ON THE IDENTITY GRAPH OF SYMMETRIC GROUP OF DEGREE FOUR (S_4)

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Abstract
Let G be a finite group. The identity graph of G denoted by $\Gamma(G)$ is a rooted tree where the centre (root) of the graph is the vertex which corresponds to the identity in G. In this paper, we represent the symmetric group S_4 in the form of an identity graph with 24 vertices and 30 edges. This identity graph turned out to be union of nine lines and seven triangles. From this graph we derive so many properties of S_4 by using the idea of general graph properties such as degree of vertex, clique, colouring, independent and dominating sets. We have shown by colouring vertices of the associated identity graph of S_4, elements having the same order, nature of the elements, conjugate elements and centre of S_4. In each case the independent sets partitioned S4. Finally, we were able to establish some results on the identity graph related to the symmetric group of degree n (S_n) and any finite group G in general.

Keywords: *Identity graph, Symmetric group of degree four, graph colouring, clique, vertex.*

1. Introduction

Recently many studies tend to find a relation between group theory and graph theory and that led to the introduction of numerous type of graphs.

In [7] the notion of graph related to finite group called identity graph was introduced. An identity graph is a rooted tree consisting of lines or triangles or both where the centre (root) of the graph is the vertex which corresponds to the identity element of a group. Two distinct vertices are joined by an edge if they correspond to mutual inverse elements in the group. Some types of finite groups in terms of identity graphs were represented as examples and from the structure of the graphs some properties of finite groups were studied.

In this research we represent the symmetric group S_4 in the form of an identity graph with 24 vertices and 30 edges. This identity graph turned out to be union of nine lines and seven triangles. From this graph we derive so many properties of S_4 by using the idea of general graph properties such as degree of vertex, clique, coloring, independent and dominating sets.

2. Preliminaries

In this section we give definition of some basic terms and relevant results needed for the understanding of this paper. For the definitions of the basic terms and results given here refer to (Chalaphathi and Kumar (2018), Vitaly (2009), Beineke and Wilson (2004), David and Richard (2004), Herstein (2003), Godsil and Boyle (2001) and Bondy and Murty (1982)).

Definition (Graph): A graph Γ is a mathematical structure (V, E), where V is the set of vertices viewed as points and E is the set of edges viewed as line joining the points.
Definition (Degree of Vertex): The degree of a vertex v in a graph denoted by $\delta(v)$ is the number of edges incident to it. That is, the number of edges connecting it. If the degree of the vertex is odd the vertex is then called an odd vertex. While if the degree of the vertex is even then the vertex called an even vertex.
Definition (Complete Graph): A complete graph denoted by K_n is a graph with n vertices where each pair of distinct vertices is connected by an edge.
Definition (Simple Graph): A graph is simple if it has no loops or multiple edges. A loop is an edge (directed or undirected) that connects a vertex to itself. Multiple edges refer to more than one edge connecting same pair of vertices. That is a graph is simple if no two edges have the same endpoints.
Definition (Tree): A tree is a simple graph in which two vertices are connected by exactly one edge. It is a special class of graphs.
Definition (Rooted Tree): A tree in which one vertex called the root is distinguished from all the others is said to be a rooted tree.
For example;

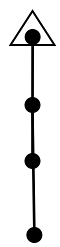


Fig. 1(a)

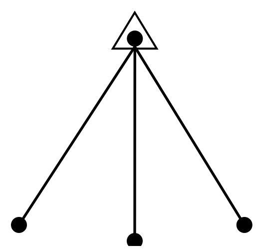


Fig. 1(b)

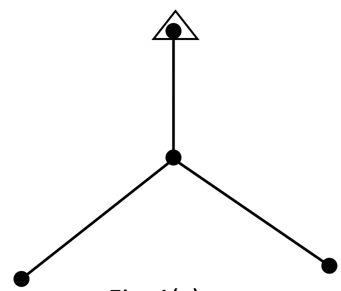


Fig. 1(c)

Figure 1(a)-(c) above gives rooted trees with four vertices.

Definition (Centre of a Graph): A vertex is said to be the centre of graph if every vertex of the graph has an edge with that vertex. For rooted trees the special vertex (root) is the centre.
For example;

x

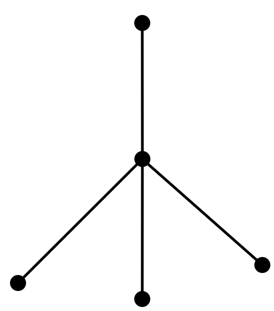


Figure 2 x is the centre of the graph

Definition (Subgraph): A subgraph of a graph Γ is a graph whose sets of vertices and edges are respectively subsets of the set of vertices and edges of Γ.

Definition (Clique): A subgraph of a graph that is a complete graph is called a clique.

Definition (Graph Coloring): A coloring of a graph is an assignment of colours to the vertices of a graph so that no two adjacent vertices have the same colour.

Definition (Independent Set): A subset S of set of vertices V in a graph that are assigned a particular color form an independent set.

Definition (Dominating Set): A dominating set is a subset D of a set of vertices V such that every vertex not in D is adjacent to at least one member of D.

Definition (**Group) :** A non-empty set G is said to form a group if $*$ is an associative binary operation on G such that:

(i) There exists unique element $e \in G$ called identity such that $\forall g \in G$, $g*e = e*g = g$ (existence of an identity in G).

(ii) For each $g \in G$ there is an element $g^{-1} \in G$ called inverse such that $g*g^{-1} = g^{-1}*g = e$ (existence of unique inverse for each element in G).

Definition (**Order of a Group)**: The number of elements in a group G is called the order of the group and is denoted by |G|. When the order is finite the group is called finite group.

Definition (**Order of an Element):** Let G be a group with an identity e and $g \in G$. A smallest positive integer m such that $g^m = e$ is the order of g denoted by o(g).

Definition (**Self and Mutual Inverse Elements in a Group**): Let $(G, *)$ be a finite group with identity e. Then an element $g \in G$ is called a self-inverse element if $g = g^{-1}$, where g^{-1} is the inverse of g in G. The set of self-inverse elements in G is denoted by SI(G). An element $h \in G$ is called mutual inverse element if there exists $k \in G$ such that $h*k = k*h = e$. The set of mutual-inverse elements in G is denoted by MI(G). In particular, $MI(G) = \{h \in G: h \neq h^{-1}\}$.

Definition (Abelian Group): A group $(G, *)$ is called abelian if $g_1*g_2 = g_2*g_1$ for all $g_1, g_2 \in G$.

Definition (Cyclic Group): Let G be a group with an identity e and $g \in G$. Then G is called cyclic if every element in G can be expressed in terms of g. That is $G = \langle g \rangle = \{g^m: m \in \mathbb{Z}\}$. If G is finite then $G = \{e, g, g^2, \ldots g^{m-1}\}$.

Definition (Symmetric Group): Let $X = \{1,2,\ldots,n\}$ and S_n be the set of all one to one mappings of X onto itself. S_n is a group under the composition of mappings and is called the permutation group or symmetric group of degree n.

Definition (Subgroup): Let $(G, *)$ be a group and H be a non-empty subset of G. If $(H, *)$ is a group then we call H a subgroup of G.

Definition (Centre of a Group): The set of all elements of a group G that commute with every element in G is called centre of G and is denoted by Z(G).

Theorem (Sylow Theorem): Let G be a finite group of order $p^\alpha. m$, where p is prime and p does not divide m. Then (i) G has Sylow p-subgroups of order p^α (ii) the number of Sylow p-subgroups of G is of the form $r = 1 + kp$ and $r \mid |G|$.

(ii) Any two Sylow p-subgroups of G are conjugate.

Theorem (THEOREM 2.1 in [7]**):** Let G be a group and G_i denote the identity graph related to G. Every subgroup of G has an identity graph which is a special identity subgraph of G_i but a subgraph of G_i need not in general be associated with a subgroup of G.

3. The Symmetric Group S_4

Symmetric group of degree four is a non-abelian group of order twenty four with 30 subgroups. We list the elements in form of cycle type and denote them by α_i for i = 1, 2. ..., 24 in Table 1:

Table 1

Elements	Type	Number
α_1= (1) (2) (3) (4) = identity	1-cycles	1
α_2 = (1 2), α_3 = (1 3), α_4 = (1 4) α_5 = (2 3), α_6 = (2 4), α_7 = (3 4)	2-cycles	6
α_8 = (1 2) (3 4), α_9 = (1 3) (2 4), α_{10} = (1 4) (2 3)	Product of 2-cycles (double transpositions)	3
α_{11} = (1 2 3), α_{12} = (1 2 4), α_{13} = (1 3 2) α_{14} = (1 3 4), α_{15} = (1 4 2), α_{16} = (1 4 3) α_{17} = (2 3 4), α_{18} = (2 4 3)	3-cycles	8
α_{19} = (1 2 3 4), α_{20} = (1 2 4 3), α_{21} = (1 3 2 4) α_{22} = (1 3 4 2), α_{23} = (1 4 2 3), α_{24} = (1 4 3 2)	4-cycles	6

Observe that: $\alpha_1^{-1} = \alpha_1$, $\alpha_2^{-1} = \alpha_2$, $\alpha_3^{-1} = \alpha_3$, $\alpha_4^{-1} = \alpha_4$, $\alpha_5^{-1} = \alpha_5$, $\alpha_6^{-1} = \alpha_6$, $\alpha_7^{-1} = \alpha_7$, $\alpha_8^{-1} = \alpha_8$, $\alpha_9^{-1} = \alpha_9$, $\alpha_{10}^{-1} = \alpha_{10}$, $\alpha_{11}^{-1} = \alpha_{13}$, $\alpha_{12}^{-1} = \alpha_{15}$, $\alpha_{14}^{-1} = \alpha_{16}$, $\alpha_{17}^{-1} = \alpha_{18}$, $\alpha_{19}^{-1} = \alpha_{24}$, $\alpha_{20}^{-1} = \alpha_{22}$, $\alpha_{21}^{-1} = \alpha_{23}$.

Hence, SI (S_4) = $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}\}$ and

MI(S_4) = $\{(\alpha_{11}, \alpha_{13}), (\alpha_{12}, \alpha_{15}), (\alpha_{14}, \alpha_{16}), (\alpha_{17}, \alpha_{18}), (\alpha_{19}, \alpha_{24}), (\alpha_{20}, \alpha_{22}), (\alpha_{21}, \alpha_{23})\}$

4. The Identity Graph of S_4 ($\Gamma(S_4)$)

The identity graph associated with the group S_4 is given below (Fig. 3):

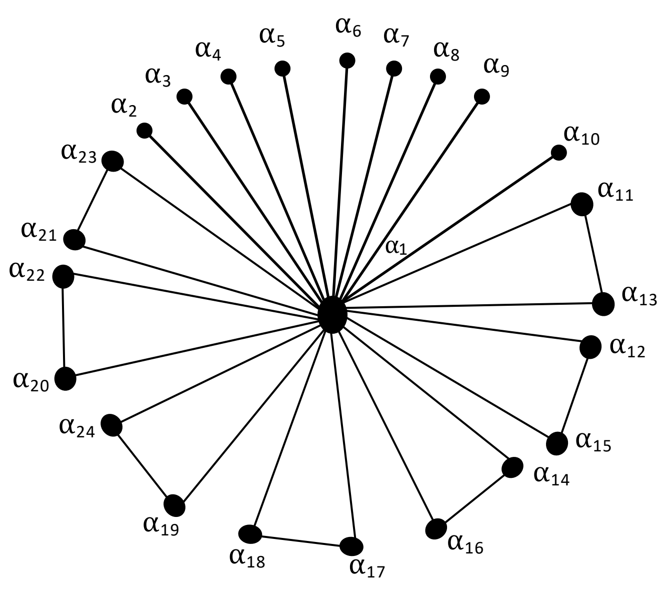


Figure 3: The identity graph of S_4

$\Gamma(S_4) = K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_2 \cup K_3 \cup K_3 \cup K_3 \cup K_3 \cup K_3 \cup K_3 \cup K_3$.
As we have seen this graph consists of 24 vertices and 30 edges, and it is a component of nine lines and seven triangles.

Now from this graph we derive so many properties of S_4 by using the idea of general graph properties such as degree of vertex, clique, colouring, independent and dominating sets.

4.1 Degree of vertex in $\Gamma(S_4)$

The order of each non-identity element in S_4 is either two or three or four. Therefore we check to see whether the order of each element in S_4 is related to its corresponding degree of vertex in $\Gamma(S_4)$.

We found that $\delta(\alpha_1) = 23$, $\delta(\alpha_i) = 1$, for $i = 2,3,...,10$ and $\delta(\alpha_i) = 2$ for $i = 11,12,...,24$.
Hence
α_1 and α_i for $i = 2,3,...,10$ are odd vertices while α_i for $i = 11,12,...,24$ are even vertices
We now generalise this for any finite group G.

Theorem (Result 1): Let G be a finite group and $g \in G$. The degree of vertex g in $\Gamma(G)$ is

$$\delta(g) = \begin{cases} |G|-1 & \text{if } g \text{ is an identity element} \\ 1 & \text{if } g \text{ is self inverse element} \\ 2 & \text{if } g \text{ is mutual inverse element} \end{cases}$$

Proof

Let G be a finite group and g∈G. We prove this by cases.

Case 1: g is the identity element.
The vertex corresponding to the identity element in any identity graph associated to G is the centre of the graph. Therefore it is connected to all other vertices corresponding to the non-identity elements in the graph. It follows that $\delta(g) = |G|-1$.

Case 2: g is self inverse.
The vertex corresponding to g is connected to the centre of the graph only and hence in this case $\delta(g) = 1$.

Case 3: g is a mutual inverse.
In an identity graph every vertices corresponding to the mutual inverse elements are adjoined by an edge apart from connecting to the centre. Hence the vertex corresponding to g is incident to two edges and thus $\delta(g) = 2$.

4.2 Clique of $\Gamma(S_4)$

Here we determine whether or not each of the cliques K_2 or K_3 yields a subgroup of S_4.

Clearly each of the subgraphs K_2 is representing a subgroup of order two, since for each $i = 2, 3, ..., 10$; $(\alpha_i)^2 = \alpha_1$.

We now have the following result.

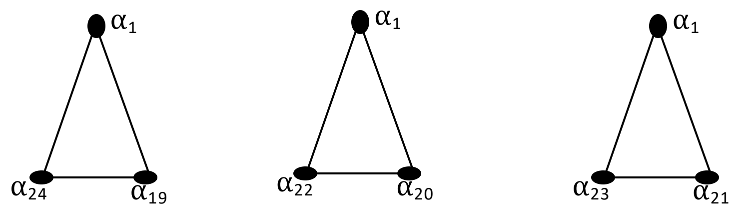
Theorem (Result 2) : In any identity graph of a finite group G, the subgraph K_2 is representing a subgroup of order two.

Proof

A line subgraph of any identity graph consists of only two vertices and a single edge, where the vertices are the identity and self inverse elements in G. Let e denote the identity element and g be a self inverse element in (G, $*$), that is $g = g^{-1}$, then $g * g^{-1} = g * g = g^2 = e$, hence g is of order two. Thus the subset W = {e, g} is a subgroup of order two generated by g and it follows that the clique K_2 is representing W.

Next we consider the subgraphs K_3 (triangles). It is very easy to verify that the composition of a 4-cycle element and itself is a double transposition but not 4-cycle.

Thus from $\Gamma(S_4)$ each of the following subgraphs does not yield a subgroup of S_4.



We now introduce the notion of graphically nice group.

Definition 4.1: In an identity graph associated with a finite group G, if each of the subgraphs represented by a triangle yield a subgroup of G then we call G a graphically nice group otherwise it is said to be graphically unpleasant group.

For example, the symmetric group S_3 is a graphically nice group because each of the cliques K_2 and K_3 in its associated identity graph is representing a subgroup of S_3.

Theorem (Result 3): The symmetric group S_4 is graphically unpleasant group.

Proof

This is obvious from the above discussion; three of the cliques K_3 in $\Gamma(S_4)$ are not representing subgroups of S_4.

Corollary (Result 4): The symmetric group S_n ($n \geq 4$) is graphically unpleasant group.

Proof

This is clearly true due to the fact that for $n \geq 4$, composition of two k-cycle elements is not order k for $k \in \{4, \ldots, n\}$. Thus some cliques K_3 in $\Gamma(S_n)$ for $n \geq 4$ are not representing subgroups of S_n.

4.3 Colouring and Independent Sets of $\Gamma(S_4)$

In this subsection we try to see how is't possible to successfully colour the vertices of $\Gamma(S_4)$, this is mainly to make easy understanding of the properties of S_4.

4.3.1 Order of an element in S_4

Let us give a particular colour to vertices corresponding to elements of the same order. Since S_4 consists of non-identity elements of order two, three and four while the identity element is having order one. In this case four different colours are required to colour all the vertices of $\Gamma(S_4)$. Hence we call S_4 4-colourable by element order group.

In $\Gamma(S_4)$, we give white colour to the vertex corresponding the identity element, and then colour the vertices corresponding to elements of order two, order three and order four with blue, red and green respectively. The coloured $\Gamma(S_4)$ is as follows (Figure 4).

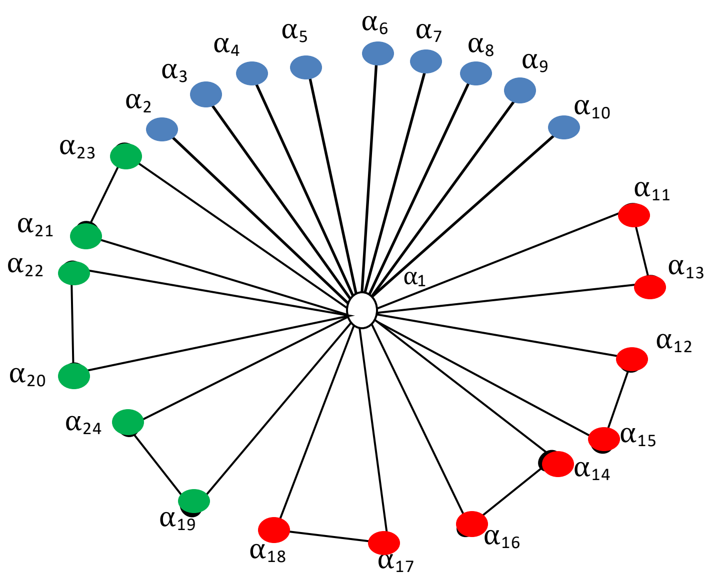


Figure 4: The associated coloured $\Gamma(S_4)$

Hence the independent sets are:

$\{\alpha_1\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}\}$, $\{\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}\}$ and $\{\alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$.

We now generalize the notion of colourable by element order in symmetric group in the following theorem.

Theorem (Result 5): Every symmetric group of degree *n* is *n*-colourable by element order.

Proof

$|S_4| = n! = 1 \cdot 2 \cdot \ldots \cdot n$ .Since each element in S_n is of order k where $k \in \{1, 2, \ldots, n\}$.
Therefore there are n different orders for elements in S_n and hence it follows that n different colours are needed to colour all .the vertices of $\Gamma(S_n)$ in terms of order of the corresponding elements, Hence every symmetric group of degree n is n-colourable element order.

4.3.2 Nature of an element in S_4

Since every permutation in S_n is either odd or even then we give green colour to the vertices corresponding to the odd permutations and purple colour to the vertices corresponding to even permutations in $\Gamma(S_4)$. Below is the associated coloured $\Gamma(S_4)$.

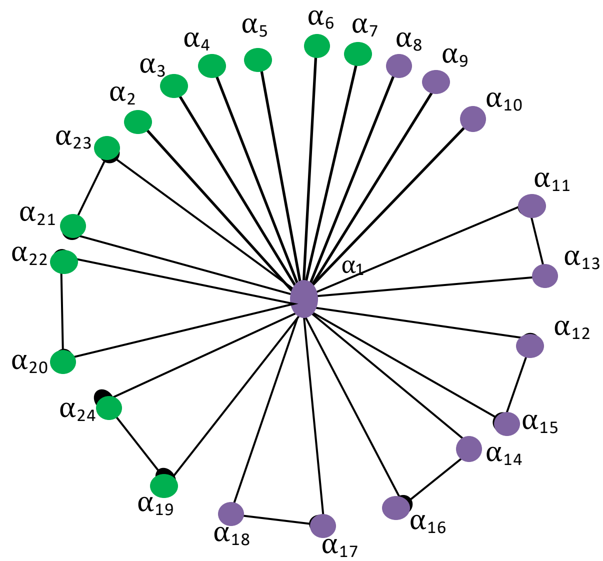


Figure 5L The associated coloured $\Gamma(S_4)$

Thus the independent sets are: $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$ and $\{\alpha_1, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}\}$.

4.3.3 Centre of S_4

Another interesting way of colouring the vertices of $\Gamma(S_4)$ is by colouring the centre of S_4 since it is a non-abelian group. So by looking at the graph of $\mathbf{S_4}$ one can know the centre Z(**S₄**) = {α_1}, hence in $\Gamma(S_4)$ only the vertex α_1 is given a colour and the rest of the vertices are remained uncoloured. In this case the $\Gamma(S_4)$ showing the centre of S_4 is as follows.

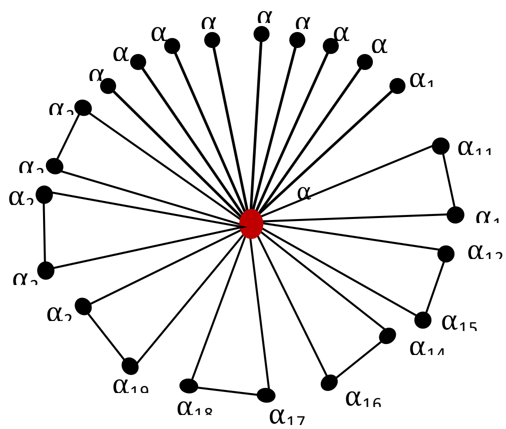


Figure 6 the associated coloured $\Gamma(S_4)$

It follows that the independent set is **{**α_1}.

4.3.4 p-Sylow Subgroups of S_4

$|S_4| = 4! = 1{\cdot}2{\cdot}3{\cdot}4$, according to theorem() S_4 is a 2-colourable by p-sylow subgroups.
That is 2 colours are enough for colouring of conjugates 2-sylow and3-sylow subgroups respectively.
According to sylow theorem (ii) there are three 2-sylow subgroups of order eight as follows:

$N_1 = \{\alpha_1, \alpha_2, \alpha_7, \alpha_8, \alpha_9, \alpha_{10,} \alpha_{21}, \alpha_{23}\}$

$N_2 = \{\alpha_1, \alpha_3, \alpha_6, \alpha_8, \alpha_9, \alpha_{10,} \alpha_{19}, \alpha_{24}\}$

$N_3 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_9, \alpha_{10,} \alpha_{20}, \alpha_{22}\}$

The identity subgraphs representing N_1, N_2 and N_3 are given below (Figs. 7(a)-(c)).

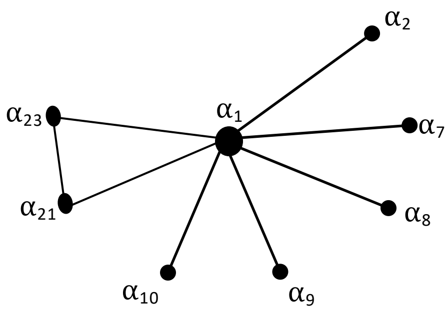


Fig. 7(a)

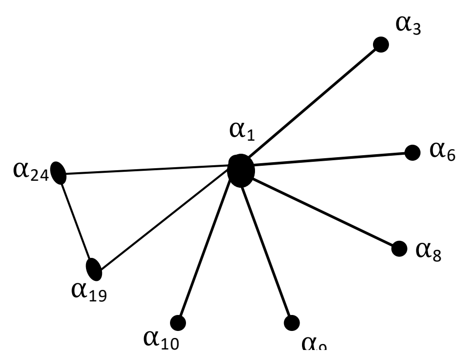


Fig. 7(b)

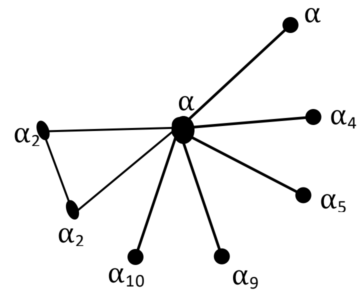


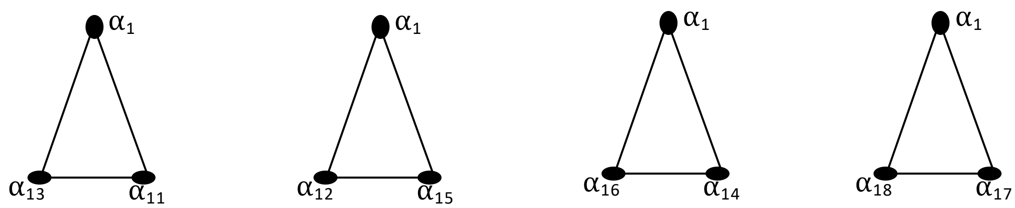
Fig. 7(c)

Note

Observe that these subgraphs are isomorphic because of conjugacy. The distinct vertices $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}$ and α_{24} are given pink colour in Fig. 8 below.

Next we consider the 3-sylow subgroups. According to sylow theorem (ii) there are four 3-sylow subgroups of order three. They are: $H_1 = \{\alpha_1, \alpha_{11}, \alpha_{13}\}$, $H_2 = \{\alpha_1, \alpha_{12}, \alpha_{15}\}$, $H_3 = \{\alpha_1, \alpha_{14}, \alpha_{16}\}$, $H_4 = \{\alpha_1, \alpha_{17}, \alpha_{18}\}$.

Below are the identity subgraphs associated with H_1, H_2, H_3 and H_4 respectively.



Hence the vertices $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}$ and α_{18} are given blue colour in $\Gamma(S_4)$ as follows (Fig. 8).

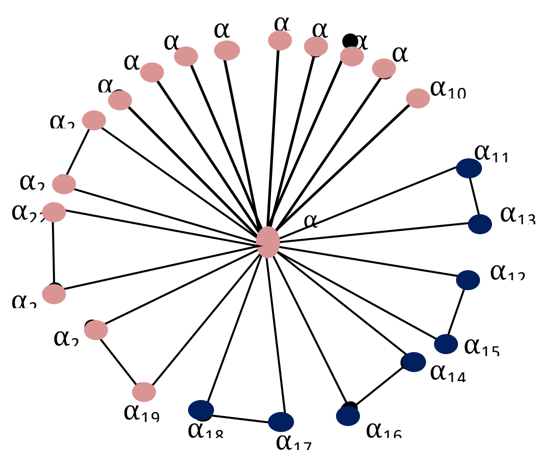


Figure 8 the associated coloured $\Gamma(S_4)$

It follows that the independent sets in this case are:
$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$ and
$\{\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}\}$.

4.4 Dominating Set in $\Gamma(S_4)$

There are many dominating sets in $\Gamma(S_4)$ but the smallest is $\{\alpha_1\}$ which corresponds with the centre of S_4.

Theorem (Result 6): In any identity graph of a group G with e as the identity element, the smallest dominating set is the trivial subgroup $\{e\}$.

Proof

Since every vertex in $\Gamma(G)$ is adjacent to the vertex corresponding the identity element in G. Hence by definition the smallest dominating set consists of only the identity element of G. Thus the trivial subgroup $\{e\}$ is the smallest dominating set in $\Gamma(G)$.

5. Conclusion

Group and Graph theory provide meaningful and interesting ways of examining relationships between elements of a given set. Identity graph makes easy understanding of group properties. By looking at the identity graph of any finite group G, one can easily determine the type of the group, nature of the elements in the group and also form subgroups of G.

In this paper the symmetric group S_4 is represented in the form of an identity graph with 24 vertices and 30 edges. From this graph we derive so many properties of S_4 by using the idea of general graph properties such as degree of vertex, clique, coloring, independent and dominating sets. We have shown by colouring vertices of the associated identity graph of S_4, elements having the same order, nature of the elements, conjugate elements and centre of S_4. In each case the independent sets partitioned S4. Finally, we were able to establish and proved some results on the identity graph related to the symmetric group of degree n (S_n) and any finite group G in general.

References

Beineke L.W and Wilson R. J. (2004). *Topics in Algebraic Graph Theory*. Cambridge University, Press. 102.

Bondy J. and Murty G. (1982). *Graph Theory with Applications,* 5th edition. North Holland, Boston New York.

Chalapathi T. and Kumar K. (2018). Invertible Graphs of Finite Groups. *Computer Science Journal of Moldova*. 2(77): 26

David S.D. and Richard M.F. (2004). *Abstract Algebra,* 3th edition. John Wiley and Son Inc.

Godsil C. and Boyle G. (2001). *Algebraic Graph Theory*. 5th edition. Springer, Boston New York.

Herstein I.N. (2003). *Topics in Algebra*. 3rd edition. Wiley Eastern Limited.

Vasantha W.B. and Smarandache F. (2009). Groups as Graphs, e-book.

Vitaly I.V. (2009). *Introduction to Graph Theory*. New York Science. Publishers. Inc. p.144 ISBN -10: 1606923749, ISBN-13: 978- 1606923740.