ON THE EXISTENCE AND UNIQUENESS OF 2π –PERIODIC SOLUTIONS OF DUFFING'S EQUATION USING ABSTRACT IMPLICIT FUNCTION THEOREM IN BANACH SPACES

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Abstract
We applied the conditions of Implicit function theorem to investigate the existence and uniqueness of 2π –periodic solution of Duffing's equation of the form
$\ddot{x} - c\dot{x} - bx - 2x^3 = h(t)$
The solutions were found to exist and are unique under some appropriate assumptions because many systems are periodic in nature, yet they do not obey periodic solutions.

Keywords: *Implicit function theorem, Duffing's equation, Periodic solutions, Banach spaces, Homeomorphism.*

1. Introduction

Our main aim in this paper is to use the hypothesis of implicit function theorem to investigate the existence and uniqueness of 2π –periodic solutions of second order Duffing's equation of the form

$$\ddot{x} - c\dot{x} - bx - 2x^3 = h(t)$$

1.0 Duffing's Equation

The Duffing's equation describes the motion of a classical particle in a double well potential. They chose units of length so that the minimum are at ± 1 at the units of energy so that the depth of each well is at $-\frac{1}{4}$ and the potential is given by $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ (Duffing's number). M. Bhatti [1] opined that the differential equation which describes a non-linear oscillator introduced first by Duffing with cubic stiffness constant has become a very common example of a non-linear oscillator. This equation permits the description of a hard spring and remains of continuous interest for example; in a family of planar maps, depending on parameters, the onset of chaos typically occurs at the parameter values for which the stable and unstable manifolds of a stable point come into contact tangentially. This method of creation of transversal homoclinic points and related issues can be established by the general form of which is

$$\ddot{x}(t) + \delta\dot{x}(t) - x(t) - \beta x^3(t) = f(t)$$

Where $f(t)$ is one of the following two functions:-

$$f(t) = Y coswt, f(t) = Y sinwt$$

This provides a model, Sana Gasmi and Alain Haraux [2].

The problem of existence as well as multiplicity of periodic solutions of forced Duffing's equation $\ddot{x} + g(x) + c\dot{x} = f(t)$ has been object of many works in both undamped (case $c = 0$) and damped case.

Our approach is slightly different in that our search is on the employment of the conditions of implicit function theorem to prove the existence and uniqueness of the 2π −periodic solutions of Duffing's equation of the form
$\ddot{x} + c\dot{x} + ax + bx + 2x^3 = h(t)$ in a Banach space.
Now the Duffing's equation of the form
$\ddot{x} + c\dot{x} + ax + bx + 2x^3 = h(t)$ (1.1)
Where a,b,c are real constants and $h: [0,2\pi] \to E$
Is continuous. The existence of 2π-periodic solutions of (1.1) will be investigated ie the solutions defined on $[0,2\pi]$ such that
$x(0) = x(2\pi)$ (1.2)
And
$\dot{x}(\dot{0}) = \dot{x}(2\pi)$ (1.3)
Let $C^2_{\pi=}(x: [0,T] \to E^n, x\ is\ a\ class\ of\ C^2)$ with the usual uniformed norm
$\|x\|_2 = max\{t \in J_{2\pi}\ |x(\dot{t})|\ and\ t \in J_{2\pi}\ |\dot{x}(t)|\}$ (1.4)
And
$C = \{x: [0,\pi] \to E: x\ is\ continous\}$ (1.5)
With the usual norm $J = [0,T] J_{2\pi} = [0,2\pi]$ with this norm $C^2_{2\pi}$ which is a Banach space. Then all we are saying is that the problem (1.1),(1.2), (1.3) are equivalent to
$F(x,h) = 0\ in\ C^2_{2\pi}Xc$
Where $F: C^2_{2\pi}Xc \to C$ is defined by
$F(x,h) \to \ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 - h = 0$ (1.6)
Let us state this lemma;

Lemma 1.1: Let X be a Banach space and K be a cone of X. Assume Ω_1, Ω_2 are open subsets of X with $0\epsilon\Omega_1 \leq \Omega_2$ and let $A: K \cap (\Omega_2 - \Omega_1) \to K$ be a completely continuous operator such that:-

i. $\|AX\| \leq \|X\|\ for\ every\ x \in K \cap \delta\Omega_1$ and $\|AX\| \geq \|X\|\ for\ every\ x \in K \cap \delta\Omega_2$ or
ii. $\|AX\| \geq \|X\|\ for\ every\ x \in K \cap \delta\Omega_1$ and $\|AX\| \leq \|X\| for\ every\ x \in K \cap \delta\Omega_2$

Then A has at least one fixed point in $K \cap (\Omega_2 - \Omega_1)$ (Yuji Liu and Weigao, G [3])

This fixed point will basically coincide with a fixed solution and a unique fixed point. We now state the abstract implicit function theorem.

Theorem 2.1: Implicit Function Theorem
Let E, F, G, be Banach spaces. Let $U = U_1 X U_2 C \in X F$ be an open set with $(X_0,Y_0) \in U$. Let U→G be a mapping satisfying the following conditions:

i. $F(X_0,Y_0) = 0$
ii. F is strongly Frechet differentiable with respect to the first variable at (X_0,Y_0)
iii. $F_x(0,0)$; E → G is a linear homomorphism.

2. Main Results

The implicit function theorem will be applied to investigate the Duffing's equation. We first notice that

i. F(0,0) = 0 (2.1)

Two other conditions will be further checked.

ii. That F is strongly Frechet differentiable with respect to x at (0,0)
iii. That $C_{2\pi}^{2} \rightarrow$ C defined by:
$F_x(0,0)$: $z \rightarrow \ddot{z} + c\dot{z} + az$ is a linear homeomorphism (2.2)

Proof:

$|F(x,h)-F(\bar{x},h)-F_x(0,0)(x-\bar{x})|=|\ddot{x}+c\dot{x}+ax+bx^2+2x^3-h-(\ddot{x}-c\dot{x}-a\bar{x}+b\bar{x}^2+2\bar{x}^3-h)-(\ddot{x}-\ddot{\bar{x}}-C(\dot{x}-\bar{x})-a(x-\bar{x})|$ (2.3)

Let

$=|b(x^2-\bar{x}^2) + 2(x^3 -\bar{x}^3)|$ (2.4)
$\leq|x-\bar{x}||b(x+\bar{x}) + 2(x^2-|x||\bar{x}|-\bar{x}^2)|$
$\leq|x-\bar{x}|b(|x| + |\bar{x}| + 2(|x|^2 - |x||\bar{x}| - |\bar{x}|^2)|$
$\leq|x-\bar{x}||(2b\delta - 6\delta^2).(|x|\leq\delta|\bar{x}|\leq\delta)\rightarrow 0$ as $|x-\bar{x}|\rightarrow 0$
$F(0,h)= -h$ (2.5)

Which is continuous. Hence F is Frechet differentiable with respect to the first variable at (0,0)
The mapping $F_x(0,0)$: $z \rightarrow \ddot{z}+c\dot{z}+az$ (2.6)
Where $C_{2\pi}^{2} \rightarrow$ C is clearly linear, continuous and hence bounded. It is also an onto-mapping. Linear homeomorphism would have been shown to be one-to-one. This is equivalent to requiring that:

$\ddot{z} + c\dot{z} + az = 0$ (2.7)

With

$Z(0) = z(2\pi)$ (2.8)
$\dot{z}(0) = \dot{z}(2\pi)$ (2.9)

Is non-critical

We now employ some conditions on the constants a *and* c such that what will be realized will be investigated.

The auxiliary equations of (2.7) is

$\lambda^2 + c\lambda + a = 0$ (2.10)

Evoking the general quadratic formula, we have

$$\frac{-c \pm \sqrt{\square^2 - 4ac}}{2a}$$

If $c = 0$ *and* $a = k^2$, where k is a natural number, then $\lambda = \pm ik$ where $i\lambda = -1$ and
$z(t) = C_1 coskt + C_2 sinkt$ (2.11)
For arbitrary constants C_1 *and* C_2, clearly
$Z(0) = Z(2\pi)$ (2.12)
$\dot{Z}(0) = \dot{Z}(2\pi)$ (2.13)
And the solution is non-trivial.
If on the other h and $C = 0$ *and* $a = k^2$, then condition
$\dot{Z}(0) = \dot{Z}(2\pi)$ (2.14)
Is satisfied only by the trivial solution
$Z = 0$ (2.15)
If $C \neq 0$ *and* $C^2 \leq 4a$, the solution is trivial. If $C \neq 0$ *and* $a = 0$, only the trivial solution exists. Most generally, put

$\lambda = \frac{-C \pm \sqrt{C^2 - 4ac}}{2a} = u + iv$ (2.16)

For some u, v in real numbers.

i. Choose $c\ and\ a$ such that $u \neq 0$

ii. Choose $c\ and\ a$ such that $u = 0, v \neq 0$, then $\ddot{z} + \dot{z} + az$ is non-critical Hale [4]

We state the following lemma to help us strengthen our argument.

Lemma 2.1: Thus with any of these conditions imposed, one deduces the one-to-one ness of $f'_x(0,0)$,

Hence by Theorem 4.1 of Schecheter.M [5].

$[f'_x(0,0)]^{-1}$ exists as a bounded operator. Linear homomorphism of $F'_x(0,0)$ follows.

Existence of a unique solution is now assured by the implicit Function Theorem.

3. Further Results

The last result will be as follows:

Theorem 3.1:

Let us consider the more general second order differential equation of the form

$\ddot{x} + c\dot{x} + ax + g(x) = h(t)$ (3.1)

Where $x, c, g\ and\ h$ are as previously defined and if $g: E \to E$ is continuous and $|g(x) - g(\dot{x})| \leq \eta(R)|x - \dot{x}|\ for\ |x| \leq R, |\dot{x}| \leq R\ where\ \eta(R) \to 0$ as $R \to 0^+$, then there exists a unique 2π −periodic solution satisfying the conditions of the Implicit function theorem.

Proof:

Lets define

$f(x, h) = \ddot{x} + c\dot{x} + ax + g(x) = h9t)$ (3.2)

Type equation here.

and

$F'_x(0,0) := Z \to \ddot{\ } + c\dot{\ } + ax + g(x) = h(t)$ (3.3)

Then linear homomorphism $F'_x(0,0)$ follows as already established.

$F(0, h) = g(0) - h(t)$ (3.4)

Which is continuous

i. Clearly $F(0,0) = 0$ (3.5)

ii. $\frac{|F(\mathrm{x}h) - F(\bar{x}h) - F\mathrm{x}'(0,0)(\mathrm{x} - \bar{x})|}{|x - \bar{x}|} = \frac{|g(x) - g(\overline{x})|}{|x - \bar{x}|}$ (3.6)

iii. $\frac{|g(x) - g(\overline{x})|}{|x - \bar{x}|} \leq \eta(R) \frac{|x - \bar{x}}{|x - \bar{x}|} = \eta(R) \to 0\ as\ R \to 0^+$ (3.7)

Thus, taking the limit we obtain

$$\lim_{(xxhR)\to 0} \frac{\left|F(x,h) - F(\bar{x},h) - F'_{x(0,0)(x-\bar{x})}\right|}{|x - \bar{x}|} = 0 \quad (3.8)$$

The hypothesis of the Implicit function theorem are established.

4. Conclusion

We have succeeded in establishing the existence and uniqueness of 2π −periodic solution of Duffing's equation using the hypothesis of implicit function theorem in Banach Spaces under the appropriate conditions. This work is still open for further research on the asymptotic stability, boundedness and convergences of solutions of Duffing's Equation.

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