ADOMIAN DECOMPOSITION METHOD FOR APPROXIMATING THE SOLUTIONS OF HYPERBOLIC ES

by

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Abstract

Most physical systems can be modelled intodifferential equations (hyperbolic, parabolic, elliptic equations, etc.) So the solutions of such equations are of interest. In this paper, the Adomian decomposition method for approximating the solutions of hyperbolic equations is implemented. The approximate solutions are calculated in the form of convergent series with easily computable components. In comparison with existing techniques, the decomposition method is highly effective in terms of accuracy and rapid convergence. The numerical results obtained by this method have been compared with the exact solutions to show the accuracy of the method.

Keywords*: Adomian decomposition method (ADM), hyperbolic equations.*

1. Introduction

Prior to the advent of digital computers, engineers relied on analytical or exact solutions of differential equations. Aside from the simplest cases these solutions often required a great deal of effort and mathematical sophistication. In addition, many of such physical systems could not be solved directly but had to be simplified using linearization, simple geometric representations, and other idealizations for solutions. Although these solutions are elegant and yield insight, they are limited with respect to how faithfully they represent real systems-especially those that are highly nonlinear and irregularly shapedSteven and Raymond (1988). The decomposition method was introduced by Adomian (1989, 1994) for solving linear and nonlinear functional equations(algebraic, ordinary and partial differential equations, etc.) Manjak and Kwami (2008),Mustafa (2005), Shaher (2008). This method leads to computable, accurate, approximately convergent solutions to linear and non-linear deterministic and stochastic operation equations. The solution can be verified to any degree of accuracy Manjak and Kwami (2008). The technique has many advantages over the classical techniques, mainly, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions Shaher (2008), Javidi and Golbabai (2007).

2. Analysis of the method

In this section, we demonstrate the main algorithm of ADM for linear and nonlinear hyperbolic equations with initial conditions, we consider the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} + N(u) + g(x, y) \qquad ...(2.1)$$

with the following initial conditions

$$u(x, 0) = f(x) \qquad ...(2.2)$$

$$\frac{\partial u(x,0)}{\partial y} = g(x) \qquad ...(2.3)$$

Where, N is a function of u. We are looking for the solution satisfying equation (2.1), and conditions (2.2) and (2.3). The decomposition method consists of approximating the solution of (2.1)- (2.3) as an infinite series

$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)$ ...(2.4)

Decomposing N (the nonlinear operator) as

$N(u)=\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n)$ ...(2.5)

Where the A_n's are the Adomian polynomials Adomian (1989), and are calculated owing to the basic formula, Adomian et al (1994), Adomian (1991).

$$A_n = A_n(u_0,u_1,\ldots,u_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0} \quad n=1,\ 2,\ 3\ldots$$ --(2.6)

From which

$$A_0 = Nu_0$$

$$A_1 = u_1\left(\frac{d}{du_0}\right)N(u_0)$$

$$A_2 = u_2\left(\frac{d}{du_0}\right)N(u_0)+\left(\frac{u_1^2}{2!}\right)\left(\frac{d^2}{du_0^2}\right)N(u_0)$$

$$A_3 = u_3\left(\frac{d}{du_0}\right)N(u_0)+u_1u_2\left(\frac{d^2}{du_0^2}\right)N(u_0)+\left(\frac{u_1^3}{3!}\right)\left(\frac{d^3}{du_0^3}\right)N(u_0)$$

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The A_n can be written in the following convenient way

$A_n=\sum_{v=1}^{n}c(v,n)f^{v}(u_0),\ \ n\geq 1$ Manjak (2006).

Applying the decomposition method, Mustafa (2004), Manjak (2006) it is convenient to re-write Eq. (1) in the standard operator formas $L_{xx}u = L_{yy}u + Nu + g(x,y)$ ...(2.7)

Where, $L_{xx}=\frac{\partial^2}{\partial x^2}$, $L_{yy}=\frac{\partial^2}{\partial y^2}$ and N is the nonlinear operator.

The decision as to which operator to solve in a multidimensional problem is made on the basis of the best- known conditions and possibly also on the basis of the operator of the lowest order to minimize integration Bellman and Adomian (1984).

We therefore solve for $L_{yy}u$in (2.7) to obtain

$L_{yy}u=-g(x,y)-Nu+l_{xx}u$ ...(2.8)

The inverse operator $L_{yy}^{-1}$of L_{yy} exists and it can conveniently be taken as the definite integral with respect to y from 0 to y, i.e., $L_{yy}^{-1}(.)=\int_0^y\int_0^y(.)dydy$ which is a two-fold definite integral since $L_{yy}$is a second-order operator.

Operating with L_{yy}^{-1} on both sides of (2.8) yields

$L_{yy}^{-1}L_{yy}u=-L_{yy}^{-1}g(x,y)-L_{yy}^{-1}N(u)+L_{yy}^{-1}L_{xx}u$ ...(2.9)

So that

$u(x,y)=u(x,0)+\frac{\partial u(x,0)}{\partial y}y-L_{yy}^{-1}g(x,y)-L_{yy}^{-1}N(u)+L_{yy}^{-1}L_{xx}u$ ...(2. 10)

Substitute initial conditions (2.2) and (2.3) into Eq. (2.10) to have

$u(x,y)=f(x)+g(x)y-L_{yy}^{-1}g(x,y)-L_{yy}^{-1}N(u)+L_{yy}^{-1}u$ ...(2.11)

Substituting (2.4) and (2.5) into (2.11) we have

$u(x,y) = f(x) + g(x)y - L_{yy}^{-1}g(x,y) - L_{yy}^{-1}(\sum_{n=0}^{\infty} A_n) + L_{yy}^{-1}\, L_{xx}(\sum_{n=0}^{\infty} u_n)$ ...(2.12)

From (2.12) the Adomian decomposition scheme is defined by the recurrent relation

$$u_0 = f(x) + g(x)y - L_{yy}^{-1}g(x,y)$$

and $u_{n+1} = -L_{yy}^{-1}A_n + L_{yy}^{-1}L_{xx}u_n$, for n=0,1,2,...

From which $u_1 = -L_{yy}^{-1}A_0 + L_{yy}^{-1}L_{xx}u_0$

$$u_2 = -L_{yy}^{-1}A_1 + L_{yy}^{-1}L_{xx}u_1$$

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We can determine the components $u_n$as many as is necessary to enhance the desired accuracy for the approximation. So, the n-terms approximation$\varphi_n = \sum_{i=0}^{n-1} u_i$can be used to approximate the solution.

3. Applications

In this section, we consider the application of the Adomian decomposition method to the Eq. (2.1) with initial conditions by considering two examples.

Example 3.1

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + (1-2x)\frac{\partial^2 u}{\partial x \partial y} + (x^2 - x - 2)\frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(3.1)$$

Subject to initial conditions, $u(x,0) = x, \frac{\partial u(x,0)}{\partial y} = 1$ Biazar and Ebrihimi(2005)

Rewrite (3.1) in the operator form as

$$L_{xx}u + (1 - 2x)L_xL_yu + (x^2 - x - 2)L_{yy}u = 0$$

From which we can have

$$(x^2 - x - 2)L_{yy}u = - L_{xx}u - (1 - 2x)L_xL_yu \qquad ....(3.2)$$

Where $L_{yy} = \frac{\partial^2}{\partial y^2}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_xL_y = \frac{\partial^2}{\partial x \partial y}$

From equation (3.2)

$$L_{yy}u = -\left(\frac{1}{x^2 - x - 2}\right)L_{xx}u - \left(\frac{1-2x}{x^2 - x - 2}\right)L_xL_yu \qquad ...(3.3)$$

Operate with L_{yy}^{-1} on equation (3.3) to have

$$L_{yy}^{-1}L_{yy}u = -L_{yy}^{-1}\left(\frac{1}{x^2 - x - 2}L_{xx}u\right) + L_{yy}^{-1}\left(\frac{2x-1}{x^2 - x - 2}\right)L_xL_yu \qquad .(3.4)$$

Evaluate L.H.S. of equation (3.4) to have

$$L_{yy}^{-1}\, L_{yy}u = u(x,y) - u(x,0) - yu_y(x,0)$$

and substitute back into equation (3.4) which yields

$$u(x,y) = u(x,0) + yu_y(x,0)$$

$$- L_{yy}^{-1}\left(\frac{1}{x^2 - x - 2}L_{xx}\sum_{n=0}^{\infty}u_n\right) + L_{yy}^{-1}\left(\frac{2x-1}{x^2 - x - 2}L_xL_y\sum_{n=0}^{\infty}u_n\right) \qquad ...(3.5)$$

From (3.5) and by the Adomian decomposition scheme,

$u_0 = u(x,0) + yu_y(x,0)$

And

$$u_{n+1} = -\, L_{yy}^{-1}\left(\frac{1}{x^2-x-2}L_{xx}u_n\right) + L_{yy}^{-1}\left(\frac{2x-1}{x^2-x-2}\right)L_xL_yu_n, \qquad n \geq 0$$

From which,

$$u_1 = -\, L_{yy}^{-1}\left(\frac{1}{x^2-x-2}\right)L_{xx}u_0 + L_{yy}^{-1}\left(\frac{2x-1}{x^2-x-2}\right)L_xL_yu_0$$

$u(x,0) = x, \quad u_y(x,0) = 1$

Therefore $\quad u_0 = x + y(1) = x + y$

$$u_1 = -\, L_{yy}^{-1}\left(\frac{1}{x^2-x-2}\right)L_{xx}u_0 + L_{yy}^{-1}\left(\frac{2x-1}{x^2-x-2}\right)L_xL_yu_0$$

$u_1 = \quad 0 \quad + \quad 0$

$u_1 = 0$

The solution is $u(x,y) = x + y$, which is the exact solution.

Example 3.2

Consider the first order hyperbolic nonlinear problem of the form:

$u_t(x,t)= uu_x(x,t)$, in $0 < x \leq 1$, $0 \leq t \leq T$ ...(3.6)

With the initial condition $u(x,0) = g(x)$, for $0 < x \leq 1$

We let $g(x) = 0.1x$ Bellman and Adomian (1984).

So that the exact solution is (Bellman and Adomian (1984)

$$u(x,t) = {}^{x}/_{(t-10)}$$

Rewrite equation (3.6) in the operator form as

$L_t u = Nu$

...(3.7)

Where, $L_t = \frac{\partial u}{\partial t}$ $\quad and \;\; Nu = \sum_{n=0}^{\infty} A_n = uu_x \quad [Nu \; is \; a \; product \; nonlinearity]$

Operate with L_t^{-1} on (3.7) to have

$L_t^{-1}L_t u = L_t^{-1}Nu$

...(3.8)

Evaluate L.H.S of (3.8) to have

$L_t^{-1}L_t u = u(x,t) - u(x,0)$

And substitute back into (3.8) to have

$$u(x,t) = u(x,0) + L_t^{-1}\sum_{n=0}^{\infty} A_n \qquad ...(3.9)$$

By using the Adomian scheme, we get the recurrent relation

$u_0 = u(x,0) = 0.1x$ and

$u_{n+1} = L_t^{-1}A_n$, $\quad n = 0, 1, 2,...$

From which,

$u_1 = L_t^{-1} A_0$

$u_2 = L_t^{-1} A_1$

$u_3 = L_t^{-1} A_2$

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Therefore $u_1 = L_t^{-1}A_0$
But, $Nu = f(u, u_x) = uu_x$, $Nu_0 = f(u, u_{ox)} = u_0 u_{0x} = A_0$
Thus $u_1 = L_t^{-1}A_0$

$$= L_t^{-1}(u_0 u_{0x})$$

$=L_t^{-1}(0.1x)(0.1)$
$= L_t^{-1}(0.01x) \qquad = 0.01xt$

To compute the remaining Adomian polynomials which will enable us compute the remaining u terms, we employ the relation
$A_n = \sum_{i=1}^{n} C(i,n) f^i(u_0)$, n = 1, 2, 3...
From which $A_1 = \sum_{i=1}^{1} C(i,1) f^i(u_0)$
$= C(1,1) f'(u_0)$
But this is a case of product non linearity, therefore,

$$A_1 = C(1,1) f'(u_o u_{0x})$$
$$A_1 = u_1(u_0 + u_{0x})$$
$$= (0.01x)(0.1x) + (0.01xt)(0.1)$$
$$= 0.001x^2 t + 0.001xt$$

Thus $\quad u_2 = L_t^{-1}A_1 = L_t^{-1}(0.001x^2 t + 0.001xt)$

$$= \frac{0.001x^2 t^2}{2} + 0.001xt^2/2$$

$$A_2 = \sum_{i=1}^{2} C(i,2) f^i(u_0\, u_{0x})$$
$$= C(1,2) f'(u_0 u_{0x}) + C(2,2) f''(u_0 u_{0x})$$
$$= u_2(u_0 + u_{0x}) + u_1^2/2!\,(1+1)$$
$$= u_2 u_0 + u_2 u_{0x} + u_1^2$$
$$= \left(\frac{0.001x^2 t^2}{2} + \frac{0.001xt^2}{2}\right)(0.1x) + \left(\frac{0.001x^2 t^2}{2} + \frac{0.001xt^2}{2}\right)(0.1) + (0.01xt)^2$$
$$= \frac{0.0001x^3 t^2}{2} + 2(0.0001x^2 t^2) + \frac{0.0001xt^2}{2}$$

But $\quad u_3 = L_t^{-1}A_2$
Substitute A_2 to have $u_3 = L_t^{-1}(\frac{0.0001x^3 t^2}{2} + 2(0.0001x^2 t^2) + \frac{0.0001xt^2}{2}$

$$= \frac{0.0001x^3 t^3}{6} + \frac{0.0002x^2 t^3}{3} + 0.0001xt^3/6$$

And $\quad u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$
$= 0.1x + 0.01xt + \frac{0.001xt^2}{2} + \frac{0.001x^2t^2}{2} + \frac{0.0001xt^3}{6} + \frac{4(0.0001x^2t^3)}{6} + \frac{0.0001x^3t^3}{6} +\ldots \qquad = \frac{0.1x}{0!} + \frac{0.01xt}{1!} + (0.001x + 0.00x^2)\frac{t^2}{2!} + \frac{(0.0001x + 0.0004x^2 + 0.0001x^3)t^3}{3!} +\ldots$
serves as the analytic solution. In a closed form the solution can be derived as $u(x,t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{n!}$, where $f_n(x) = \sum_{i=1}^{n} (0.1)^{n+1} x^i$ and take $f_0(x) = (0.1)x$.

4. Numerical implementation of ADM

In order to verify numerically whether the proposed methodology lead to accurate solutions, we will evaluate the absolute ADM solutions using the n-terms approximations for example 3.2 and the results compared with the exact solutions.We report absolute error which is defined by$|u_E(x_i,y_i)-u_A(x_i,y_i)|$as shown in the table below, where u_E is the exact solution and $u_A=\sum_{n=0}^{\infty}u_n(x,y)$is the Adomian solution. For n=4 we achieved a very good approximation with the exact solution. However, many other iterates can be generated using Matlab in order to achieve a high level of accuracy of the decomposition method.

Table: Absolute error for test problem 3.2 for various values of x and t with n=4

X	**T**	**Absolute Adomian Solution (u_A)**	**Exact Solution (u_E)**	**Absolute Error ($\lvert u_E-u_A\rvert$)**
0.1	0.001	0.010001	0.010001	4.50076E-11
0.2	0.002	0.020004	0.020004001	3.20111E-10
0.3	0.003	0.030009002	0.030009003	9.45501E-10
0.4	0.004	0.040016004	0.040016006	1.92138E-09
0.5	0.005	0.050025009	0.050025013	3.12787E-09
0.6	0.006	0.060036017	0.060036022	4.32485E-09
0.7	0.007	0.070049029	0.070049034	5.15186E-09
0.8	0.008	0.080064046	0.080064051	5.12795E-09
0.9	0.009	0.090081069	0.090081073	3.65151E-09
1	0.01	0.1001001	0.1001001	1.00073E-13

5. Conclusion

The goal of this work had been to derive an approximation for the solutions of hyperbolic equations with initial conditions and the results compared with the exact solutions. We have achieved this goal by applying Adomian decomposition method.

Example 3.1 is a linear hyperbolic equation. After some steps the remaining terms would vanish and sum of the non-zero terms gives exactly the exact solution, which shows the reliability of the method. In example 3.2, a closed form of the analytic solution was obtained. As indicated in the table comparisons with exact solutions were made in terms of absolute errors and small error of the method in comparison with the exact solution using only four terms of the approximations were obtained, which shows the accuracy of the method.

We have demonstrated that the decomposition methodology displays a fast convergence of the solution. In addition, the numerical results obtained by this method have high degree of accuracy.

It may be concluded that the Adomian methodology is a very powerful and efficient technique in finding exact and approximate solutions for hyperbolic physical problems and also a wide class of other problems. One hopes therefore, that physically more realistic, accurate results and predictions can be obtained using the Adomian method for solving physical problems with initial/boundary conditions.

References

Adomian G. (1989). Nonlinear Stochastic Systems. The Theory and Application to Physics. Kluwer Academic Publ.

Adomian G. (1991). A Review of the Decomposition Method and Some Recent Results for Nonlinear Equations Comp. Math. Appl. 21(5):101-127.

Adomian G. (1991). An Analytic Solution of Stochastic Navier-Stokes Problem. Foundation Physics 21(7):831-843.

Adomian G. (1994). Solving Frontier Problems on Physics. The Decomposition Method. Kluwer Academic publisher.

Adomian G. (1994). Solution of Physical Problems by Decomposition. Comp. Math. Appl. 27(9-10):145-154.

Biazar J.and Ebrahimi (2005). An Approximation to the Solution of Hyperbolic Equations by Adomian Decomposition Method and Comparison with Characteristics Method. Applied Mathematics and Computation 163(2005):633-638.

Javidi M. and Golbabai A. (2007). Adomian Decomposition Method for Approximating the Solution of Parabolic Equations. Applied Mathematical Sciences 1(5):219-225.

Manjak N. Haggai (2006). Ph.D. Thesis. Solutions of Physical Problems using Adomian Decomposition Method.

Manjak N. H. and Kwami A. M. (2008). On Numerical Solutions for Partial Differential Equations by Decomposition Method. Science Forum. J. Pure and Appl. Sci. 11(1):53-57.

Mustafa INC (2005). Decomposition Method for Solving Parabolic Equations in Finite Domains. Inc/J Zhejiang Univ. SCI. 6A(10):1058-1064.

Richard Bellman and George Adomian (1984). Partial Differential Equations .New Methods for Their Treatment and Solutions. Mathematics and Its Applications. D. Reidal Publishing Company.Dordrecht/Boston/Lancaster.

Shaher Momani (2008). A decomposition Method for Solving Unsteady Convection-Diffusion Problems. Turk J. Math 32(2008):51-60.

Steven C.C.and Raymond P.C. (1988). Numerical Methods for Engineers. MGraw-Hill, Inc.