OPTIMAL ASSET ALLOCATION POLICY FOR A DEFINED CONTRIBUTION PENSION FUND WITH REFUND CLAUSE OF PREMIUM WITH PREDETERMINED INTEREST UNDER HESTON'S VOLATILITY MODEL

by

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Abstract

We study optimal asset allocation policy for a Defined Contribution (DC) pension fund with refund of premium clauses under Heston's volatility model using mean-variance utility function. In this model, death members' next of kin are allowed to withdraw their member's accumulations with predetermined interest. Next, we considered investments in one risk free asset (cash) and a risky asset (equity) to help increase the accumulated funds of the remaining members to meet their retirement needs. Also, the actuarial symbol is used to formulize the problem as a continuous time mean-variance stochastic optimal control problem. We establish an optimization problem from the extended Hamilton Jacobi Bellman equations using the game theoretic approach and solve the optimization problem to obtain the optimal allocation strategy for the two assets and also the efficient frontier of the pension members. Furthermore, we analysenumerically the effect of some parameters on the optimal allocation strategy. We deduce that as the initial wealth, predetermined interest rate, voluntary contribution and risk averse level increases, the optimal allocation policy for the risky asset (equity) decreases.

Keywords: *DC pension fund,extended HJB equation, optimal allocation policy, refund of contribution clause, interestrate.*

1. Introduction

In general, a man's life cycle is made up of two crucial phase; first, the person's working years and his retirement years. Based on this, Individuals need to create a definite consumption plan for their needs from when income is earned (working life) to when there might be no other funds available, except for a possible survival level of support from the members' accumulated contribution during working life or members employers contributions (retirement life). The funds given to a retiree monthly from the accumulated contributions after their working years is referred to as pension. The important of pension scheme cannot be over emphasized in the life of retirees.

Already in existence, are two types of pension plan whereby members can take advantage of; this are the Defined Benefit (DB) pension plan and the Defined Contribution pension plan (DC). The defined benefit pension plan is a type in which members benefits are determined in advance following some basic requirements which include age, years in service, members' salary histories etc. These benefits depend basically on the contributions made by the employers and because of the mode of contributions, most private organizations found it difficult to develop a pension plan for their members as a result, this plan was limited to members in government organizations. Although most members are pleased with this plan since only the employers contribute, it has over the years generate controversies and delay in implementation after retirement and these has led to the introduction of the alternative plan known as the Defined Contribution (DC) pension plan which is mostly members dependent. It also requires that members contribute a certain proportion of their income into the members' Retirement Serving

Account (RSA). The DC pension plan is much more attractive and reliable than the DB pension plan since members are fully involved in the contribution and investment process and depend mostly on the returns of the investment during the accumulation period and this expected return is influenced by some factors such as investment efficiency, inflation, mortality risk, etc. Although the DC pension plan seems attractive, it requires investment knowledge in different assets available in the financial market. These assets include the cash, bond and stock etc.Since investment in stock involve risk, there is need to study the best possible way to invest for optimal returns. This lead to the study of optimal allocation policy by financial institutions and this explains the proportion of the members' wealth to be invested in various assets available in the financial market for optimal returns with less risk.

In Cairns et al, (2006) optimal investment strategy to DC members with asset, salary and interest rate risk was studied, and proposed a novel form of terminal utility function by incorporating habit formulation. Giacinto (2011),proposed and investigated a model of optimal allocation for DC pension plan with a minimum guarantee in the continuous-time setting. In Gao (2008), asset allocation problem under a stochastic interest rate was studied, Boulier (2001),worked on optimal investment strategy for a DC pension with stochastic interest rate. and Battocchio and Menoncin (2004), investigated a case where the interest rate was of Vasicek model, Chubing and Ximing (2013) studied Optimal investment strategies for DC pension with a stochastic salary under affine interest rate model which includes the Cox- Ingersoll- Ross (CIR) model and Vasicek model. lately, the study of constant elasticity of Variance (CEV) model in DC pension fund investment strategies have taken centre stage in modelling the stock price. Xiao and Hong (2007) studied the constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts. Gao (2009), obtained explicit solutions of the optimal investment strategy for investor with CRRA and CARA utility function by extending the work of Xiao and Hong (2007). Osu, et al (2017) studied optimal investment strategies in DC pension fund with multiple contributions using Legendre transformation method to obtain the explicit solution for CRRA and CARA. Akpanibah and Samaila (2017), studied stochastic strategies of optimal investment for DC pension fund with multiple contributors where they considered the rate of contribution to be stochastic. Osu, et al (2018), studied optimization problem with return of premium in a DC pension with multiple contributors

In recent years, some authors studied the optimal investment strategy with refund of contributions clause some of which include He and Liang (2013), who investigated optimal investment strategy for a defined contribution pension scheme with the return of premiums clauses in a mean-variance utility function. Li (2017), investigated equilibrium investment strategy for DC pension plan with default risk andreturn of premiums clauses under constant elasticity of variance model. Sheng and Rong (2014), investigated the optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts.

Throughout the literatures, there has not been any work done on optimal allocation policy with refund of contributions clause that considers the refund contributions with predetermined interest. This form the basis of our discussion in this paper where we study optimal asset allocation strategy in a DC pension scheme with refund of contributions clauses under Heston volatility model, we assume the refund contribution is with predetermined interest and the price of the equity satisfies Heston's volatility model and voluntary contribution is stochastic.

2. Mathematical formulation

In this section, we consider a financial market which is complete and frictionless and is continuously open over a fixed period of time interval $0 \leq t \leq T$, where T is the retirement age of any given plan member.

Assume the market is made of a risk-free asset (cash) and a risky asset (equity) and suppose $(\Omega, \mathcal{F}, \mathbb{p})$ is a complete probability space such that Ω is a real space and $\mathbb{p}$ a probability measure satisfying the condition $0 \leq t \leq T$.$\{W_s(t), W_l(t) : \mathrm{t} \geq 0\}$are two standard Brownian motions. $\mathcal{F}$is the filtration and denotes the information generated by the two Brownian motions

Let $A_t(t)$ denote the price of the risk free asset and $B_t(t)$the price of the risky asset which satisfies the Heston's stochastic volatility model. Their price models are described as follows:

$$\frac{dA_t(t)}{A_t(t)} = rdt,\ A_0(0) = 1 \tag{2.1}$$

$$\frac{dB_t(t)}{B_t(t)} = \big(r + \delta L(t)\big)dt + \sqrt{L(t)}dW_s B_0(0) = b_0 (2.2)$$

$$\frac{dL_t(t)}{L_t(t)} = k\left(\frac{\vartheta}{L_t} - 1\right)dt + \sigma\frac{\sqrt{L_t}}{L_t}dW_l,\ L_0(0) = l_0$$

Where r is the predetermined interest rate of the risk free asset and $k, \vartheta, \delta, \sigma$ are positive constants and the two Brownian motions $W_s(t), W_l(t)$ are such that $\mathrm{E}[W_s(t), W_l(t)] = \rho$ where ρrepresent the correlation coefficient of the two Brownian motions and satisfies the condition $-1 \leq \rho \leq 1$.

Let the premium received at a given time be represented as b, letπ_0 represent the initial age of accumulation phase, T, the period of the accumulation phase such that $\pi_0 + T$ is the end age. The actuarial symbol $\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}$ is the mortality rate from time t to $t + \frac{1}{n}$, bt is the premium accumulated at time t, $tb\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}$ is the premium returned to the death members. Secondly, we assume that apart from the accumulated fund of the death member, a certain interest is paid as well from the investment in the risk-free asset since the interest rate is predetermined which is our main contribution in this paper and also we assume ξ to be a voluntary contribution which will help balance the system.

Let φ represent the proportion of the wealth invested in risky asset and $1 - \varphi$, is the proportioninvested in riskless asset.

Considering the time interval $[t, t + \frac{1}{n}]$, the differential form associated with the fund size is given as:

$$Z\left(t + \frac{1}{n}\right) = Z(t)\left(\varphi\frac{B_{t+\frac{1}{n}}}{B_t} + (1-\varphi)\frac{A_{t+\frac{1}{n}}}{A_t}\right) + b\left(\frac{1}{n}\right) + \xi dW_s - tb\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t} - Z(t)(1-\varphi)\frac{A_{t+\frac{1}{n}}}{A_t}\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}$$

$$Z\left(t + \frac{1}{n}\right) = Z(t)\left(\varphi\Big(\frac{B_{t+\frac{1}{n}}}{B_t} - \frac{B_t}{B_t} + \frac{B_t}{B_t}\Big) + (1-\varphi)\left(\frac{A_{t+\frac{1}{n}}}{A_t} - \frac{A_t}{A_t} + \frac{A_t}{A_t}\right)\right) + b\left(\frac{1}{n}\right) + \xi dW_s - tb\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}$$
$$- Z(t)(1-\varphi)\left(\frac{A_{t+\frac{1}{n}}}{A_t} - \frac{A_t}{A_t} + \frac{A_t}{A_t}\right)$$

$$Z\left(t+\frac{1}{n}\right)=Z(t)\left(\varphi+1-\varphi+\varphi(\frac{B_{t+\frac{1}{n}}}{B_t}-\frac{B_t}{B_t})+(1-\varphi)(\frac{A_{t+\frac{1}{n}}}{A_t}-\frac{A_t}{A_t})\right)+b\left(\frac{1}{n}\right)+\xi dW_s-tbM_{\left(\frac{1}{n}\right),\pi_0+t}$$
$$-Z(t)(1-\varphi)\left(\frac{A_{t+\frac{1}{n}}}{A_t}-\frac{A_t}{A_t}+1\right)\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}$$

$$Z\left(t+\frac{1}{n}\right)-Z(t)=Z(t)\left(\varphi(\frac{B_{t+\frac{1}{n}}-B_t}{B_t})+(1-\varphi)(\frac{A_{t+\frac{1}{n}}-A_t}{A_t})\right)+b\left(\frac{1}{n}\right)+\xi dW_s-tb\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}-Z(t)(1-\varphi)\left(\frac{A_{t+\frac{1}{n}}-A_t}{A_t}+1\right)\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t} \quad (2.3)$$

$$\left\{\begin{array}{c}\mathfrak{M}_{\left(\frac{1}{n}\right),\pi_0+t}=1-\exp\{-\int_0^{\frac{1}{n}}\alpha(\pi_0+t+s)ds\}=\alpha(\pi_0+t)\frac{1}{n}+O\left(\frac{1}{n}\right),\\ Z\left(t+\frac{1}{n}\right)-Z(t)=dZ(t)\\ n\to\infty,\delta_{\left(\frac{1}{n}\right),\pi_0+t}=\alpha(\pi_0+t)dt,\\ b\left(\frac{1}{n}\right)\to bdt,\frac{B_{t+\frac{1}{n}}-B_t}{B_t}\to\frac{dB_t(t)}{B_t(t)},\qquad \frac{A_{t+\frac{1}{n}}-A_t}{A_t}\to\frac{dA_t(t)}{A_t(t)}\end{array}\right\} \quad (2.4)$$

Substituting eqns. (2.1), (2.2) and (2.4) into eqn.(2.3) we have

$$dZ(t)=Z(t)\left(\varphi\left(\frac{dB_t(t)}{B_t(t)}\right)+(1-\varphi)\left(\frac{dA_t(t)}{A_t(t)}\right)\right)+bdt+\xi dW_s-tb\alpha(\pi_0+t)dt-Z(t)(1-\varphi)\left(1+\frac{dA_t(t)}{A_t(t)}\right)\alpha(\pi_0+t)dt$$

$$dZ(t)=\{Z(t)(\varphi[\delta L+\alpha(\pi_0+t)]+r-\alpha(\pi_0+t))+b(1-t\alpha(\pi_0+t))\}dt+(\varphi\sqrt{L}Z(t)+\xi)dW_s,$$
$$Z(0)=Z_0 \quad (2.5)$$

Where π is the maximal age of the life table and $\alpha(t)$is the force function given by

$$\alpha(t)=\frac{1}{\pi-t},\qquad 0\le t<\pi$$

$$\alpha(\pi_0+t)=\frac{1}{\pi-\pi_0-t} \quad (2.6)$$

Substituting eqn.(2.6) into eqn.(2.5), we have

$$dZ(t)=\left\{Z(t)\left(\varphi\left[\delta L+\left(\frac{1}{\pi-\pi_0-t}\right)\right]+r-\left(\frac{1}{\pi-\pi_0-t}\right)\right)+b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right\}dt+(\varphi\sqrt{L}Z(t)+\xi)dW_sZ(0)=Z_0 \quad (2.7)$$

3. Methodology

If we consider the pension wealth and the volatility of the accumulations, the remaining members of the pension scheme will want to increase their total wealth and minimize the risk as much as possible. Hence there is need toformulate the optimal portfolio problem under the mean-variance criterion as follows:

$$\sup_{\varphi}\{E_{t,z,l}Z^{\varphi}(T)-Var_{t,z,l}Z^{\varphi}(T)\} \quad (3.1)$$

Our main aim is to obtain the optimal investment strategies for both the risk-free and risky asset using the mean-variance utility function.

Applying game theoretic method in Björk and Murgoci,(2009) and He and Liang(2013), the mean-variance control problem in eqn.(3.1) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $C(t,z,l)$

$$\begin{cases} D(t,z,l,\varphi) = E_{t,z,l}[Z^{\varphi}(T)] - \frac{\gamma}{2} Var_{t,z,l}[Z^{\varphi}(T)] \\ D(t,z,l,\varphi) = E_{t,z,l}[Z^{\varphi}(T)] - \frac{\gamma}{2}\left(E_{t,z,l}[Z^{\varphi}(T)^2] - (E_{t,z,l}[Z^{\varphi}(T)])^2\right) \\ C(t,z,l) = \sup_{\varphi} D(t,z,l,\varphi) \end{cases}$$

Following Sheng and Rong 2014), the optimal portfolio policyφ^* satisfies:

$C(t,z,l) = \sup_{\varphi} D(t,z,l,\varphi^*)$

$\gamma$is a constant representing risk aversion coefficient of the members.

Let $p^{\varphi}(t,z,l) = E_{t,z,l}[Z^{\varphi}(T)]$, $q^{\varphi}(t,z,l) = E_{t,z,l}[Z^{\varphi}(T)^2]$ then

$C(t,z,l) \quad = \sup_{\varphi} u\left(t,z,l,p^{\varphi}(t,z,l),q^{\varphi}(t,z,l)\right)$

Where,

$u(t,z,l,p,q) = p - \frac{\gamma}{2}(q-p^2)$

Theorem 3.1 (verification theorem). If there exist three real functions $U,V,W : [0,T] \times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\begin{cases} \sup_{\varphi} \begin{Bmatrix} U_t - u_t + (U_z - u_z)\left[z\left(\varphi\left[\delta l + \left(\frac{1}{\pi-\pi_0-t}\right)\right] + r - \left(\frac{1}{\pi-\pi_0-t}\right)\right) + b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right] \\ +(U_l - u_l)k(\vartheta - l) + \\ \frac{1}{2}(U_{zz} - P_{zz})(\xi + \varphi z\sqrt{l})^2 + \frac{1}{2}(U_{ll} - P_{ll})\sigma^2 l + (U_{zl} - P_{zl})(\xi + \varphi z\sqrt{l})\rho\sigma\sqrt{l} \end{Bmatrix} = 0 \\ U(T,z,l) = u(t,z,l,z^2) \end{cases} \qquad (3.2)$$

Where,

$$\begin{cases} P_{zz} = u_{zz} + 2u_{zp}p_z + 2u_{zq}q_z + u_{pp}p_z^2 + 2u_{pq}p_z q_z + u_{qq}q_z^2 = \gamma V_z^2 \\ P_{ll} = u_{ll} + 2u_{lp}p_l + 2u_{lq}q_l + u_{pp}p_l^2 + 2u_{pq}p_l q_l + u_{qq}q_l^2 = \gamma V_l^2 \\ P_{zl} = u_{zl} + u_{zp}p_l + u_{zq}q_l + u_{pl}p_z + u_{ql}q_z + u_{pp}p_z p_l + u_{pq}p_z q_l + u_{pq}p_l q_z + u_{qq}q_l q_z = \gamma V_z V_l \end{cases}$$

$$\begin{cases} \begin{Bmatrix} V_t + V_z\left[z\left(\varphi\left[\delta l + \left(\frac{1}{\pi-\pi_0-t}\right)\right] + r - \left(\frac{1}{\pi-\pi_0-t}\right)\right) + b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right] \\ +V_l k(\vartheta - l) + \frac{1}{2}V_{zz}(\xi + \varphi z\sqrt{l})^2 + \frac{1}{2}V_{ll}\sigma^2 l + V_{zl}(\xi + \varphi z\sqrt{l})\rho\sigma\sqrt{l} \end{Bmatrix} = 0 \\ V(T,z,l) = z \end{cases} \qquad (3.3)$$

$$\begin{cases} \begin{Bmatrix} W_t + W_z\left[z\left(\varphi\left[\delta l + \left(\frac{1}{\pi-\pi_0-t}\right)\right] + r - \left(\frac{1}{\pi-\pi_0-t}\right)\right) + b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right] \\ +W_l k(\vartheta - l) + \frac{1}{2}W_{zz}(\xi + \varphi z\sqrt{l})^2 + \frac{1}{2}W_{ll}\sigma^2 l + W_{zl}(\xi + \varphi z\sqrt{l})\rho\sigma\sqrt{l} \end{Bmatrix} = 0 \\ W(T,z,l) = z^2 \end{cases} \qquad (3.4)$$

Then $C(t,z,l) = U(t,z,l), p^{\varphi^*} = V(t,z,l), q^{\varphi^*} = W(t,z,l)$for the optimal investment strategy φ^*

Proof:

The details of the proof can be found in He and Liang(2009), Liang, J. Huang(2011) and Zeng and Li (2011).

Our focus now is to obtain the optimal investment strategies for both risky and riskless asset as well as the efficient frontier by solving eqns.(3.2), (3.3)and eqn.(3.4).

Proposition 3.2

The optimal allocation policy for equity is given as

$$\varphi^* = \left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right)\left[\frac{\left(\delta+\frac{1}{l}\left(\frac{1}{\pi-\pi_0-t}\right)\right)}{z\gamma}\right]e^{r(t-T)} - \frac{\xi}{z\sqrt{l}}$$

Proof

Recall that $u(t,z,l,p,q) = p - \frac{\gamma}{2}(q-p^2)$

$u_t = u_z = u_l = u_{zz} = u_{ll} = u_{zl} = u_{pl} = u_{ql} = u_{zp} = u_{zq} = u_{pq} = u_{qq} = 0, u_p = 1+\gamma p,$

$u_{pp} = \gamma, u_q = -\frac{\gamma}{2}$

(3.5)

Substituting eqn.(3.5) into eqn.(3.2) and differentiating eqn.(3.2) with respect to φ and solving for φ we have:

$$\varphi^* = -\left[\frac{\left(\delta+\frac{1}{l}\left(\frac{1}{\pi-\pi_0-t}\right)\right)U_z+(U_{zl}-\gamma V_zV_l)\rho\sigma+(U_{zz}-P_{zz})\frac{\xi}{\sqrt{l}}}{z\left(U_{zz}-\gamma V_z^2\right)}\right] \quad (3.6)$$

Since $\rho \in [-1,1]$, we assume a case where $\rho = 0$. Hence eqns.(3.2), (3.3), and eqn.(3.6) reduce to

$$\begin{cases} \sup\limits_{\varphi}\left\{\begin{array}{l} U_t + U_z\left[z\left(\varphi\left[\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)\right]+r-\left(\frac{1}{\pi-\pi_0-t}\right)\right)+b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right] \\ +U_lk(\vartheta-l)+\frac{1}{2}(U_{zz}-\gamma V_z^2)(\xi+\varphi z\sqrt{l})^2+\frac{1}{2}\left(U_{ll}-\gamma V_l^2\right)\sigma^2 l \end{array}\right\} = 0 \\ U(T,z,l) = u(t,z,l,z^2) \end{cases} \quad (3.7)$$

$$\begin{cases} \left\{\begin{array}{l} V_t + V_z\left[z\left(\varphi\left[\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)\right]+r-\left(\frac{1}{\pi-\pi_0-t}\right)\right)+b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right] \\ +V_lk(\vartheta-l)+\frac{1}{2}V_{zz}(\xi+\varphi z\sqrt{l})^2+\frac{1}{2}V_{ll}\sigma^2 l \end{array}\right\} = 0 \\ V(T,z,l) = z \end{cases} \quad (3.8)$$

$$\varphi^* = \left[\frac{-\frac{1}{l}\left(\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)\right)U_z+(U_{zz}-P_{zz})\frac{\xi}{\sqrt{l}}}{z\left(U_{zz}-\gamma V_z^2\right)}\right] \quad (3.9)$$

Substituting eqn.(3.9) into eqn.(3.7) and eqn.(3.8) we have

$$\begin{cases} U_t + U_z\left[z\left(r-\frac{1}{\pi-\pi_0-t}\right)+b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right]+U_lk(\vartheta-l)+\frac{1}{2}\left(U_{ll}-\gamma V_l^2\right)\sigma^2 l \\ \quad -\frac{U_z^2}{2l}\frac{\left(\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)\right)^2}{(U_{zz}-\gamma V_z^2)}-\frac{\xi}{\sqrt{l}}\left(\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)\right)U_z = 0 \\ U(T,z,l) = u(t,z,l,z^2) \end{cases} \quad (3.10)$$

$$\begin{cases} V_t + V_z\left[z\left(r-\frac{1}{\pi-\pi_0-t}\right)+b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right]+V_lk(\vartheta-l)++\frac{1}{2}V_{ll}\sigma^2 l \\ -\delta U_zV_z\left(\frac{\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)}{(U_{zz}-\gamma V_z^2)}\right)-\frac{\xi}{\sqrt{l}}\left(\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)\right)V_z+\frac{1}{2l}V_{zz}U_z^2\left(\frac{\delta l+\left(\frac{1}{\pi-\pi_0-t}\right)}{(U_{zz}-\gamma V_z^2)}\right)^2 = 0 \\ V(T,z,l) = z \end{cases} \quad (3.11)$$

Next, we assume a solution for $U(t,z,l)$and $V(t,z,l)$ as follows:

$$\begin{cases} U(t,z,l) = X_1(t)z+\frac{X_2(t)l}{\gamma}+\frac{X_3(t)}{\gamma}, X_1(T)=1,\ X_2(T)=0,\ X_3(T)=0 \\ V(t,z,l) = Y_1(t)z+\frac{Y_2(t)l}{\gamma}+\frac{Y_3(t)}{\gamma}, Y_1(T)=1,\ Y_2(T)=0,\ Y_3(T)=0 \\ U_t = zX_{1t}(t)+\frac{X_{2t}(t)l}{\gamma}+\frac{X_{3t}(t)}{\gamma}, U_z = X_1(t), U_{zz}=0, U_l=\frac{X_2(t)}{\gamma}, U_{ll}=0 \\ V_t = zY_{1t}(t)+\frac{Y_{2t}(t)l}{\gamma}+\frac{Y_{3t}(t)}{\gamma}, V_z = Y_1(t), V_{zz}=0, V_l=\frac{Y_2(t)l}{\gamma}, V_{ll}=0 \end{cases} \quad (3.12)$$

Substituting eqn.(3.12) into eqn.(3.10) and eqn.(3.11)

$$\begin{cases} X_{1t}(t) + \left(r - \frac{1}{\pi-\pi_0-t}\right) X_1(t) = 0 \\ X_{2t}(t) - kX_2 - \frac{\sigma^2 {Y_2}^2}{2} + \frac{{X_1}^2}{2{Y_1}^2}\left(\delta + \frac{1}{l}\left(\frac{1}{\pi-\pi_0-t}\right)\right)^2 = 0 \\ X_{3t}(t) + k\vartheta X_2 + X_1 b\gamma\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right) - \frac{\xi}{\sqrt{l}}\left(\delta l + \left(\frac{1}{\pi-\pi_0-t}\right)\right) X_1 = 0 \end{cases} \quad (3.13)$$

$$\begin{cases} Y_{1t}(t) + \left(r - \frac{1}{\pi-\pi_0-t}\right) Y_1(t) = 0 \\ Y_{2t}(t) - kY_2 + \frac{X_1}{Y_1}\left(\delta + \frac{1}{l}\left(\frac{1}{\pi-\pi_0-t}\right)\right)^2 = 0 \\ Y_{3t}(t) + k\vartheta Y_2 + Y_1 b\gamma\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right) - \frac{\xi}{\sqrt{l}}\left(\delta l + \left(\frac{1}{\pi-\pi_0-t}\right)\right) Y_1 = 0 \end{cases} \quad (3.14)$$

Solving eqns.(3.13) and (3.14), we have

$X_1(t) = \left(\frac{\pi-\pi_0-T}{\pi-\pi_0-t}\right) e^{r(T-t)}$

$X_2(t) = \frac{\sigma^2\delta^4}{2k^2}\left\{\frac{1}{k} + 2(T-t)e^{k(t-T)} - e^{2k(t-T)}\right\} + \frac{\delta^2}{2k}\left\{e^{k(T-t)} - 1\right\} - \frac{\delta}{l} I(t) e^{kt} - \frac{e^{kt}}{2l^2}\int_t^T \frac{e^{-k\tau}}{(\pi-\pi_0-t)^2} d\tau + \frac{\sigma^2\delta^2}{2l^2}\int_t^T I(\tau)^2 e^{k\tau}\, d\tau + \frac{\sigma^2\delta^3}{2l^2}\int_t^T I(\tau)\left\{e^{k(\tau-T)} - 1\right\} d\tau$

$$X_3(t) = -k\vartheta \int_t^T X_2(\tau)\, d\tau - (\delta\xi\sqrt{l} + b\gamma)(\pi-\pi_0-T)\int_t^T \frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau + b\gamma(\pi-\pi_0-T)$$

$\int_t^T \left(\frac{\tau}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau - \frac{r\xi}{\sqrt{l}}(\pi-\pi_0-T)\int_t^T \left(\frac{1}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau$

$Y_1(t) = \left(\frac{\pi-\pi_0-T}{\pi-\pi_0-t}\right) e^{r(T-t)}$

$Y_2(t) = \frac{\delta^2}{k}\left\{1 - e^{k(T-t)}\right\} - \frac{\delta}{l} e^{kt} I(t),$ where $I(t) = \int_t^T \frac{e^{-r\tau}}{\pi-\pi_0-\tau} d\tau$

$$Y_3(t) = -k\vartheta \int_t^T Y_2(\tau)\, d\tau - (\delta\xi\sqrt{l} + b\gamma)(\pi-\pi_0-T)\int_t^T \frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau + b\gamma(\pi-\pi_0-T)$$

$\int_t^T \left(\frac{\tau}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau - \frac{r\xi}{\sqrt{l}}(\pi-\pi_0-T)\int_t^T \left(\frac{1}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau$

$U(t,z,l) =$
$z\left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right) e^{-r(T-t)} + \frac{1}{\gamma}\left\{\frac{\sigma^2\delta^4 l}{2k^2}\left\{\frac{1}{k} + 2(T-t)e^{k(t-T)} - e^{2k(t-T)}\right\} + \frac{\delta^2 l}{2k}\left\{e^{k(T-t)} - 1\right\} - \delta I(t) e^{kt} - \frac{e^{kt}}{2l}\int_t^T \frac{e^{-k\tau}}{(\pi-\pi_0-\tau)^2} d\tau + \frac{\sigma^2\delta^2}{2l}\int_t^T I(\tau)^2 e^{k\tau}\, d\tau + \frac{\sigma^2\delta^3}{2l}\int_t^T I(\tau)\left\{e^{k(\tau-T)} - 1\right\} d\tau\right\} -$
$\frac{1}{\gamma}\left\{\begin{matrix} k\vartheta\int_t^T X_2(\tau)\, d\tau \\ +b\gamma(\pi-\pi_0-T)\int_t^T \frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau - \left(\delta\xi\sqrt{l} + b\gamma\right)(\pi-\pi_0-T)\int_t^T \left(\frac{\tau}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau \\ -\frac{r\xi}{\sqrt{l}}(\pi-\pi_0-T)\int_t^T \left(\frac{1}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau \end{matrix}\right\}$ $V(t,z,l) = z\left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right) e^{r(T-t)} +$
$\frac{1}{\gamma}\left\{\frac{l\delta^2}{k}\left\{1 - e^{k(t-T)}\right\} - \delta I(t)\right\} - \frac{1}{\gamma}\left\{\begin{matrix} k\vartheta\int_t^T Y_2(\tau)\, d\tau \\ +b\gamma(\pi-\pi_0-T)\int_t^T \frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau - \left(\delta\xi\sqrt{l} + b\gamma\right)(\pi-\pi_0-T)\int_t^T \left(\frac{\tau}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau \\ -\frac{r\xi}{\sqrt{l}}(\pi-\pi_0-T)\int_t^T \left(\frac{1}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau} d\tau \end{matrix}\right\}$

$U_z = X_1(t) = \left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right) e^{r(T-t)}$ and $V_z = Y_1(t) = \left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right) e^{r(T-t)}$ $\quad U_{zz} = 0$ (3.15)

Substituting eqn.(3.15) into eqn.(3.9), we have

$$\varphi^* = \left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right)\left[\frac{\left(\delta + \frac{1}{l}\left(\frac{1}{\pi-\pi_0-t}\right)\right)}{z\gamma}\right] e^{r(t-T)} - \frac{\xi}{z\sqrt{l}}. \quad (3.16)$$

Proposition 3.3

The efficient frontierof the pension members is given by

$$E_{t,z,l}[Z^{\varphi^*}(T)] =$$

$$z\left(\frac{\pi-\pi_0-T}{\pi-\pi_0-t}\right)e^{r(T-t)} - b(\pi-\pi_0-T)[\int_t^T \frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau}d\tau + \int_t^T \left(\frac{\tau}{\pi-\pi_0-\tau}\right)\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau}d\tau] +$$

$$\frac{\left\{\frac{l\delta^2}{k}\{1-e^{k(t-T)}\}-r\delta I(t)-k\vartheta\int_t^T Y_2(\tau)d\tau\right\}\sqrt{Var_{t,z,l}[Z^{\varphi^*}(T)]}}{\left\{\begin{array}{c}\frac{l\delta^2}{k}\{1-e^{k(T-t)}\}+2\delta I(t)(e^{kt}-1)+2k\vartheta\int_t^T[X_2(\tau)-Y_2(\tau)]d\tau-\frac{\sigma^2\delta^4 l}{k^2}\left\{\frac{1}{k}-2(T-t)e^{k(t-T)}-e^{2k(t-T)}\right\} \\ +\frac{2e^{kt}}{l}\int_t^T(\frac{1}{(\pi-\pi_0-\tau)^2}-\delta^2 I^2)e^{-k\tau}d\tau-\frac{4\delta^3}{k}\int_t^T I(\tau)\{e^{-kT}-e^{-k\tau}\}d\tau\end{array}\right\}}$$

Proof

$$Var_{t,z,l}[Z^{\varphi^*}(T)] = E_{t,z,l}[Z^{\varphi}(T)^2] - (E_{t,z,l}[Z^{\varphi}(T)])^2 = \frac{2}{\gamma}(V(t,z,l) - U(t,z,l))$$

$$Var_{t,x}[Z^{\varphi^*}(T)] = \frac{2}{\gamma^2}\left\{\frac{l\delta^2}{2k}\{1-e^{k(T-t)}\} + \delta I(t)(e^{kt}-1) + k\vartheta\int_t^T[X_2(\tau)-Y_2(\tau)]\,d\tau - \frac{\sigma^2\delta^4 l}{2k^2}\left\{\frac{1}{k} - 2(T-t)e^{k(t-T)} - e^{2k(t-T)}\right\} + \frac{e^{kt}}{l}\int_t^T(\frac{1}{(\pi-\pi_0-\tau)^2} - \delta^2 I^2)e^{-k\tau}d\tau - \frac{2\delta^3}{k}\int_t^T I(\tau)\,\{e^{-kT} - e^{-k\tau}\}d\tau\right\}$$

$$\frac{1}{\gamma} = \frac{\sqrt{Var_{t,z,l}[Z^{\varphi^*}(T)]}}{\left\{\begin{array}{c}\frac{l\delta^2}{k}\{1-e^{k(T-t)}\}+2\delta I(t)(e^{kt}-1)+2k\vartheta\int_t^T[X_2(\tau)-Y_2(\tau)]d\tau-\frac{\sigma^2\delta^4 l}{k^2}\left\{\frac{1}{k}-2(T-t)e^{k(t-T)}-e^{2k(t-T)}\right\} \\ +\frac{2e^{kt}}{l}\int_t^T(\frac{1}{(\pi-\pi_0-\tau)^2}-\delta^2 I^2)e^{-k\tau}d\tau-\frac{4r\delta^3}{k}\int_t^T I(\tau)\{e^{-kT}-e^{-k\tau}\}d\tau\end{array}\right\}} \qquad .17)$$

$$E_{t,z,l}[Z^{\varphi^*}(T)] = V(t,z,l)$$

$$E_{t,z,l}[Z^{\varphi^*}(T)] = z\left(\frac{\pi-\pi_0-T}{\pi-\pi_0-t}\right)e^{r(T-t)} - b(\pi-\pi_0-T)[\int_t^T\left(\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau}\right)d\tau + \int_t^T\left(\frac{\tau}{\pi-\pi_0-\tau}\right)\left(\frac{e^{r(T-\tau)}}{\pi-\pi_0-\tau}\right)d\tau] +$$
$$\frac{1}{\gamma}\left\{\frac{l\delta^2}{k}\{1-e^{k(t-T)}\} - r\delta I(t) - k\vartheta\int_t^T Y_2(\tau)\,d\tau\right\} \qquad (3.18)$$

Substituting (3.17) in (3.18), we have

$$E_{t,z,l}[Z^{\varphi^*}(T)] = z\left(\frac{\pi-\pi_0-T}{\pi-\pi_0-t}\right)e^{r(T-t)} - b(\pi-\pi_0-T)[\int_t^T\left(\frac{e^{(T-\tau)}}{\pi-\pi_0-\tau}\right)d\tau + \int_t^T\left(\frac{\tau}{\pi-\pi_0-\tau}\right)\left(\frac{e^{(T-\tau)}}{\pi-\pi_0-\tau}\right)d\tau] +$$

$$\frac{\left\{\frac{l\delta^2}{k}\{1-e^{k(t-T)}\}-r\delta I(t)-k\vartheta\int_t^T Y_2(\tau)d\tau\right\}\sqrt{Var_{t,z,l}[Z^{\varphi^*}(T)]}}{\left\{\begin{array}{c}\frac{l\delta^2}{k}\{1-e^{k(T-t)}\}+2\delta I(t)(e^{kt}-1)+2k\vartheta\int_t^T[X_2(\tau)-Y_2(\tau)]d\tau-\frac{\sigma^2\delta^4 l}{k^2}\left\{\frac{1}{k}-2(T-t)e^{k(t-T)}-e^{2k(t-T)}\right\} \\ +\frac{2e^{kt}}{l}\int_t^T(\frac{1}{(\pi-\pi_0-\tau)^2}-\delta^2 I^2)e^{-k\tau}d\tau-\frac{4\delta^3}{k}\int_t^T I(\tau)\{e^{-kT}-e^{-k\tau}\}d\tau\end{array}\right\}}$$

Proposition 3.4

The optimal fund size $Z^{\varphi^*}(t)$corresponding to the optimal allocation strategy $\varphi^*$is given as

$$\begin{aligned}Z(t) = \frac{l}{\gamma}\left(\frac{e^{-rT}}{(\pi-\pi_0-T)}\right)&\left[(\pi-\pi_0-t)\left(l^2\delta^2 t + \frac{1}{\pi-\pi_0-t} + 2l\delta\ln\left(\frac{1}{(\pi-\pi_0-t)}\right)\right)\right. \\ &\left.-(\omega-\omega_0)\left(\frac{1}{\pi-\pi_0} + 2l\delta\ln\left(\frac{1}{(\pi-\pi_0)}\right)\right)\right] + \xi\delta\sqrt{l}(\pi-\pi_0-t) \\ &- b\int_t^T\left(\frac{e^{-r\tau}}{(\pi-\pi_0-\tau)} - \frac{\tau e^{-r\tau}}{(\pi-\pi_0-\tau)^2}\right)d\tau - \frac{\xi e^{rt}(\pi-\pi_0-t)}{\sqrt{l}}\int_t^T\left(\frac{e^{-r\tau}}{\pi-\pi_0-t}\right)d\tau\end{aligned}$$

Proof

From equation (2.7), we have

$$\begin{aligned}dZ(t) = &\left\{Z(t)\left(\varphi\left[\delta L + \left(\frac{1}{\pi-\pi_0-t}\right)\right] + r - \left(\frac{1}{\pi-\pi_0-t}\right)\right) + b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right)\right\}dt \\ &+ \left(\varphi\sqrt{L}Z(t) + \xi\right)dW_s Z(0) = Z_0\end{aligned}$$

Divide the above equation by $dt$and Substitute (3.16) into the equation, we have

$$\frac{dZ(t)}{dt} = Z(t)\left(\left(\left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right)\left[\frac{\left(\delta+\frac{1}{l}\left(\frac{1}{\pi-\pi_0-t}\right)\right)}{z\gamma}\right]e^{r(t-T)} - \frac{\xi}{z\sqrt{l}}\right)\left[\delta l + \left(\frac{1}{\pi-\pi_0-t}\right)\right] + r - \left(\frac{1}{\pi-\pi_0-t}\right)\right) + b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right) Z(0) = z_0. \tag{3.19}$$

Simplifying eqn.(3.19), we have

$$\frac{dZ(t)}{dt} - \left(r - \left(\frac{1}{\pi-\pi_0-t}\right)\right)Z(t) = \frac{l}{\gamma}\left(\delta l + \frac{1}{\pi-\pi_0-t}\right)^2\left(\frac{\pi-\pi_0-t}{\pi-\pi_0-T}\right)e^{r(t-T)} - b\left(\frac{\pi-\pi_0-2t}{\pi-\pi_0-t}\right) - \left[\xi\delta\sqrt{l} + \frac{\xi}{\sqrt{l}}\left(\frac{1}{\pi-\pi_0-t}\right)\right] \tag{3.20}$$

Eqn.(3.20) is a first order ODE, and finding the integrating factor and multiplying through by it, we have

$$\frac{d}{dt}\left[\frac{Ze^{-rt}}{(\pi-\pi_0-t)}\right] = \frac{l}{\gamma}\left(\delta l + \frac{1}{\pi-\pi_0-t}\right)^2\left(\frac{e^{-rT}}{(\pi-\pi_0-t)}\right) - b[\left(\frac{e^{-rt}}{(\pi-\pi_0-t)}\right) - \left(\frac{te^{-rt}}{(\pi-\pi_0-t)^2}\right) - \left[\xi\delta\sqrt{l} + \frac{\xi}{\sqrt{l}}\left(\frac{1}{\pi-\pi_0-t}\right)\right]e^{-rt} \tag{3.21}$$

Integrating eqn.(3.21) w.r.t t, we have

$$Z(t) = \frac{l}{\gamma}\left(\frac{e^{-rT}}{(\pi-\pi_0-T)}\right)\left[(\pi-\pi_0-t)\left(l^2\delta^2 t + \frac{1}{\pi-\pi_0-t} + 2l\delta\ln\left(\frac{1}{(\pi-\pi_0-t)}\right)\right) - (\omega-\omega_0)\left(\frac{1}{\pi-\pi_0} + 2l\delta\ln\left(\frac{1}{(\pi-\pi_0)}\right)\right)\right] + \xi\delta\sqrt{l}(\pi-\pi_0-t) - b\int_t^T\left(\frac{e^{-r\tau}}{(\pi-\pi_0-\tau)} - \frac{\tau e^{-r\tau}}{(\pi-\pi_0-\tau)^2}\right)d\tau - \frac{\xi e^{rt}(\pi-\pi_0-t)}{\sqrt{l}}\int_t^T\left(\frac{e^{-r\tau}}{\pi-\pi_0-t}\right)d\tau$$

4. Numerical Simulations

In this section we present numerical simulations of the optimal investment strategy with respect to time and observed the effect of the various parameters of the optimal investment strategy on it using math lab programming language.

The following parameters were used unless otherwise stated $\gamma = 0.2$, $r = 0.06$, $\delta = 0.05$, $z = 0.1$, $T = 40$, $l = 0.7$, $t = 0:5:20$, $\xi = 0:05$, $\pi = 100$, $\pi_0 = 20$.

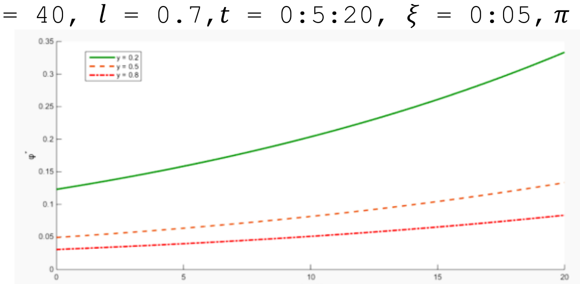

Fig. 1 Evolution of optimal allocation strategywith different risk averse level

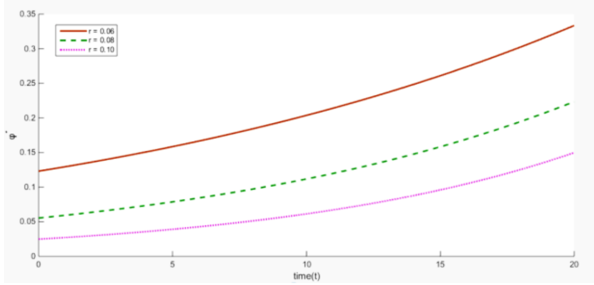

Fig. 2 Evolution of optimal allocation strategywith different predetermined interest rates

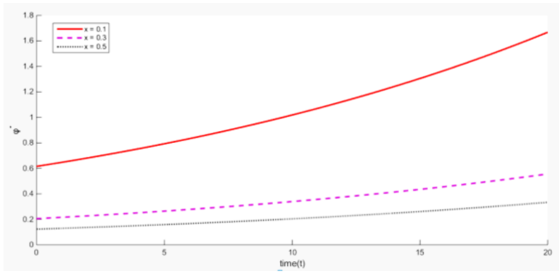

Fig. 3 Evolution of optimal allocation strategywith different initial wealth

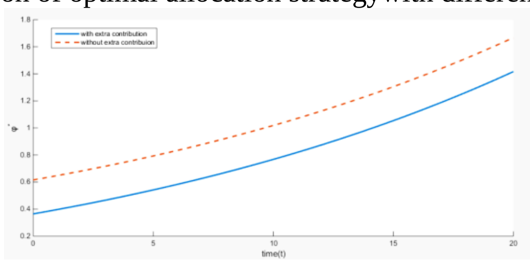

Fig. 4 Evolution of optimal allocation strategy with and without extra contribution

5. Discussion

From figure 1, the optimal allocation strategy increases with a decrease in the risk-averse coefficient. The implication is that members with high risk averse will prefer to invest more in cash and will reduce that of the equity.

Figure 2, shows that the optimal allocation strategy increases with a decrease in the interest rate of cash. This implies that if the interest rate of the cash is high, the members will increase the proportion of its wealth to be invested in cash thereby reducing the proportion invested in equity and vice versa.

Figure 3, shows that the optimal allocation strategydecreases with increase in the initial wealth. The implication here is that if the initial wealth of the plan member is high, the member will prefer to invest more in cash to minimize risk instead of investing more in equity but if the initial wealth is low, the member prefers taking the risk to grow the wealth by investing in equity.

Figure 4, shows that the optimal allocation strategy decreases when there is extra voluntary contribution and vice versa.

In general, we observe that at the beginning of the accumulation phase, the pension manager will invest more in cash because there is no return initially, but once refund is made to the death members, the fund manager will increase its investment inthe equity to meet the retirement needs of the remaining members.

6. Conclusion

We investigated asset allocation strategy in a defined contribution (DC) pension plan with refund of contribution clauses under Heston's Volatility model using mean-variance utility function. We assumed that the refund contributions are with predetermined interest and considered investments in cash and equity to help increase the accumulated funds of the remaining members to meet their retirement needs. We established an optimized problem from the extended Hamilton Jacobi Bellman equations and solved the optimized problem and obtained the optimal allocation strategy for both cash and equity and also the efficient frontier of the members. Next, we analysed numerically the effect of some parameters on the optimal allocation strategy. Our conclusion is that as the initial wealth, predetermined interest rate, and risk averse level increases, the optimal allocation strategy for the risky asset (equity) decreases.

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STABILITY ANALYSIS AND SYNTHESIS OF STOCHASTIC OSCILLATOR SYSTEMS DESCRIBED BY PERTURBED DUFFING EQUATION

by

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Abstract

The stability analysis and synthesis of stochastic oscillator systems described by perturbed Duffing equation is studied in this paper. The Lyapunov stability is a tool for the study of dynamical behaviours (stability) of nonlinear systems. The results show the points where the system is Lyapunov stable and where the point is asymptotically stable. This stability analysis is used to checkmate the volatility of the stock and their prices (fluctuations). We recommend the use of other methods of stability to study the market model. These methods can also be compared with each other to see the best method.

Keywords: *stability analysis, asymptotic stability, perturbed Duffing equation, Lyapunov stability, asymptotic stability dynamical behaviours.*

INTRODUCTION

The Stability analysis of nonlinear systems has been uninterruptedly investigated in many fields by so many researchers such as in control theory and engineering (see for instance Khalil and Grizzle (2002) and their references). Stability of stochastic equations (SDE) was investigated by literatures in stochastic stability in probability almost sure stability, etc. (see for example Kozin (1969)).

For deterministic and stochastic systems, several articles have developed different methods to address the stability issue. The Lyapunov's stability methods have been successfully applied for long years by engineers and scientists (Slotine and Li 1991, Khalil, 1992). Once the Lyapunov function is obtained for the system of interest, the next practical issue becomes the region of attraction. In order to do this, some computational approaches, such as, geometrical, numerical methods etc. have been applied. For previous works have been proposed for the construction of Lyapunov functions based on conventional methods (Golub et al, 1979), numerical methods (Zhaolu and Chuanqing 2008, Sorensen and Zhou 2003) and artificial intelligent methods (Grosman and Lewin ,2008, Banks).

The Hessian term that exists in the Itô formula is difficult to interpret physically and is hard to handle for stability analysis. There is also a difficulty in the selection between two well-known descriptions of SDE, an Itô integral equation and a Stratonovich integral equation for a specific application. Moreover since a white noise is unbounded, it fails to describe the model of some applications; therefore, other stochastic processes such as a stationary process are required. Because of these limitations and difficulties, SDE model is not accurate enough to model all application that contains a stochastic disturbance (Sanjari and Tahmasebi). To address the above problem, nonlinear random models have been required to alleviate the problems mentioned above (Wu, 2015). Moreover, nonlinear random model enable some deterministic analysis tools to be applied (Jiao et al, 2015).

Stability results of nonlinear random differential equation (RDE) have been presented in (Bertram and Sarachik, 1959), but some restrictive assumptions and constrains confine the extension of RDE to the range of applications dealing with a stochastic disturbance. However, recently, (Wu, 2015) constructs a general framework to address the stability criteria of nonlinear RDE and presents theorems employing mild assumptions that conclude the stability of RDE based on a Lyapunov approach, which renders extending the applications that nonlinear RDE, especially in the control theory (Jiao et al, 2016, Xia, et al, 2015). However, to the best of the authors' knowledge no work on stability analysis and synthesis of stochastic oscillator system described by perturbed doffing equation especially in Mathematical Finance has been done until now.

However, the rest of this paper is organised as follows; section II presents problem statement and system description and section III Lyapunov methods, IV, gives example using the model of study, section V, Results and Conclusion.

II. PROBLEM STATEMENT AND SYSTEM DESCRIPTION

The volatility of stock and their prices (fluctuations) are stochastic in nature. They are modelled with stochastic oscillators and also are described by perturbed Duffing equation. The stability analysis and synthesis of stochastic oscillators are then required to checkmate the fluctuations in the market.

SDE's have been proven to be an appropriate model to fit the data in many applications, but in some situation, they provide inappropriate model to describe systems that contain stochastic disturbance and therefore suffer some disadvantages (Wu, 2015). For example white noise is driven Wiener process which does not have derivative anywhere, so it is unsuitable to model Fluctuation in practical applications.

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Consider the following nonlinear random system (Osu et al ,2019);

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2qu + 2w^2pu^2 + \varepsilon w\gamma u^3 = \varepsilon\mu f(t;\omega) + g(t)N(t), \qquad (1.1)$$

Where $u\epsilon R^n$ is the state vector, $N\epsilon R^1$ is $\digamma_t$-adapted and piecewise continuous stochastic process, $f(t;\omega): R^n \times R_+ \rightarrow R^n$ is a known nonlinear function and $g(u,t): R^n \times R_+ \rightarrow R^{n-1}$ is an envelope function.

The following assumptions were made;

Assumption 1; The Stochastic process $N(t)$ is $\digamma_t$-adapted, piecewise continuous such that there exists a positive constant k satisfying

$$Sup_{t\geq t_0} E\{|N(t)|^2\} < k$$

It means that the mean-square of the stochastic process $N(t)$ is bounded by a constant.

Assumption II; The solution $u(t)$ of eq(1.1) is $\digamma_t$-adapted and satisfies all $t\epsilon[t_0, T]$

$$u(t) = u(t_0) + \int_{t_0}^{T} f(u,s)ds + \int_{t_0}^{T} g(u,s)N(s)ds$$

Assumption III; Nonlinear functions $f(.)g(.)$ vanish at the origin, i.e. $f(0,t) = g(0,t) = 0$ for all $t\epsilon[t_0, \infty]$.

III. LYAPUNOV METHODS

The Lyapunov theory of dynamic systems is the most useful general theory for studying the stability of nonlinear systems. It includes two methods; *(i)* Lyapunov's indirect (reduced) method and *(ii)* Lyapunov's direct method.

i) Lyapunov's indirect (reduced) method or First Lyapunov criterion

Lyapunov's Indirect method states that the dynamical system

$$\dot{u} = f(u)$$

Where $f(0) = 0$, has a locally exponentially stable equilibrium point at the origin if and only if the real parts of the eigenvalues 0f the Jacobian matrix of f at zero are all strickly negative. Considering the autonomous system above, the Jacobian at the equilibrium point can can be defined as:

$$A = \frac{\partial f(u)}{\partial u} \Big|_{u_e=0}$$

For Lyapunov's Indirect method,

- If all eigenvalues of A are strictly in the left-half complex plane (negative real part), then the asymptotic of the linearized system is concluded.
- If at least one eigenvalue of A is strictly in right-half complex plane (positive real part), then the instability of the linearized system is concluded.
- If all eigenvalues of A are in the left-hand complex plane but at least one of them is on the jw-axis or imaginary part, then the linearized system is said to be marginally stable but one cannot conclude anything about the stability of the nonlinear system from the linear approximation. (Panikhom and Sujitjiorn 2010}

In the indirect method, the quadratic Lyapunov function can be generally applied. It can be expressed as;

$$V(u) = u^t P u > 0$$

Where u is the state vector and$P$ is a symmetrically scalar matrix. The following equations must be satisfied:

$$\dot{u} = Au$$
$$\dot{u}^t = u^t A^t$$
$$\dot{V}(u) = u^t P \dot{u} + \dot{u}^t P u$$
$$\dot{V}(u) = u^t PAu + u^t A^t P u$$
$$\dot{V}(u) = u^t (PA + A^t P) u$$
$$\dot{V}(u) = u^t Q u$$

Where $Q = PA + A^t P$ and $Q = Q^t$

Finding the Lyapunov function of eqn (1.1).

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 + \varepsilon w\gamma u^3 = \varepsilon\mu f(t;\omega) + g(t)N(t),$$

With the third assumption above and $\varepsilon = 0$ eqn (1.1) becomes:

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 = 0$$

With linearization the last equation becomes;

$$\dot{u}_{\mathbf{1}} = u_2 \dot{u}_{\mathbf{2}} = -\delta u_2 - w^2 q u_1 - 2w^2 p {u_1}^2$$

In matrix form is

$$\begin{bmatrix} \dot{u}_{\mathbf{1}} \\ \dot{u}_{\mathbf{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2 q - 2w^2 p u_1 & -\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

At the origin, $u = 0$, and $[u_1 \quad u_2] = [0 \quad 0]$, the matrix above becomes;

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2 q & -\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Now, $A = \begin{bmatrix} 0 & 1 \\ -w^2 q & -\delta \end{bmatrix}$

And choose, $P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, a symmetric matrix.

Checking the definiteness of P by

$$u^t P u = [u_1 \quad u_2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = {u_1}^2 + (u_1 - u_2)^2$$

The last equation shows that P is positive definite. So

$$V(u) = u^t P u > 0$$

Then find the value of $Q = PA + A^t P$ as

$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -w^2 q & -\delta \end{bmatrix} + \begin{bmatrix} 0 & -w^2 q \\ 1 & -\delta \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2w^2 q & 2 + \delta - w^2 q \\ 2 - w^2 q + \delta & -2 - 2\delta \end{bmatrix}$$

$$\begin{aligned} \dot{V}(u) = u^t Q u &= [u_1 \quad u_2] \begin{bmatrix} 2w^2 q & 2 + \delta - w^2 q \\ 2 - w^2 q + \delta & -2 - 2\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \dot{V}(u) \\ &= 2w^2 q {u_1}^2 + 2(1 + \delta - w^2 q) u_1 u_2 - 2\delta {u_2}^2 \dot{V}(u) \\ &= -2(-2w^2 q {u_1}^2 - 2(-1 - \delta + w^2 q) u_1 u_2 + 2\delta {u_2}^2) \end{aligned}$$

Let's choose the values of the scalar variables in the equation to be as follows; $w = 1, q = -2 \; and \; \delta = 3$

Then substituting in the last equation becomes:

$$\dot{V}(u) = -2(+2{u_1}^2 + 6u_1 u_2 + 3{u_2}^2)$$

The last equation shows the derivative of $V(u)$ is negative definite in the chosen values of the scalar variables and so the system is asymptotically stable at that point.

ii) Lyapunov's direct method or second Lyapunov criterion

Lyapunov's direct method is a mathematical extension of fundamental physical observation that an energy dissipative system must eventually settle down to an equilibrium point. Lyapunov's direct method states that if there is an energy-like function V of

$$\dot{u} = f(u)$$

that is strictly decreasing along its trajectories, then the equilibrium at the origin is asymptotically stable. The function V is said to be a Lyapunov function for the system. A Lyapunov function provides via its pre-images a lover-bound of the region of attraction of the equilibrium. This bound is non-conservative in the sense that it extends to the boundary of the domain of the Lyapunov function. (Törner andFreiling, 2002).

A nonlinear system can be represented by $\dot{u} = f(u, t)$ for a non- autonomous one, and $\dot{u} = f(u)$ for an autonomous system. At equilibrium, $u_e = 0$, the following condition holds $f(u_e) = 0$ and $\dot{u}_e = 0$.

For the Lyapunov's direct method, the stability analysis of an equilibrium point u_0 is done using proper scalar functions called Lyapunov functions defined in the state space. The second Lyapunov function $V(u)$ must be found and used to conclude the stability region of the system for a nonlinear system without knowing the solution of the governing equations of the system. $V(u)$ most be scalar, positive definite and differentiable.

For a nonlinear system to have a globally asymptotically stable equilibrium, the Lyapunov function $V(u)$ must have the following properties;

- $V(u) > 0$
- $\dot{V}(u) < 0$
- $V(u) \to \infty$ as $\|u\| \to \infty$.

Consider the Duffing equation in eqn (1.1) if the quadratic nonlinear term is perturbed instead of the cubic and $\varepsilon = 0$, the equation becomes;

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2qu + w\gamma u^3 = 0$$

The energy function used as the Lyapunov function candidate is

$$V(u) = \frac{1}{2}(\dot{u}^2 + w^2qu^2 + \frac{1}{2}w\gamma u^4)$$

It can be seen that $V(u)$ is scalar, differentiable, positive definite and unbounded. If $V(u)$ satisfies all the properties, it is said to be the Lyapunov function of the system OF eqn (1.1).

$$\dot{V}(u) = \dot{u}(-a\dot{u}|\dot{u}|) = -a|\dot{u}^2|,$$

the derivative of the Lyapunov function $V(u)$, is negative definite, then the global asymptotic stability of the system is concluded.

Computing the equilibrium points and checking their stability

Consider the equation:

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2qu + 2w^2pu^2 + \varepsilon w\gamma u^3 = \varepsilon\mu f(t;\omega) + g(t)N(t),$$

With the conditions stated earlier, the unperturbed system becomes

$$\ddot{u}(t;\omega) = -\delta\dot{u} - w^2qu - 2w^2pu^2$$

The equivalent system of the above unperturbed system is

$$\dot{u}_{\mathbf{1}} = u_2\dot{u}_{\mathbf{2}} = -\delta u_2 - w^2qu_1 - 2w^2p{u_1}^2$$

The equilibrium points of the system are;

$\dot{u}_{\mathbf{1}} = \dot{u}_{\mathbf{2}} = 0$, i.e. (0, 0)

$-\delta u_2 - w^2qu_1 - 2w^2p{u_1}^2 = 0$

$-w^2qu_1 - 2w^2p{u_1}^2 = 0$

Since $-\delta u_2 = 0$

$\lambda_{1,2} = \frac{w^2q \pm \sqrt{(-w^2q)^2}}{-4w^2p}$, $\lambda_1 = 0, \lambda_2 = \frac{q}{-2p}$

So the system has only two equilibrium points; (0, 0) and $(\frac{q}{-2p}, 0)$

Let

$$\dot{u}_{\mathbf{1}} = u_2 = f_1(u_1, u_2)$$

$$\dot{u}_{\mathbf{2}} = -\delta u_2 - w^2qu_1 - 2w^2p{u_1}^2 = f_1(u_1, u_2)$$

The Jacobian matrix,

$$J(u_1, u_2) = \begin{bmatrix} 0 & 1 \\ -w^2q - 4w^2pu_1 & -\delta \end{bmatrix}$$

With $\varepsilon = 0$, the system has centres at $(u_1, u_2) = (0, 0) and\ (\frac{q}{2p}, 0)$

Examining the stability using eigenvalues approach

$$|A - I\lambda| = \begin{vmatrix} 0 - \lambda & 1 \\ -w^2q - 4w^2pu_1 & -\delta - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + \delta\lambda + w^2q + 4w^2pu_1 = 0$$

$$\lambda_1 = \frac{-\delta + \sqrt{\delta^2 - 4w^2(q + 4pu_1)}}{2}$$

$$\lambda_2 = \frac{-\delta - \sqrt{\delta^2 - 4w^2(q + 4pu_1)}}{2}$$

Now, the values of λ for which the system achieves stability is specified.

For $u_1 = 0, \lambda_1 = \frac{-\delta+\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_1 = 0$, if, $w = 0$ *and* δ, q *and* $p = R$, the real line

$\lambda_1 > 0$if $q \leq 0, \delta \leq 0$and $p = R$, the real line

$\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \leq 0$

$\lambda_1 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$

For $u_1 = \frac{q}{-2p}, \lambda_1 = \frac{-\delta+\sqrt{\delta^2+4w^2q}}{2}$

$\lambda_1 = 0$, if, $w = 0$ *or* $q = 0$ *and* $\delta = R$, the real line

$\lambda_1 > 0$if$q > 0$,and$\delta = R$, the real line

$\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \leq 0$

$\lambda_1 = a \pm ib$, complex numbers if $q < 0$, $\delta > 0$ and $\delta^2 < 4w^2q$

For $u_1 = 0, \lambda_2 = \frac{-\delta-\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_2 = 0$, if, $w = 0$ *or* $q = 0$ *and* $\delta = 0$

$\lambda_2 > 0$, if, $\delta \leq 0$ and $\sqrt{\delta^2 - 4w^2q} \leq 0$

$\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \geq 0$

$\lambda_2 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$

For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta-\sqrt{\delta^2+4w^2q}}{2}$

$\lambda_2 = 0$, if, $w = 0$ *or* $q = 0$ *and* $\delta = 0$

$\lambda_2 > 0$, if, $\delta < 0$, *and*$\sqrt{\delta^2 + 4w^2q} \leq 0$

$\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \geq 0$

$\lambda_2 = a \pm ib$, complex numbers, if,$q < 0$, $\delta > 0$ and $\delta^2 < 4w^2q$

Since the system $S: R^n \rightarrow R^n$ is nonlinear, with continuous first derivative and $\bar{u}_e$ is a critical point of the nonlinear system $u' = f(u)$

1) If all eigenvalues of the Jacobian matrix $J(\bar{u}_e)$ have negative real parts, then the critical point $\bar{u}_e$ is asymptotically stable.
2) If any eigenvalue of the Jacobian matrix $J(\bar{u}_e)$ has positive real part or zero, then the critical point $\bar{u}_e$ is unstable[Ledder,2005].

Analysis

The interest is on the eigenvalues of the Jacobian matrix that have negative real parts for which the critical points $\bar{u}_e$ is asymptotically stable. They are,

[A] For $u_1 = 0, \lambda_1 = \frac{-\delta+\sqrt{\delta^2-4w^2q}}{2}$

[A]i) $\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \leq 0$

$\Rightarrow \delta^2 \leq 4w^2q$, since $\delta > 0$, then $q > 0$ and $w > R$

Since w is the natural frequency of the system, it will always be positive and so $w\epsilon R^{+}$, q being the extent of resistance to the deformation in response to the external force that causes the push or pull in the market shows resistance and then keeps the market prices stable. δ,the economic damping due to speculations is kept in check since $\delta^2 \leq 4w^2q$.

[A]ii) $\lambda_1 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$ and $a < 0$

Since $\delta^2 < 4w^2q$, it implies like in [A]i) that the extent of economic damping due to speculations is also kept in check and so the stability of the market prices hold.

[B] For $u_1 = \frac{q}{-2p}, \lambda_1 = \frac{-\delta+\sqrt{\delta^2+4w^2q}}{2}$

[B]i) $\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \leq 0$

$\Rightarrow \delta^2 \leq -4w^2q$

From the last equation, since $\delta > 0$, then $q < 0$. It means that with the state vector as $\frac{q}{-2p}$, there is damping due to the speculations and negative resistance to the deformation in response to the external force that causes the push or pull in the market shows resistance and so stability cannot be achieved.

[B]ii) $\lambda_1 = a \pm ib$, complex numbers if $q < 0$, $\delta > 0$, $\delta^2 < 4w^2q$ and $a < 0$

With $q < 0$and $\delta > 0$, then $\delta^2 < 4w^2q$is undefined.

[C]For $u_1 = 0, \lambda_2 = \frac{-\delta-\sqrt{\delta^2-4w^2q}}{2}$

[C]i) $\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \geq 0$

This gives the same result as [B]i).

[C]ii) $\lambda_2 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$and $a < 0$

The result here is the same as that of [A]i)

[D] For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta-\sqrt{\delta^2+4w^2q}}{2}$

[D]i) $\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \geq 0$

It gives the same result as [B]i)

[D]ii) $\lambda_2 = a \pm ib$, complex numbers, if, $q < 0$, $\delta > 0$, $\delta^2 < 4w^2q$ and $a < 0$

This holds the same result as [A]i) and [A]ii).

Summary

The summary of the only eigenvalues, λ_i's and the state vector u_1 the achieved stability are as follows;

i) For $u_1 = 0, \lambda_1 = \frac{-\delta+\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_1 < 0$and$\lambda_1 = a + ib$

ii) For $u_1 = 0, \lambda_2 = \frac{-\delta-\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_2 = a + ib$

iii) For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta-\sqrt{\delta^2+4w^2q}}{2}$

$\lambda_2 = a + ib$

Synthesis of the Stochastic Oscillator system described by perturbed Duffing Equation

The Lyapunov exponent and the eigenvalues of the Jacobian matrix are used to perform the synthesis. The Lyapunov exponent is used to test the convergence of the nearby trajectories while the eigenvalues are used with the same parameter values as in the Lyapunov exponent to test the stability of the system. The values of the parameters for which convergence is achieved using the Lyapunov exponent,

$$\lambda_1(t) = \frac{\varepsilon\delta\sin(\theta)^2}{2} - \frac{\varepsilon\delta\cos(\theta)^2}{2} + \frac{\varepsilon\delta\cos(\theta)^2 t}{2} + \frac{\varepsilon\delta\sin(\theta)^2 t}{2} - \frac{\sin(2\theta)(t + \frac{\mu\sin(\omega t)}{\omega} - w^2 qt - 3w\gamma u^2 t - 4\varepsilon w^2 ut}{2}$$

$$\lambda_2(t) = \frac{-\varepsilon\delta\sin(\theta)^2}{2} + \frac{\varepsilon\delta\cos(\theta)^2}{2} + \frac{\varepsilon\delta\cos(\theta)^2 t}{2} + \frac{\varepsilon\delta\sin(\theta)^2 t}{2} - \frac{\sin(2\theta)(t + \frac{\mu\sin(\omega t)}{\omega} - w^2 qt - 3w\gamma u^2 t - 4\varepsilon w^2 ut}{2}$$

are;

$\varepsilon = 0.01, \delta = 0.01, \theta = 30, t = 90, w = 0.4, p = 240, \mu = 1, q = 0.7, and\ \ \omega = 180, u, \gamma$

With the above values of the parameters, $\lambda_1$and λ_2, are negative for all values of $u\ and\ \gamma$ except where u is zero. Here $\lambda_1$and λ_2, are positive showing non-convergence.

Now using the eigenvalues and the same parameters of the Lyapunov exponent $\delta = 0.01, q = 0.7\ and\ \ w = 0.4$

For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta - \sqrt{\delta^2 + 4w^2 q}}{2}$

$\lambda_2 = -0.6794$, with the above parameter values.

Since λ_2 of the Jacobian matrix $J(u_e)$ has negative real part, then the critical point u_e is asymptotically stable.

Conclusion:

This paper has demonstrated the stability analysis of the market price fluctuations using the Lyapunov's direct and indirect methods. The derivative of the Lyapunov's function was excited parametrically and it was found that with some values, the derivative achieved negative definiteness which shows complete asymptotic stability. The Lyapunov exponent and the eigenvalues of the Jacobian matrix are used to perform the synthesis.

Recommendation;

We recommend the use of other methods of stability to study the market model. These methods can also be compared with each other to see the best method.

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