# SHORT HISTORY ON DETERMINANT OF A MATRIX

**By**

**Babarinsa Olayiwola**

Department of Mathematical Sciences, Federal University Lokoja, P.M.B 1154, Kogi, Nigeria.

olayiwola.babarinsa@fulokoja.edu.ng

**Abstract:**

*Determinant was discovered over two centuries before the term "matrix" was coined. The determinant has played important role in matrix theory because it provides information about the matrix.The history of determinant of a matrix is discussed from the sixteenth century to nineteenth century andthe review gives the core contributors of determinants in chronological order. Then a modified form of condensation methodis suggested to compute the determinant of hourglass matrix.*

***Keyword:*** *Matrix, Determinant, History*

## 1. INTRODUCTION

To begin with the history of determinant, one cannot underemphasize what brought determinant into existence, the matrix. Although the word matrix has not been coined for over four millennia, yet the history of matrices goes back to ancient times. Matrices can be classified to be square or not square and sometimes special. Special square matriceswith zero and nonzero entries have many applications and properties, but one of theiruseful properties is the existence of determinant. Many related methods highlight the determinant of some (special) square matrices with zero and nonzero entries, see (Babarinsa & Kamarulhaili, 2018; Garnett, Olesky, Shader, & Van Den Driessche, 2014; Golub & Van Loan, 1996) and the reference therein. Although, determinant of all square matrices can be computed from general determinant method such as Laplace expansion, every special square matrix has a special method of computing its determinant (Turnbull, 1960).

In Section 2, the history of determinant will be discussed in chronological order including their contributors.While in Section 3, suggestion is raised on how to use a modified form of Dodgson condensation to compute determinant of hourglass matrix.

## 2. THE HISTORY OF DETERMINANTS

Determinants (resultants) were discovered over two centuries before matrices were coined. Matrices and determinant are the backbone of linear algebra (Bernstein, 2009). Determinant provides information about a matrix: Geometrically, it provides absolute value of area and volume in $n$-dimensional space, preserving transformationand can be used to produce equations of lines, planes and other curves; and algebraically, determinant **determines** whether the system of $n$-linear equations in $n$-unknowns has a unique solution and a well indicator whether a square matrix has an inversethe matrix represents the coefficients of a system of $n$-linear equations in $n$-unknowns(Karim, 2013; Rice & Torrence, 2006). The discussion of the history of determinants will be from 16$^{th}$ century to 19$^{th}$ century including their contributors.

### 2.1 Determinant in 16$^{th}$ Century

Based on history, the theory of determinant started by an Italian mathematician GerolamoCardano in 16$^{th}$ century. He gave a rule for solving only a system of two linear equations which he termed*regula de modo*-mother of rules(Cardano, 1993). The rule later gave what we called Cramer's rule.His determinants were practically for $2 \times 2$ determinants and larger ones were discussed by Leibniz (Cardano, 1993; Eves, 1969).

### 2.2 Determinant in 17$^{th}$ Century

In 1683, theidea of determinant appeared in Japan through Seki Takakasu. He published his findings but without having word which corresponds to determinant(Martzloff, 2008). His concept of determinant gave the general methods for calculating $2 \times 2$determinant through a process he called *tatamn (folding)*. On the other hand, a German mathematician, Leibniz Gottfriedworked independently on determinant. Leibniz used the word "*resultant*" for certain combinatorial sums of terms of a determinant(Muir, 1906). Though Leibniz and Seki did not publish the findings, they are both aware that determinant could be expanded using any column(Debnath, 2013).

## 2.3 Determinant in 18$^{th}$ Century

Over a half century, improvement on resultant (determinant) was out of site to mathematicians until a Scottish mathematician, Colin Maclaurin in 1748 gave the first published results on resultants on solving two, three and four simultaneous equations. Though, the publication of his findings was made two years after his death (Boyer, 1966; MacLaurin, 1748).

In 1750,a Swiss mathematician called Gabriel Cramer hinted that resultants might be useful in analytical geometry. His rule later became what we called Cramer' s rule(Cramer, 1750; Robinson, 1970).Though the Cramer's rule has many disadvantages as it fails when the determinant of the coefficient matrix is zero, requires many calculations of determinants and also numerically unstable, see(Brunetti & Renato, 2014; Debnath, 2013; Habgood & Arel, 2012; Higham, 2002).

In 1771, a French mathematician Alexandre-ThéophileVandermondemade known of the widely general theorem described as the theorem for expressing a determinant as an aggregate of products of complementary minors. His method can evaluate determinant of order $n$(Hadamard, 1897; Vandermonde, 1772). Thus, the only one fit to be viewed as the founder of theory of determinants is Vandermonde. He was the first to recognize determinants as independent functions(Campbell, 1980).

In 1772, a French mathematician Pierre-Simon marquis de Laplace gave a notation for a resultant or determinant(De Laplace, 1772). He gave a rule for expressing a resultant as an aggregate of terms composed of factors (minors) which are themselves resultants and made a mode of finding the number of terms in this aggregate. His theorem may be described as giving an expression of a resultant in the form of an aggregate of terms each of which is a product of lower degree (Brualdi & Schneider, 1983). Hegave the general method of expanding a determinant in terms of its complementary minors. The minor $M_{ij}$ is defined to be the determinant of the $(n-1) \times (n-1)$matrix that results from the matrix by removing the $i$th row and the $j$th column. The expression $(-1)^{i+j}M_{ij}$ is known as cofactor (Lancaster & Tismenetsky, 1985; Salomon, 1998). The determinant ofmatrix $A$ is given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij,}$$

where

$$(-1)^{i+j} = \begin{cases} + & \text{when } i = j \text{ or } i+j \text{ is even} \\ - & \text{when } i \neq j \text{ or } i+j \text{ is odd} \end{cases} \quad \text{for } i, j \in \mathbb{Z}^{+}.$$

In addition, Laplace expansion is a building block for other methods of determinant. Laplace expansion is the best for computing determinant as it works for all forms of square matrices except it has high complexity time(Cormen, 2009).When determinants are defined in terms of permutations then this theorem can be obtained by a suitable rearrangement of summands(Janjia, 2005).

In 1773, an Italian mathematician Joseph Louis Lagrange for the first time gave volume interpretation of a determinant(Lagrange, 1775). While other contributors focus on problem of elimination, Langrage workconsists of several incidentally obtained algebraic identities. He further gave a theorem that a minor of determinant adjugated to another determinant (Weld, 1893).

In 1784, a German mathematician Carl Friedrich Hindenburg worked on Cramer and Bezout point of view. He wrote his permutation, calculating determinant, in a definite order. He regards the sequence of signs by successfully combine rule of term formation and rule of signs(Muir, 1881) .

### 2.4 Determinant in 19th Century

In 1801, the term "determinant " was first introduced by a German mathematician Johann Carl Friedrich Gausswhile discussing quadratic forms(Gauss, 1966). He used the term because the determinant determines the properties of quadratic forms. In fact, the new term introduced by Gauss was not 'determinant" but "determinant of a form"(Knobloch, 1994). Nowadays, determinant of a form is referring to discriminant of a quantic. Gauss also arrived at the notion of reciprocal or inverse determinants(Kani, 2011).

In 1812, a French mathematician Augustin-Louis Cauchy used "determinant" in its modern sense as was the most complete work on determinant(Cauchy, 1812). He gave new results of his own minors and adjoints, and multiplication theorem for determinants.

In 1826, Cauchy in the context of quadratic forms in $n$ variables, used the term "tableau" for the matrix of coefficients. His approach led to eigenvalues and eigenvectors that provided new method of dealing with $n$ variable quadratic expressions(Knobloch, 1994). He introduced the $2 \times 2$ determinant involving partial derivatives – known today as Jacobian determinant**.**

In 1866,an English writer Charles Lutwidge Dodgson gave A method of computing the determinant of a square matrix by condensation. The method proves to be effective as well as minimize error before arriving at the solution (D. Leggett, Perry, & Torrence, 2009). Dodgson condensation reduces the matrix into $2 \times 2$ submatrices for easy computation of determinant. The method reduces the risk of miscalculation as it is bound to divide the determinant of the submatrices by interior elements. The fatal of Dodgson condensation defect is that the determinant of interior matrix must not be zero because dividing the determinant of the minors by zero makes the solution indeterminate. The advantage of Dodgson condensation is that the determinant of a square matrix is a rational function of all its connected minors of any two consecutive sizes (Schmidt & Greene, 2011).

## 3. DODGSON CONDENSATION AND ITS POSSIBLE APPLICATION IN COMPUTING DETERMINANT OF HOURGLASS MATRIX

Let A be an $n \times n$ matrix. After k successful condensation, Dodgson produces the matrix

$$A^{(n-k)} = \begin{pmatrix} |A_{1\dots k+1,1\dots k+1}| & |A_{1\dots k+1,2\dots k+2}| & \cdots & |A_{1\dots k+1,n-k\dots n}| \\ |A_{2\dots k+2,1\dots k+1}| & |A_{2\dots k+2,2\dots k+2}| & \cdots & |A_{2\dots k+2,n-k\dots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-k\dots n,1\dots k+1}| & |A_{n-k\dots n,2\dots k+2}| & \cdots & |A_{n-k\dots n,n-k\dots n}| \end{pmatrix}$$

Whose entries are the determinants of all $(k+1) \times (k+1)$ contiguous submatrices of A. For an $n \times n$matrix $A$, let $A_r(i,j)$ denote the $r$ by $r$ minor consisting of $r$ contiguous rows and columns of $A$, beginning with row $i$, column $j$. note that $A_{n-2}(2,2)$ is the central minor or interior elements; $A_{n-1}(1,1)$, $A_{n-1}(2,2)$, $A_{n-1}(1,2)$ and $A_{n-1}(2,1)$ are the northwest, southwest, southeast, northeast, and southwest minors, respectively (Abeles, 2014; Amdeberhan, 2001).Then,

$$detA = detA_n(1,1) = \frac{detA_{n-1}(1,1)\, detA_{n-1}(2,2) - detA_{n-1}(1,2)\, detA_{n-1}(2,1)}{detA_{n-2}(2,2)}$$

Whenever other methods (except Laplace expansion and Leibniz method) are used there bound to be a division from one of the entries. With this (division), the methods may have shortcomings. Although, the division seems to identify if there is any mistake during the computation, yet it increases the number of arithmetic operations used. That is, all methods of computing determinants that require division of specific entry tend to have four arithmetic operations (Plus, minus, multiplication and division). If the entry is nonzero then round off error may become significant and if the entry is zero, then swapping takes place - which row or column to swap? Swapping may as well introduce other zero and it may not always work(D. R. Leggett, 2011). In such case, those methods will be impracticable for matrix with zeros in the entries even if the square matrix has a pattern. The shortcoming of Dodgson condensation is that it tends to make use of all elements in the computation. More so, Dodgson condensation has not been extended to special square matrices with zero and nonzero entries. Thus, this review will give readers the insight to formulate a method to compute determinant of hourglass matrix.The possibility to compute the determinant of hourglass matrix may result from the disadvantages of Dodgson condensation. That is, the method to compute determinant of hourglass matrix should avoid the use of some entries as well as division by interior matrixin the computation. Thus, any proposed method to compute the determinant of hourglass matrix will not only be easier and faster in computation than Dodgson condensation but also have better complexity time. For instance, let us consider hourglass matrix of order 4 given below

$$B=\begin{pmatrix}5&3&4&7\\0&9&3&0\\0&2&8&0\\2&1&5&6\end{pmatrix}$$

Using Dodgson condensation method, we compute the determinant of $B$ as follows

$$\det(B)=\begin{pmatrix}\begin{vmatrix}5&3\\0&9\end{vmatrix}&\begin{vmatrix}3&4\\9&3\end{vmatrix}&\begin{vmatrix}4&7\\3&0\end{vmatrix}\\\begin{vmatrix}0&9\\0&2\end{vmatrix}&\begin{vmatrix}9&3\\2&8\end{vmatrix}&\begin{vmatrix}3&0\\8&0\end{vmatrix}\\\begin{vmatrix}0&2\\2&1\end{vmatrix}&\begin{vmatrix}2&8\\1&5\end{vmatrix}&\begin{vmatrix}8&0\\5&6\end{vmatrix}\end{pmatrix}=\begin{pmatrix}45&-27&-21\\0&66&0\\-4&2&48\end{pmatrix}$$

$$=\begin{pmatrix}\begin{vmatrix}45&-27\\0&66\end{vmatrix}&\begin{vmatrix}-27&-21\\66&0\end{vmatrix}\\\begin{vmatrix}0&66\\-4&2\end{vmatrix}&\begin{vmatrix}66&0\\2&48\end{vmatrix}\end{pmatrix}=\begin{vmatrix}2970/9&1386/3\\264/2&3168/8\end{vmatrix}=\begin{vmatrix}330&462\\132&396\end{vmatrix}=\frac{69696}{66}=1056.$$

Without using all the entries of $B$, we can compute its determinant as

$$\det(B)=\begin{vmatrix}5&7\\2&6\end{vmatrix}\times\begin{vmatrix}9&3\\2&8\end{vmatrix}=1056.$$

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