ON THE APPLICATION OF POWER SERIES METHOD TO THE SOLUTION OF A SECOND ORDER AUTONOMOUS AND NON-AUTONOMOUS AIRY'S DIFFERENTIAL EQUATIONS

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Abstract
In this paper, the existence of solution of autonomous and non-autonomous Airy's equation were investigated using power series method. The differential equation for both autonomous and non-autonomous type have an ordinary point but no singular point. The result shows that the existence of ordinary point of the differential equation which is analytic at the point of definition confirms the existence of solutions. Furthermore, numerical simulations were used to describe the behaviour of the solution which extends some existing results in literature.

Keywords: *Power Series method; Autonomous; Non-autonomous; Airy's Equation.*

Mathematics Subject Classification (2010): 34B15, 34C15, 34C25, 34K13

1.0 Introduction

Consider an autonomous second order linear differential equation of the form

$$y'' + (1 - x)y = 0 \tag{1.1}$$

where y'' is the second derivative and $(1 - x)$ is the coefficient of y. In Cengel and Palm (2013), Airy's differential equation is a differential equation that arises in optics, in the study of intensity of light. Many of their special functions find its application in the Partial Differential Equations of Mathematical Physics as well as Pure Mathematics through the theory of orthogonal expansions. For a larger class of Linear Differential Equations with variable coefficients such as Airy's equations, the need for a search for solution cannot be over emphasized especially beyond the familiar elementary functions of calculus.

Olver (1974), commented that the Airy's function is a special function named after the British Astronomer George Biddel Airy (1801-1892). The function $A_i(x)$ and the related function $B_i(x)$ are linearly independent solutions of the differential equation

$$\frac{d^2y}{dx^2} - xy = 0 \tag{1.2}$$

known as the Airy's equation or the Stoke's equation. Aspenes (1966), discussed the Airy's equation as the simplest second order linear differential equation with a turning point. The character of the solution changes from oscillatory to exponential and acts as a solution to Schrondinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field. The function also serves to provide uniform semi-classical approximations near a turning point when the potential may be locally approximated by a linear function of position. The triangular potential well solution is directly relevant for the understanding of many semi-conductor devices. Furthermore Airy's function underlies the form of the intensity near an optical directional caustic such as that of the rainbow. Historically, this was the mathematical problem that led Airy's to develop this special function. Saddiq (2013), opined that the idea of solving the Airy's equation is to first assume that the solution exists and that the solutions so obtained is not in the form of elementary functions rather it is in the form of infinite power series. Boggarapu (2015) contributed that the solution of the Airy's equation has power series representation of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. The determination of the coefficients $a_n's$ help to find the solution of the differential equation which is similar to the method of undermined coefficients. However, the objectives of this paper therefore are to investigate the series solution of Airy's equation and to identify the singular and ordinary points. This study is significant because of its application to different areas of physical phenomena such as rainbows and earthquakes. In Mathematics, the applications can be found in expansion of series which are useful for approximation purposes and in understanding of polynomials and trigonometric functions much better than arbitrary functions. The application can also be found in parabolic equations, in underwater calculations of acoustics and radar propagation in the troposphere. In Engineering, the significance can be seen in spectrum analysis and especially in radios, audios and light application where the reception of wide range of frequencies can be detected. The application can be found in classical mechanics especially relativistic mechanics where the momentum and energy quantities are expressed in infinite orders of velocity. Also in quantum mechanics, where Airy's function gives uniform asymptotic approximations valid in the neighborhood of a turning point, in the context of the connection problem and the solution of one-dimensional Schrodinger equation of a particle subjected to a constant force and also used to improve the statistical atom model beyond the Thomas-farm approximation.

This work is motivated by studying the works of Knaust (1998) and Mathias and Ronald (2004), where discrete second order linear differential equation were investigated. However, our approach will be based on the autonomous and non-autonomous second order linear differential equation using power series method.

2.0 Preliminaries

Definition 2.1 (Power series): Power series method is that used to seek a power series solution to certain differential equations. In general, such a solution assumes a power series with unknown coefficients which when substituted into the differential equation yields the recurrence relation for the coefficients.

Definition 2.2 (Power series about the point zero): it is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{2.1}$$

where $a_0, a_1, a_2 \dots a_n$ are real constants

Definition 2.3 (Power series about the point x_0): it is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots \tag{2.2}$$

where $a_0, a_1, a_2 \dots a_n$ are real constants

Definition 2.4 (Convergence of Power series): The power series is said to converge at a point x if its nth partial sum $\sum_{k=0}^{n} a_k x^k$ converges; that is to say that the limits $L = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$ exists. In this case, the sum of the series is the limit and such point is called point of convergence.

Note: $x = 0$ is always a point of convergence of the power series (2.1)

For example

(i) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ converges for every value of x in $\mathbb{R}$ and

(ii) $\sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + x^3 + \ldots$ converges only for $|x| < 1$.

The point of convergence of (i) and (ii) form an interval. Moreover there exists $0 \leq R \leq \infty$ such that the power series (i) and (ii) converges for all $|x - x_0| < R$ and diverges for all $|x - x_0| >$ □. Here R is called radius of convergence. In many cases, the radius of convergence can be found by using the formula $R = \lim_{n=0}\left|\frac{a_n}{a_{n+1}}\right|$ whenever the limits exist.

Definition 2.5 (Differential of power series): Suppose that the power series (2.1) converges for $|x| < R$ with $R > 0$, denote the sum by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots. \qquad (2.3)$$

Then $f(x)$ is automatically continuous and has the derivatives of all order for $|x| < R$. Also

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \qquad (2.4)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2a_2 + 3.2a_3 x + \ldots \qquad (2.5)$$

and so on. Each of the resulting series converges for $|x| < R$. The coefficient a_n can be linked to $f(x)$ and its derivative via the following formula

$$a_n = \frac{f^{(n)}(0)}{n!} \qquad (2.6)$$

Definition 2.6 (Algebra of power series): Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two power series with radius of convergence at least $R > 0$, then these power series can be added or subtracted term wise as follows

$$f(x) \pm g(x) = \sum_{n=o}^{\infty}(a_n \pm b_n)x^n = (a_n \pm b_0) + (a_1 \pm b_1)x + \ldots \qquad (2.7)$$

They can also be multiple as they are polynomials in the sense that

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$

$f(x) = g(x)$ for $|x| < R$ if and only if $a_n = b_n$ for all n ie if both series converge to the same function for $|x| < R$ if and only if they have the same coefficients.

Theorem 2.7 Let x_0 be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2.8)$$

and let a_0 and a_1 be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at x_0 in a certain neighborhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x - x_0| < R$, $R > 0$, then the power series expansion of this solution is also valid on the same interval.

Definition 2.8 (Ordinary point): Consider $f(x) = \frac{q(x)}{p(x)}$ where $q(x)$ and $p(x)$ are real valued functions. The point x_0 at which $p(x) \neq 0$ is called ordinary point of $f(x)$. Precisely, x_0 is called an ordinary point of $f(x)$ if $p(x_0) \neq 0$.

Definition 2.9 (Singular point): Consider $f(x) = \frac{q(x)}{p(x)}$ where $q(x)$ and $p(x)$ are real valued functions. The point x_1 at which $p(x) = 0$ is called a singular point of $f(x)$. Precisely, x_1 is called a singular point of $f(x)$ if $p(x_1) = 0$.

Definition 2.10 (Regular singular point): Consider the linear differential equation

$p(x)y'' + q(x)y' + r(x)y = w(x)$ with polynomial coefficients. The point $x = x_0$ is called a regular point of $w(x)$ if $\frac{q(x)}{p(x)} = \frac{r_1(x)}{(x-x_0)^m}$, $\frac{r(x)}{p(x)} = \frac{r_2(x)}{(x-x_0)^n}$ and $\lim_{x\to x_0}(x - x_0)\frac{q(x)}{p(x)} = \lim_{x\to x_0}(x - x_0)\frac{r_1(x)}{(x-x_0)^m} = r_1(x_0)$ or zero which is finite, provided $m \le 1$.

Also $\lim_{x\to x_0}(x - x_0)^2\frac{r(x)}{p(x)} = \lim_{x\to x_0}(x - x_0)^2\frac{r_2(x)}{(x-x_0)^n} = r_1(x_0)$ or zero which is finite, provided $n \le 2$.

Definition 2.11 (Irregular singular point): Redefine $f(x) = \frac{q(x)}{p(x)} = \frac{r_1(x)}{(x-x_0)^m}$.

If $\lim_{x\to x_0}(x - x_0)^k\frac{q(x)}{p(x)} = \lim_{x\to x_0}(x - x_0)^k\frac{r_1(x)}{(x-x_0)^m} = r_1(x_0)$ or zero which is finite, provide $m \le k$ then the singularity at $x = x_0$ is removable. Such a singular point is called an irregular singular point.

Remark: k depends on the order of the linear ordinary differential equation.

3.0 Results

3.1 Power Series Solution of an Autonomous Airy's Equation

We consider an autonomous Airy's equation of the form

$$y'' + (1 - x)y = 0 \tag{3.1}$$

Eqn. (3.1) assumes the solution of the form

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n \tag{3.2}$$

Differentiating term by term gives

$$y' = a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1} \tag{3.3}$$

$$y'' = 2a_2 + 3.2a_3x + \cdots + n(n-1)a_nx^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} \tag{3.4}$$

Substitute eqn (3.3) and eqn (3.4) into eqn (3.1)

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + (1 - x)\sum_{n=0}^{\infty} a_nx^n = 0 \tag{3.5}$$

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n - x\sum_{n=0}^{\infty} a_nx^n = 0 \tag{3.6}$$

Evaluating for each term using initial value of n gives

$$2a_2 + 3.2a_3x + 4.3a_4x^2 + 5.4a_5x^3 + \cdots + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$
$$-x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots) = 0 \tag{3.7}$$

Eqn. (3.7) is further reduced to

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$
$$-xa_0 - a_1x^2 - a_2x^3 - a_3x^4 - a_4x^5 - a_5x^6 + \cdots = 0 \tag{3.8}$$

For power series to vanish identically over an interval, each coefficient in the series must be zero

For x^0: $\quad 2a_2 + a_0 = 0 \Rightarrow a_2 = \frac{-a_0}{2}$

For x^1: $\quad 6a_3 + a_1 - a_0 = 0 \Rightarrow a_3 = \frac{a_0 - a_1}{6}$

For x^2: $\quad 12a_4 + a_2 - a_1 = 0 \Rightarrow a_4 = \frac{a_0 + 2a_1}{24}$

For x^3: $\quad 20a_5 + a_3 - a_2 = 0 \Rightarrow a_5 = \frac{-4a_0 + a_1}{120}$

Hence the general solution ids of them form $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$ which gives

$$y = a_0 + a_1 x + \frac{-a_0}{2}x^2 + \frac{a_0 - a_1}{6}x^3 + \frac{a_0 + 2a_1}{24}x^4 + \frac{-4a_0 + a_1}{120}x^5 + \cdots$$
$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{30} + \cdots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{120} + \cdots\right) \quad (3.9)$$

where a_0 and a_1 are arbitrary constants. Eqn. (3.9) is the series solution for autonomous differential equation which is not time dependent. The series solution does not terminate as the power of the series increases.

3.2 Power Series Solution of Non-autonomous Airy's Equation

We consider non-autonomous Airy's equation of the form

$$\ddot{y} + kt(1 - y) = 0 \quad (3.10)$$

where $\ddot{y}$ is a second derivative of y with respect to time. The solution of eqn (3.10) is of the form

$$y = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots = \sum_{n=0}^{\infty} a_n t^n \quad (3.11)$$

Differentiating eqn (3.11) term by term gives

$$\dot{y} = a_1 + 2a_2 t + \cdots + n a_n t^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n t^{n-1} \quad (3.12)$$

$$\ddot{y} = 2a_2 + 3.2a_3 t + \cdots + n(n-1)a_n t^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} \quad (3.13)$$

Substituting for eqn (3.11) and eqn (3.13) into eqn (3.10) gives

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + kt\left(1 - \sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (3.14)$$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + kt - kt\sum_{n=0}^{\infty} a_n t^n = 0 \quad (3.15)$$

Evaluating for each values of n for each term in eqn (3.15) we have

$$2a_2 + 3.2a_3 t + 4.3a_4 t^2 + \cdots + kt - a_0 kt - a_1 kt^2 - a_2 kt^3 - a_3 kt^4 - a_4 kt^5 - \cdots = 0 \quad (3.16)$$

For the power to vanish identically over an interval, each coefficient in the series must be zero.

For t^0: $\quad 2a_2 = 0 \Rightarrow a_2 = 0$

For t^1: $\quad 6a_3 + k - a_0 k = 0 \Rightarrow a_3 = \frac{k(a_0 - 1)}{6}$

For t^2: $\quad 12a_4 - a_1 k = 0 \Longrightarrow a_4 = \frac{a_1 k}{12}$

Hence the solution $y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots$ gives

$$y = a_0 + a_1 t + \frac{k(a_0 - 1)}{6}t^3 + \frac{a_1 k}{12}t^4 + \cdots$$
$$= a_0\left(1 + \frac{kt^3}{6} + \cdots\right) + a_1\left(t + \frac{kt^4}{12} + \cdots\right) \quad (3.17)$$

where a_0 and a_1 are arbitrary constants. Eqn. (3.17) is power series solution for non-autonomous Airy's equation which is time dependent. The series solution terminates as the power of the series solution increases.

For ordinary point of eqn (1.11), we compare eqn (1.11) with

$$p(x)y'' + q(x)y' + r(x)y = f(x) \quad (3.18)$$

which gives $p(x) = 1, q(x) = 0, r(x) = (1-x)$ and $f(x) = 0. p(x), q(x), r(x)$ and $f(x)$ are real valued functions with domain as a real number. Dividing eqn (3.19) by $p(x)$ gives

$$y'' + \frac{q(x)}{p(x)}y' + \frac{r(x)}{p(x)}y = \frac{f(x)}{p(x)}$$

Let $w(x) = \frac{q(x)}{p(x)}$ where $w(x)$ is also a real valued function, then $w(x) = q(x)$ since $p(x) = 1$

Hence, eqn (1.11) has no singular point. Ordinary point of eqn (1.11) exist since $p(x) \neq 0$

4.0 Numerical Solution of Airy's Equation

$$k := 0.1$$

Define a function that determines a vector of derivative values at any solution point (t,Y):

$$D(t, X) := \begin{bmatrix} X_1 \\ -(1-t)\cdot X_0 \end{bmatrix}$$

Define additional arguments for the ODE solver:

$t0 := 0$ Initial value of independent variable

$t1 := 50$ final value of independent variable

$S := \mathrm{Rkadapt}(X0, t0, t1, N, D)$

$t := S^{\langle 0 \rangle}$ Independent variable values

$x1 := S^{\langle 1 \rangle}$ First solution function values

$x2 := S^{\langle 2 \rangle}$ Second solution function values

$X0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Vector of initial function values

$N := 1500$ Number of solution values on [t0, t1]

Solution matrix:

Table 1: Table of values for the independent variables

S =

	0	1	2
0	0	0	1
1	0.033	0.033	0.999
2	0.067	0.067	0.998
3	0.1	0.1	0.995
4	0.133	0.133	0.992
5	0.167	0.166	0.988
6	0.2	0.199	0.983
7	0.233	0.231	0.977
8	0.267	0.264	0.971
9	0.3	0.296	0.964
10	0.333	0.328	0.957
11	0.367	0.36	0.95
12	0.4	0.392	0.942
13	0.433	0.423	0.934
14	0.467	0.454	0.926
15	0.5	0.485	...

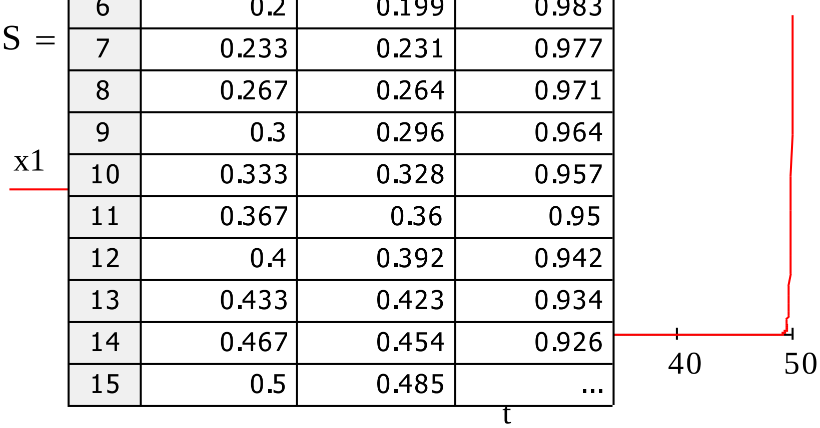

Figure 1: The relation between first solution function values and independent variable values

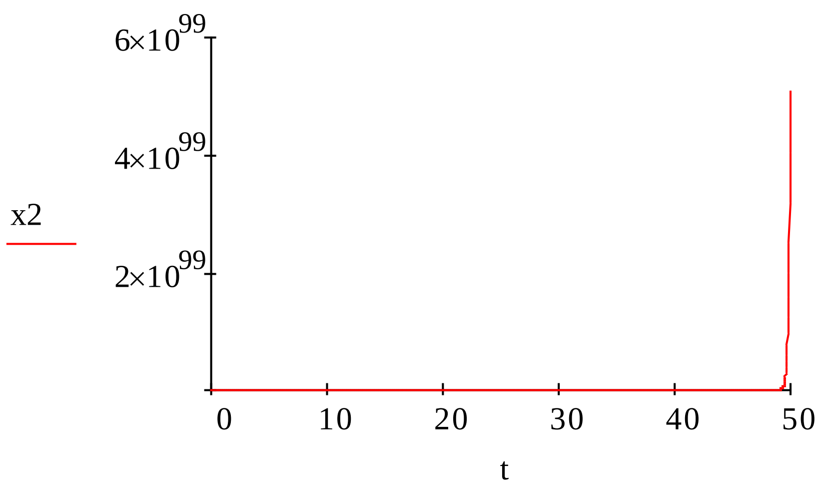

Figure 2: The relation between second solution function values and independent variable values

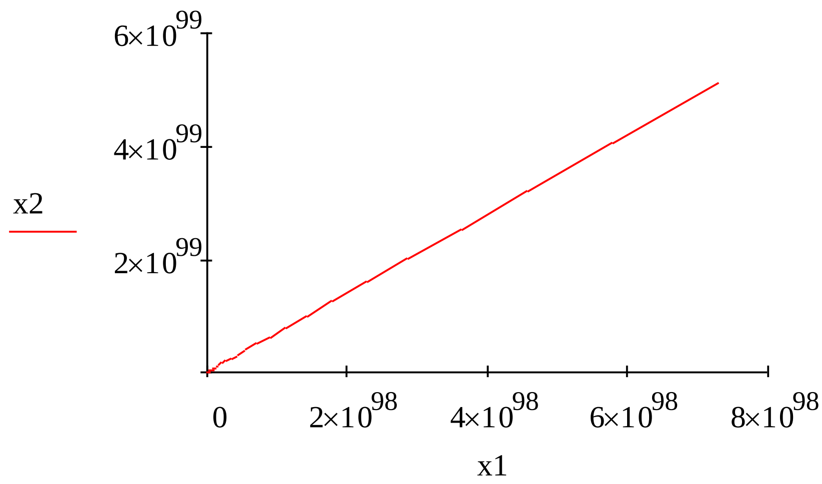

Figure 3: The relation between first solution function values and second solution function values

4.1 Discussion

From Figure 1, the increase in the first solution function values leads to a constant increase along the independent variable values. This constant increase stop at $t = 50$. At this point, the trajectory is parallel to the axis of the first solution function values. This behaviour describes the solution of non-autonomous differential equation which is time dependent.

Figure 2, shows that the increase in the second solution values leads to a constant increase along the independent variable values which terminate at $t = 50$. This invariably has a similar behaviour as since in Figure 1.

Figure 3, shows the trajectory that start from the origin. In this case, the increase in the first solution function values leads to a corresponding increase in the second solution function values. This behavior describes the solution of autonomous differential equation which is not time dependent.

5.0 Conclusion

From our result, existence of series solution of autonomous and non-autonomous Airy's equation were obtained using Power series method. This is because Airy's equation has variable coefficient with respect to the dependent variable. The differential equation has an ordinary point but no singular point. This is because there is no point x_0 for which $p(x) = 0$. Existence of ordinary point of the differential equation shows that the real valued functions are analytic at that point which confirms the existence of solution. Our conclusion is that autonomous differential equation of Airy's type is more applicable than the non-autonomous type.

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