STABILITY ANALYSIS AND SYNTHESIS OF STOCHASTIC OSCILLATOR SYSTEMS DESCRIBED BY PERTURBED DUFFING EQUATION

by

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Abstract

The stability analysis and synthesis of stochastic oscillator systems described by perturbed Duffing equation is studied in this paper. The Lyapunov stability is a tool for the study of dynamical behaviours (stability) of nonlinear systems. The results show the points where the system is Lyapunov stable and where the point is asymptotically stable. This stability analysis is used to checkmate the volatility of the stock and their prices (fluctuations). We recommend the use of other methods of stability to study the market model. These methods can also be compared with each other to see the best method.

Keywords: *stability analysis, asymptotic stability, perturbed Duffing equation, Lyapunov stability, asymptotic stability dynamical behaviours.*

INTRODUCTION

The Stability analysis of nonlinear systems has been uninterruptedly investigated in many fields by so many researchers such as in control theory and engineering (see for instance Khalil and Grizzle (2002) and their references). Stability of stochastic equations (SDE) was investigated by literatures in stochastic stability in probability almost sure stability, etc. (see for example Kozin (1969)).

For deterministic and stochastic systems, several articles have developed different methods to address the stability issue. The Lyapunov's stability methods have been successfully applied for long years by engineers and scientists (Slotine and Li 1991, Khalil, 1992). Once the Lyapunov function is obtained for the system of interest, the next practical issue becomes the region of attraction. In order to do this, some computational approaches, such as, geometrical, numerical methods etc. have been applied. For previous works have been proposed for the construction of Lyapunov functions based on conventional methods (Golub et al, 1979), numerical methods (Zhaolu and Chuanqing 2008, Sorensen and Zhou 2003) and artificial intelligent methods (Grosman and Lewin ,2008, Banks).

The Hessian term that exists in the Itô formula is difficult to interpret physically and is hard to handle for stability analysis. There is also a difficulty in the selection between two well-known descriptions of SDE, an Itô integral equation and a Stratonovich integral equation for a specific application. Moreover since a white noise is unbounded, it fails to describe the model of some applications; therefore, other stochastic processes such as a stationary process are required. Because of these limitations and difficulties, SDE model is not accurate enough to model all application that contains a stochastic disturbance (Sanjari and Tahmasebi). To address the above problem, nonlinear random models have been required to alleviate the problems mentioned above (Wu, 2015). Moreover, nonlinear random model enable some deterministic analysis tools to be applied (Jiao et al, 2015).

Stability results of nonlinear random differential equation (RDE) have been presented in (Bertram and Sarachik, 1959), but some restrictive assumptions and constrains confine the extension of RDE to the range of applications dealing with a stochastic disturbance. However, recently, (Wu, 2015) constructs a general framework to address the stability criteria of nonlinear RDE and presents theorems employing mild assumptions that conclude the stability of RDE based on a Lyapunov approach, which renders extending the applications that nonlinear RDE, especially in the control theory (Jiao et al, 2016, Xia, et al, 2015). However, to the best of the authors' knowledge no work on stability analysis and synthesis of stochastic oscillator system described by perturbed doffing equation especially in Mathematical Finance has been done until now.

However, the rest of this paper is organised as follows; section II presents problem statement and system description and section III Lyapunov methods, IV, gives example using the model of study, section V, Results and Conclusion.

II. PROBLEM STATEMENT AND SYSTEM DESCRIPTION

The volatility of stock and their prices (fluctuations) are stochastic in nature. They are modelled with stochastic oscillators and also are described by perturbed Duffing equation. The stability analysis and synthesis of stochastic oscillators are then required to checkmate the fluctuations in the market.

SDE's have been proven to be an appropriate model to fit the data in many applications, but in some situation, they provide inappropriate model to describe systems that contain stochastic disturbance and therefore suffer some disadvantages (Wu, 2015). For example white noise is driven Wiener process which does not have derivative anywhere, so it is unsuitable to model Fluctuation in practical applications.

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Consider the following nonlinear random system (Osu et al ,2019);

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 + \varepsilon w\gamma u^3 = \varepsilon\mu f(t;\omega) + g(t)N(t), \qquad (1.1)$$

Where $u\epsilon R^n$ is the state vector, $N\epsilon R^1$ *is* F_t-adapted and piecewise continuous stochastic process,$f(t;\omega)\colon R^n \times R_+ \to R^n$ is a known nonlinear function and $g(u,t)\colon R^n \times R_+ \to R^{n-1}$ is an envelope function.

The following assumptions were made;

Assumption 1; The Stochastic process $N(t)$ is F_t-adapted, piecewise continuous such that there exists a positive constant k satisfying

$$Sup_{t\geq t_0} E\{|N(t)|^2\} < k$$

It means that the mean-square of the stochastic process$N(t)$ is bounded by a constant.

Assumption II; The solution$u(t)$ of eq(1.1) is F_t-adapted and satisfies all $t\epsilon[t_0, T]$

$$u(t) = u(t_0) + \int_{t_0}^{T} f(u,s)ds + \int_{t_0}^{T} g(u,s)N(s)ds$$

Assumption III; Nonlinear functions $f(.)g(.)$ vanish at the origin, i.e. $f(0,t) = g(0,t) = 0$ for all $t\epsilon[t_0, \infty]$.

III. LYAPUNOV METHODS

The Lyapunov theory of dynamic systems is the most useful general theory for studying the stability of nonlinear systems. It includes two methods; *(i)* Lyapunov's indirect (reduced) method and *(ii)* Lyapunov's direct method.

i) Lyapunov's indirect (reduced) method or First Lyapunov criterion

Lyapunov's Indirect method states that the dynamical system

$$\dot{u} = f(u)$$

Where $f(0) = 0$, has a locally exponentially stable equilibrium point at the origin if and only if the real parts of the eigenvalues 0f the Jacobian matrix of f at zero are all strickly negative. Considering the autonomous system above, the Jacobian at the equilibrium point can can be defined as:

$$A = \frac{\partial f(u)}{\partial u} \Big|_{u_e=0}$$

For Lyapunov's Indirect method,

- If all eigenvalues of A are strictly in the left-half complex plane (negative real part), then the asymptotic of the linearized system is concluded.
- If at least one eigenvalue of A is strictly in right-half complex plane (positive real part), then the instability of the linearized system is concluded.
- If all eigenvalues of A are in the left-hand complex plane but at least one of them is on the jw-axis or imaginary part, then the linearized system is said to be marginally stable but one cannot conclude anything about the stability of the nonlinear system from the linear approximation. (Panikhom and Sujitjiorn 2010}

In the indirect method, the quadratic Lyapunov function can be generally applied. It can be expressed as;

$$V(u) = u^t P u > 0$$

Where u is the state vector and$P$ is a symmetrically scalar matrix. The following equations must be satisfied:

$$\dot{u} = Au$$
$$\dot{u}^t = u^t A^t$$
$$\dot{V}(u) = u^t P\dot{u} + \dot{u}^t P u$$
$$\dot{V}(u) = u^t PAu + u^t A^t P u$$
$$\dot{V}(u) = u^t (PA + A^t P) u$$
$$\dot{V}(u) = u^t Q u$$

Where $Q = PA + A^t P$ and $Q = Q^t$

Finding the Lyapunov function of eqn (1.1).

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 + \varepsilon w\gamma u^3 = \varepsilon\mu f(t;\omega) + g(t)N(t),$$

With the third assumption above and $\varepsilon = 0$ eqn (1.1) becomes:

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 = 0$$

With linearization the last equation becomes;

$$\dot{u}_{\mathbf{1}} = u_2 \dot{u}_{\mathbf{2}} = -\delta u_2 - w^2 q u_1 - 2w^2 p {u_1}^2$$

In matrix form is

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2q - 2w^2pu_1 & -\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

At the origin, $u = 0$, and $[u_1 \quad u_2] = [0 \quad 0]$, the matrix above becomes;

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2q & -\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Now, $A = \begin{bmatrix} 0 & 1 \\ -w^2q & -\delta \end{bmatrix}$

And choose, $P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, a symmetric matrix.

Checking the definiteness of P by

$$u^tPu = [u_1 \quad u_2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = {u_1}^2 + (u_1 - u_2)^2$$

The last equation shows that P is positive definite. So

$$V(u) = u^tPu > 0$$

Then find the value of $Q = PA + A^tP$ as

$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -w^2q & -\delta \end{bmatrix} + \begin{bmatrix} 0 & -w^2q \\ 1 & -\delta \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2w^2q & 2 + \delta - w^2q \\ 2 - w^2q + \delta & -2 - 2\delta \end{bmatrix}$$

$$\dot{V}(u) = u^tQu = [u_1 \quad u_2] \begin{bmatrix} 2w^2q & 2 + \delta - w^2q \\ 2 - w^2q + \delta & -2 - 2\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \dot{V}(u)$$

$$= 2w^2q{u_1}^2 + 2(1 + \delta - w^2q)u_1u_2 - 2\delta{u_2}^2\dot{V}(u)$$

$$= -2(-2w^2q{u_1}^2 - 2(-1 - \delta + w^2q)u_1u_2 + 2\delta{u_2}^2)$$

Let's choose the values of the scalar variables in the equation to be as follows; $w = 1, q = -2\ and\ \delta = 3$

Then substituting in the last equation becomes:

$$\dot{V}(u) = -2(+2{u_1}^2 + 6u_1u_2 + 3{u_2}^2)$$

The last equation shows the derivative of $V(u)$ is negative definite in the chosen values of the scalar variables and so the system is asymptotically stable at that point.

ii) Lyapunov's direct method or second Lyapunov criterion

Lyapunov's direct method is a mathematical extension of fundamental physical observation that an energy dissipative system must eventually settle down to an equilibrium point. Lyapunov's direct method states that if there is an energy-like function V of

$$\dot{u} = f(u)$$

that is strictly decreasing along its trajectories, then the equilibrium at the origin is asymptotically stable. The function V is said to be a Lyapunov function for the system. A Lyapunov function provides via its pre-images a lover-bound of the region of attraction of the equilibrium. This bound is non-conservative in the sense that it extends to the boundary of the domain of the Lyapunov function. (Törner andFreiling, 2002).

A nonlinear system can be represented by $\dot{u} = f(u, t)$ for a non- autonomous one, and $\dot{u} = f(u)$ for an autonomous system. At equilibrium, $u_e = 0$, the following condition holds $f(u_e) = 0$ and $\dot{u}_e = 0$.

For the Lyapunov's direct method, the stability analysis of an equilibrium point u_0 is done using proper scalar functions called Lyapunov functions defined in the state space. The

second Lyapunov function $V(u)$ must be found and used to conclude the stability region of the system for a nonlinear system without knowing the solution of the governing equations of the system. $V(u)$most be scalar, positive definite and differentiable.
For a nonlinear system to have a globally asymptotically stable equilibrium, the Lyapunov function $V(u)$ must have the following properties;

- $V(u) > 0$
- $\dot{V}(u) < 0$
- $V(u) \to \infty$ as $\|u\| \to \infty$.

Consider the Duffing equation in eqn (1.1) if the quadratic nonlinear term is perturbed instead of the cubic and $\varepsilon = 0$, the equation becomes;

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + w\gamma u^3 = 0$$

The energy function used as the Lyapunov function candidate is

$$V(u) = \frac{1}{2}(\dot{u}^2 + w^2 qu^2 + \frac{1}{2} w\gamma u^4)$$

It can be seen that $V(u)$ is scalar, differentiable, positive definite and unbounded. If $V(u)$ satisfies all the properties, it is said to be the Lyapunov function of the system OF eqn (1.1).

$$\dot{V}(u) = \dot{u}(-a\dot{u}|\dot{u}|) = -a|\dot{u}^2|,$$

the derivative of the Lyapunov function $V(u)$, is negative definite, then the global asymptotic stability of the system is concluded.

Computing the equilibrium points and checking their stability

Consider the equation:

$$\ddot{u}(t;\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 + \varepsilon w\gamma u^3 = \varepsilon\mu f(t;\omega) + g(t)N(t),$$

With the conditions stated earlier, the unperturbed system becomes

$$\ddot{u}(t;\omega) = -\delta\dot{u} - w^2 qu - 2w^2 pu^2$$

The equivalent system of the above unperturbed system is

$$\dot{u}_{\mathbf{1}} = u_2 \dot{u}_{\mathbf{2}} = -\delta u_2 - w^2 qu_1 - 2w^2 p{u_1}^2$$

The equilibrium points of the system are;

$\dot{u}_{\mathbf{1}} = \dot{u}_{\mathbf{2}} = 0$, i.e. (0, 0)

$-\delta u_2 - w^2 qu_1 - 2w^2 p{u_1}^2 = 0$

$-w^2 qu_1 - 2w^2 p{u_1}^2 = 0$

Since $-\delta u_2 = 0$

$\lambda_{1,2} = \frac{w^2 q \pm \sqrt{(-w^2 q)^2}}{-4w^2 p}$, $\lambda_1 = 0, \lambda_2 = \frac{q}{-2p}$

So the system has only two equilibrium points; (0, 0) and $(\frac{q}{-2p}, 0)$

Let

$$\dot{u}_{\mathbf{1}} = u_2 = f_1(u_1, u_2)$$
$$\dot{u}_{\mathbf{2}} = -\delta u_2 - w^2 qu_1 - 2w^2 p{u_1}^2 = f_1(u_1, u_2)$$

The Jacobian matrix,

$$J(u_1, u_2) = \begin{bmatrix} 0 & 1 \\ -w^2 q - 4w^2 pu_1 & -\delta \end{bmatrix}$$

With $\varepsilon = 0$, the system has centres at $(u_1, u_2) = (0, 0) and\ (\frac{q}{2p}, 0)$

Examining the stability using eigenvalues approach

$$|A - I\lambda| = \begin{vmatrix} 0-\lambda & 1 \\ -w^2q - 4w^2pu_1 & -\delta-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \delta\lambda + w^2q + 4w^2pu_1 = 0$$

$$\lambda_1 = \frac{-\delta + \sqrt{\delta^2 - 4w^2(q + 4pu_1)}}{2}$$

$$\lambda_2 = \frac{-\delta - \sqrt{\delta^2 - 4w^2(q + 4pu_1)}}{2}$$

Now, the values of λ for which the system achieves stability is specified.

For $u_1 = 0, \lambda_1 = \frac{-\delta+\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_1 = 0$, if, $w = 0$ and δ, q and $p = R$, the real line

$\lambda_1 > 0$if $q \leq 0$, $\delta \leq 0$and $p = R$, the real line

$\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \leq 0$

$\lambda_1 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$

For $u_1 = \frac{q}{-2p}, \lambda_1 = \frac{-\delta+\sqrt{\delta^2+4w^2q}}{2}$

$\lambda_1 = 0$, if, $w = 0$ or $q = 0$ and $\delta = R$, the real line

$\lambda_1 > 0$if$q > 0$,and$\delta = R$, the real line

$\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \leq 0$

$\lambda_1 = a \pm ib$, complex numbers if $q < 0$, $\delta > 0$ and $\delta^2 < 4w^2q$

For $u_1 = 0, \lambda_2 = \frac{-\delta-\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_2 = 0$, if, $w = 0$ or $q = 0$ and $\delta = 0$

$\lambda_2 > 0$, if, $\delta \leq 0$ and $\sqrt{\delta^2 - 4w^2q} \leq 0$

$\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \geq 0$

$\lambda_2 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$

For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta-\sqrt{\delta^2+4w^2q}}{2}$

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$\lambda_2 > 0$, if, $\delta < 0$, $and \sqrt{\delta^2 + 4w^2q} \leq 0$

$\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \geq 0$

$\lambda_2 = a \pm ib$, complex numbers, if,$q < 0$, $\delta > 0$ and $\delta^2 < 4w^2q$

Since the system $S: R^n \to R^n$ is nonlinear, with continuous first derivative and $\bar{u}_e$ is a critical point of the nonlinear system $u' = f(u)$

1) If all eigenvalues of the Jacobian matrix $J(\bar{u}_e)$ have negative real parts, then the critical point $\bar{u}_e$ is asymptotically stable.
2) If any eigenvalue of the Jacobian matrix $J(\bar{u}_e)$ has positive real part or zero, then the critical point $\bar{u}_e$ is unstable[Ledder,2005].

Analysis

The interest is on the eigenvalues of the Jacobian matrix that have negative real parts for which the critical points $\bar{u}_e$ is asymptotically stable. They are,

[A] For $u_1 = 0, \lambda_1 = \frac{-\delta+\sqrt{\delta^2-4w^2q}}{2}$

[A]i) $\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \leq 0$

$\Rightarrow \delta^2 \leq 4w^2q$, since $\delta > 0$, then $q > 0$ and $w > R$

Since w is the natural frequency of the system, it will always be positive and so $w\epsilon R^{+}$, q being the extent of resistance to the deformation in response to the external force that causes the push or pull in the market shows resistance and then keeps the market prices stable. δ,the economic damping due to speculations is kept in check since $\delta^2 \leq 4w^2q$.
[A]ii) $\lambda_1 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$ and $a < 0$
Since $\delta^2 < 4w^2q$, it implies like in [A]i) that the extent of economic damping due to speculations is also kept in check and so the stability of the market prices hold.

[B] For $u_1 = \frac{q}{-2p}, \lambda_1 = \frac{-\delta+\sqrt{\delta^2+4w^2q}}{2}$

[B]i) $\lambda_1 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \leq 0$
$\Rightarrow \delta^2 \leq -4w^2q$
From the last equation, since $\delta > 0$, then $q < 0$. It means that with the state vector as $\frac{q}{-2p}$, there is damping due to the speculations and negative resistance to the deformation in response to the external force that causes the push or pull in the market shows resistance and so stability cannot be achieved.
[B]ii) $\lambda_1 = a \pm ib$, complex numbers if $q < 0$, $\delta > 0$, $\delta^2 < 4w^2q$ and $a < 0$
With $q < 0$and $\delta > 0$, then $\delta^2 < 4w^2q$is undefined.

[C]For $u_1 = 0, \lambda_2 = \frac{-\delta-\sqrt{\delta^2-4w^2q}}{2}$

[C]i) $\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 - 4w^2q} \geq 0$
This gives the same result as [B]i).
[C]ii) $\lambda_2 = a \pm ib$, complex numbers if $\delta^2 < 4w^2q$and $a < 0$
The result here is the same as that of [A]i)

[D] For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta-\sqrt{\delta^2+4w^2q}}{2}$

[D]i) $\lambda_2 < 0$, if $\delta > 0$ and $\sqrt{\delta^2 + 4w^2q} \geq 0$
It gives the same result as [B]i)
[D]ii) $\lambda_2 = a \pm ib$, complex numbers, if, $q < 0$, $\delta > 0$, $\delta^2 < 4w^2q$ and $a < 0$
This holds the same result as [A]i) and [A]ii).

Summary
The summary of the only eigenvalues, λ_i's and the state vector u_1 the achieved stability are as follows;

i) For $u_1 = 0, \lambda_1 = \frac{-\delta+\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_1 < 0$and$\lambda_1 = a + ib$

ii) For $u_1 = 0, \lambda_2 = \frac{-\delta-\sqrt{\delta^2-4w^2q}}{2}$

$\lambda_2 = a + ib$

iii) For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta-\sqrt{\delta^2+4w^2q}}{2}$

$\lambda_2 = a + ib$

Synthesis of the Stochastic Oscillator system described by perturbed Duffing Equation

The Lyapunov exponent and the eigenvalues of the Jacobian matrix are used to perform the synthesis. The Lyapunov exponent is used to test the convergence of the nearby trajectories while the eigenvalues are used with the same parameter values as in the Lyapunov exponent to test the stability of the system. The values of the parameters for which convergence is achieved using the Lyapunov exponent,

$$\lambda_1(t) = \frac{\varepsilon\delta\sin(\theta)^2}{2} - \frac{\varepsilon\delta\cos(\theta)^2}{2} + \frac{\varepsilon\delta\cos(\theta)^2 t}{2} + \frac{\varepsilon\delta\sin(\theta)^2 t}{2} - \frac{\sin(2\theta)(t + \frac{\mu\sin(\omega t)}{\omega} - w^2qt - 3w\gamma u^2 t - 4\varepsilon w^2 ut}{2}$$

$$\lambda_2(t) = \frac{-\varepsilon\delta\sin(\theta)^2}{2} + \frac{\varepsilon\delta\cos(\theta)^2}{2} + \frac{\varepsilon\delta\cos(\theta)^2 t}{2} + \frac{\varepsilon\delta\sin(\theta)^2 t}{2} - \frac{\sin(2\theta)(t + \frac{\mu\sin(\omega t)}{\omega} - w^2qt - 3w\gamma u^2 t - 4\varepsilon w^2 ut}{2}$$

are;

$\varepsilon = 0.01, \delta = 0.01, \theta = 30, t = 90, w = 0.4, p = 240, \mu = 1, q = 0.7, and\ \ \omega = 180, u, \gamma$

With the above values of the parameters, $\lambda_1$and λ_2, are negative for all values of $u\ and\ \gamma$ except where u is zero. Here $\lambda_1$and λ_2, are positive showing non-convergence.

Now using the eigenvalues and the same parameters of the Lyapunov exponent $\delta = 0.01, q = 0.7\ and\ \ w = 0.4$

For $u_1 = \frac{q}{-2p}, \lambda_2 = \frac{-\delta - \sqrt{\delta^2 + 4w^2 q}}{2}$

$\lambda_2 = -0.6794$, with the above parameter values.

Since λ_2 of the Jacobian matrix $J(u_e)$ has negative real part, then the critical point u_e is asymptotically stable.

Conclusion:

This paper has demonstrated the stability analysis of the market price fluctuations using the Lyapunov's direct and indirect methods. The derivative of the Lyapunov's function was excited parametrically and it was found that with some values, the derivative achieved negative definiteness which shows complete asymptotic stability. The Lyapunov exponent and the eigenvalues of the Jacobian matrix are used to perform the synthesis.

Recommendation;

We recommend the use of other methods of stability to study the market model. These methods can also be compared with each other to see the best method.

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