time and complete backlogging under trade credit policy. The optimal time with positive inventory, cycle length and economic order quantity that minimise total variable cost are determined. Some numerical examples are presented to illustrate the application of the model. Sensitivity analysis is also carried out to show the effect of changes in system parameters on decision variables. The results show that the retailer can reduce total variable cost by ordering less to shorten the time with positive inventory and cycle length when deterioration sets in, unit purchasing price increases, unit selling price increases, interest charges increases, shortage cost increases and interest earn decreases respectively.

The model developed in this paper is an extension of Babangida and Baraya (2018) by allowing shortages which are completely backlogged. The proposed model can be extended to allow for partial backlogging, variable deterioration rate, quantity discounts, inflation rates, finite time horizon and so on.

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