STEADY NATURAL CONVECTION MHD FLOW IN A VERTICAL CHANNEL WITH INDUCED MAGNETIC FIELD AND VARIABLE FLUID PROPERTIES

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Abstract
In this paper, we discussed the results of the variability of viscosity and thermal conductivity on a magneto-hydrodynamic natural convection steady two dimensional flow of a viscous and electrically conducting fluid which has constant density. The boundary layer equations which govern the flow together with the boundary conditions are transformed into non-dimensional forms by suitable non-dimensional parameters to obtain a set of simultaneous ordinary differential equations which are nonlinear. The numerical solutions for the velocity, temperature and the magnetic field profiles are determined using a semi analytic numerical method called the differential transform method (DTM). The various physical parameters which are involved in the flow are tested on each of the profiles and the results are shown in graphs and tables. It is observed among others that as the viscosity parameter increases and the thermal conductivity parameters decreases, the fluid velocity reduces at the boundary layer. The induced magnetic field is enhanced with reduced viscosity away from the boundary layer. It is further observed that the temperature of the fluid vary in the same direction as the thermal conductivity.

Keywords: *Natural convection; induced magnetic field; variable viscosity; variable thermal conductivity; induced current density; skin friction; Nusselt number; differential transform method (DTM).*

1. Introduction

This study investigated the combined effect of viscosity and thermal conductivity (when both depend on temperature) as they influence the flow formation and thermodynamics of a MHD fluid. Magneto-hydrodynamics (MHD) describes the flow of electrically conducting fluids which combines the elements of electromagnetism and fluid mechanics. Natural or free convection flow happens due to the buoyancy effects which takes place in the fluid without the use of any external force but because of the density differences resulting from temperature gradients. The occurrence of natural convection of heat in a Magneto-hydrodynamic (MHD) electrically conducting viscous incompressible fluid flows driven by temperature differences has been and is still being studied extensively by many researchers because of its wide application in geophysical sciences, chemical engineering and metallurgical processes. Its uses can be found in the cooling of nuclear reactors, the design of Magneto-hydrodynamic accelerators, heat exchangers and in the building of power generators. Some of the most recent works in the literature include the work of (Jha and Ajibade, 2009). Where an analytical method was used to discuss the flow of a heat generating and absorbing free convective fluid flow which takes place between vertical porous plates where heat is periodically inputted.

They discovered that the presence of suction and injection misshapes the symmetry of the flow when the thermal boundary layer is increasing as it gets closer to the wall due to injection and reducing as it gets closer to the wall due to suction. (Shrama and Singh, 2010), investigated a steady magneto hydrodynamic free convection flow with variable electrical conductivity and thermal variation along a vertical plate that is maintained at constant temperature, by using the fourth order Runge-Kutta with double shooting techniques. They stated that the fluid velocity is enhanced either due to the presence of volumetric rate of heat generation or due to increases in the electrical conductivity parameter; but it decreases due to increase in magnetic field intensity. (Olanrewaju, Arulogun, Adebimpe, 2012), also applied the shooting technique together with fourth order Runge-Kutta integration scheme to investigate the interaction of internal heat production with a Newton boundary condition on the boundary layer flow over a flat plate. They discovered that the thermal boundary layer thickness becomes thinner with the growing of the local Prandtl number and the local Biot number.

Also (Parveen and Alim, 2012), studied the Ohmic Heating effect on Magneto-hydrodynamic free convection flow through a vertical wavy plane. They used the implicit finite difference method (Keller-box scheme), to solve the boundary value equations which governs the problem and drew the conclusions that the velocity and the thermal boundary layer become thicker, the local rate of heat transfer decreases and the skin friction increases when Ohmic heating is increased. (Goyal and Choudhary, 2015) studied the magneto-hydrodynamic free convective flow over a vertical porous plate with Joule heating, thermal radiation and chemical reaction. The solutions of the governing equations were obtained by the use of perturbation method. They deduce that the velocity increase with decrease in viscosity and increase in thermal conductivity. The primitive variable method and the Keller box method were used by (Roy *et al* 2014) to discuss the unsteady magneto-hydrodynamic natural convection flow along a vertical plate with thermal radiation. They discovered that the amplitude of skin friction decreases and the phase of the skin friction increases with the increase of the Prandtl number.

Mixed convective magneto hydrodynamic flow of a micro polar fluid with Ohmic heating, radiation and viscous dissipation over a chemically reacting porous plate subjected to a constant heat flux and concentration gradient was carried out by (Hitesh Kumar, 2014). They also used the Keller box scheme to obtain the results that the Schmidt number and chemical reaction reduces the concentration, the magnetic field decreases the velocity of the boundary layer while it increases the thermal boundary layer. (Sarveshanand and Singh, 2015) studied a magneto-hydrodynamic natural convection flow through two vertical parallel porous plates with induced magnetic field. They solved the boundary value problems involved analytically and deduced that increasing the Prandtl number, the Hartmann, the Magnetic Prandtl numbers and the suction parameter, decrease the velocity profile. That the induced magnetic profile decreases when the magnetic Prandtl number is enhanced, while it increases when the suction parameter, the Prandtl number and the Hartmann number increases. They further deduced that increasing the Prandtl number and the Hartmann number causes a decline in the induced current density profile. A corresponding increase is observed when the magnetic Prandtl number is increased. Combined natural convection and radiation heat transfer was analyzed by (Mehdi and Muhammad, 2015) using a similarity solution together with the fourth order Runge-Kutta algorithm and shooting procedure to solve the governing equations simultaneously. They observed that the thickness of the thermal and momentum boundary layer is enhanced due to the radiation heat transfer. And that for increase in the radiation parameter, the velocity at the boundary layer increases. The induced magnetic field effect on magneto-hydrodynamic natural convection flow in infinite vertical and electrically non-conducting parallel plates was studied by (Jha and Aina, 2016). The equation involved is solved analytically and they found that the effect of Hartmann number and magnetic Prandtl number reduces the induced current density at the central region of the micro channel. In all the aforementioned works, the effects of viscosity and thermal conductivity were assumed as constant. Meanwhile, in reality a change in temperature produces change in viscosity and thermal conductivity.

The role of temperature dependent viscosity on convective heat and mass transfer by free convection from vertical plane in porous medium was investigated by (Moorthy and Vadivu, 2012). In which they used the Runge - Kutta - Gill method along with shooting technique to solve the equations involved and conclude that the heat transfer increases and mass transfer decreases as Lewis number increases for both positive and negative values of the viscosity parameter. (Devi and Gururaj, 2012) worked on the roles of temperature dependent viscosity and nonlinear radiation on magneto-hydrodynamic flow with heat transfer through a surface stretching with a power-law velocity. They solved the governing boundary value equations by the Nachtsheim-Swigert iteration shooting technique for satisfaction of the boundary conditions by Runge-kutta fourth order method by which they deduce that the thermal boundary layer of the fluid decreases sharply with increasing Prandtl number; the increase in the values of viscosity measuring parameter increases the velocity and the skin friction coefficient. Whereas its effect is to decrease the temperature and dimensionless Nusselt number; the velocity and skin friction are decreased by the velocity exponent parameter. On the other hand, heat transfer rate grows by the growing velocity parameter. The combine effects of viscous dissipation and variable viscosity on free convection flow past a sphere with Ohmic heating and thermal conductivity was considered by (Alam M. M**,** Ruhul I, Sumita G, and Raihanul H, 2018) using the Keller box scheme. They discovered among others that the velocity profiles, the temperature profiles and the coefficient of the skin friction increase significantly when the values of the viscosity variation parameter and that of the dissipation parameter increases. While the local heat transfer increases with increase in the viscosity parameter and it reduces with the enhancement in the dissipation parameter. Most of the studies mentioned above avoided taking the effect of variable viscosity and variable thermal conductivity together in a problem in other to simplify the mathematical analysis of the problem. Meanwhile, in real life situations when temperature varies the viscosity of fluids together with the thermal conductivity also varies.

The following literatures are works which examined almost similar problems with the present study. Maxwell fluid flow and heat transfer with temperature dependent viscosity and thermal conductivity above a rapidly stretching sheet was investigated by (Singh and Agarwal, 2013). Where they also used the Keller box method to solve the governing equation and observed that the velocity profiles decreases when the value of the fluid viscosity parameter increases. (Hazarika and Konch, 2016) considered the roles of temperature dependent viscosity and thermal conductivity on magneto-hydrodynamic natural convection of a dusty fluid flowing in a vertical porous plate with heat generation. They used a numerical method called the shooting method to solve the governing equations and discovered that viscosity and species concentration decrease and temperature increases with the increasing value of the viscosity parameter of the fluid and the dust particles while it enhances with increase in the thermal conductivity parameter.

In this present work, we consider the Natural convection flow of a fluid with induced magnetic field, temperature dependent viscosity together with variable thermal conductivity between two very long vertical parallel plates. The governing boundary value problems involving the velocity, temperature and the induced magnetic fields are nondimensionalized and solved by the differential transform method. Also the expression for the induced current density, the skin friction and the Nusselt number are obtained. The results of the influence of the parameters affecting the flow are shown in graphs and some in tables.

2. A General Description of DTM

Differential transform of function y(x) is a numerical method conceptualized by (Zhou, 1986). Its basic idea is derived from Taylor expansion. The concept of this method involves trying to find coefficients of series expansion of the function y(x) by using the known initial condition on the particular problem.

The one-dimensional differential transform of a function y(x) at the point $x = x_0$ is defined by (Zhou, 1986) and (Rebenda J, Zdenek S, Yasir K, 2017) as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{1}$$

In eq. (1), y(x) is the function to be transformed and Y (K) is the transformed function. The differential inverse transformation of Y (K) is given as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(K), \tag{2}$$

Putting Eq. (1) in Eq. (2), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{3}$$

From Eq. (3) the idea of differential transform can be seen to be obtained from expansion by Taylor series, where the derivatives are not evaluated symbolically. In essence, the method is an approximation of the analytic solution of equations which are sufficiently differentiable. The approximation always takes the form of polynomials. Unlike Taylor series, much effort in order to find all the necessary derivatives of functions is not required. The differential transformation method involves an iterative procedure of which high-order Taylor series can be obtained without rigorous computation efforts.

Table 1: **Basic operational rules of the differential transform method**

Original function	Transformed function
$y(x) = u(x) \pm v(x)$	$Y(K) = U(K) \pm V(K)$
$y(x) = au(x)$	$Y(K) = aU(K)$
$y(x) = du/dx$	$Y(K) = (K+1)U(K+1)$
$y(x) = d^n u/dx^n$	$Y(K) = (K+1)(K+2)...(K+n)U(K+n)$
$y(x) = u(x)v(x)$	$Y(K) = \sum_{r=0}^{k} u(r)v(K-r)$
$y(x) = x^n$	$Y(K) = \partial(K-n) = \begin{cases} 1, K = n \\ 0, K \neq n \end{cases}$

In this study as it is the case in many other studies lower case letter of the English alphabet is used to represent the original function and the upper case letter to represent the transformed function. The mathematical operations shown in Table 1 represent the transformed functions obtained from the definitions of Eq. (1) and Eq. (2). For the purpose of implementation, the function y (x) is expressed by a finite series so that Eq. (2) becomes

$$y(x) = \sum_{k=0}^{n} x^k Y(k), \tag{4}$$

Where the positive integer n is obtained based on the convergence of natural frequency of the equation. Also, the sum of further terms is taken to be negligibly small; that is

$$\sum_{k=n+1}^{\infty} x^k Y(k) \to 0 \tag{5}$$

Unlike the other semi-analytic methods, the differential transform method transforms the nonlinear problems into algebraic equations. So, we do not get into difficulties such as linearization, integral equations, perturbations and calculations of the Adomian polynomials.
It has been established by (Oke, 2017) and (Rebenda J, Zdenek S, Yasir K, 2017) that the differential transform converges efficiently to exact solution. As the solution is actually approximated by the finite Taylor polynomial, it is possible to use the criteria for convergence of the Taylor series.

3. Description of the Problem

We consider a magneto-hydrodynamic steady natural convection flow of a viscous fluid of constant density and electrically conducting with applied induced magnetic field between two very long vertical parallel plates as shown in Figure 1 below.

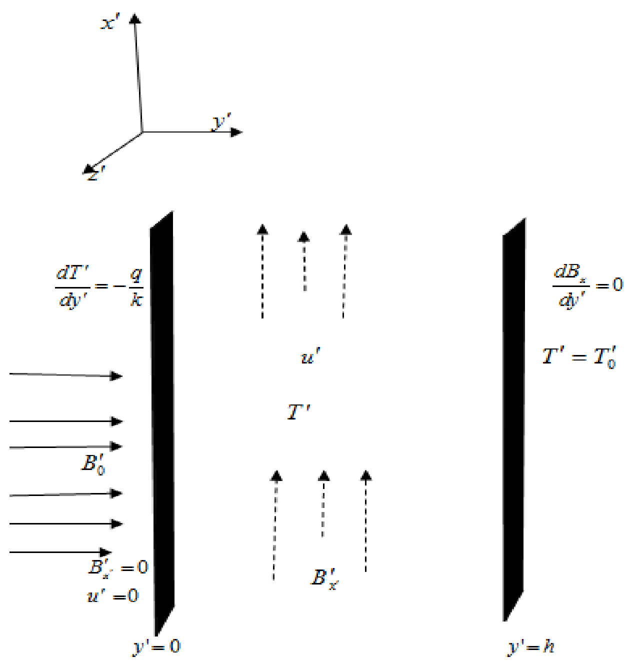


Figure 1: Geometry of the flow

The two plates are positioned vertically parallel and at a distance h to each other. A constant heat flux is employed at the plate $y' = 0$ while the other plate at $y' = h$ is maintained at an isothermal condition. Also, u' is the component of the velocity x' (along the plates) and v' is the component of the velocity y' (normal to the plates). The velocity of the fluid has only one nonzero component in the x' direction since the variables describing the flow depend only on the transverse coordinate y'. This is the case because of the geometry of the flow. Furthermore, a transverse uniform magnetic field of strength B_0' is applied to the plates at the direction of flow x'. The plate at $y' = 0$ is taken to be electrically non- conducting while the plate at the position $y' = h$ is taken to be conducting. The electrical conductivity of the flowing fluid (σ) induces a magnetic field $B_{x'}'$ along the $x' - axis$. Also, ρ is the fluid density and C_p is the specific heat when the pressure is constant.

The governing equations following (Sarveshanand and Singh, 2015), with the additional assumption that the viscosity and the thermal conductivity vary with temperature and considering the induced magnetic field, in a no slip regime are given as:

$$\frac{1}{\rho}\frac{d}{dy'}(\mu^*\frac{du'}{dy'})+\frac{\mu_e B_0}{\rho}\frac{dB'_{x'}}{dy'}+g\beta(T'-T'_0)=0 \quad (6)$$

$$\frac{1}{\sigma\mu_e}\frac{d^2B'_{x'}}{dy'^2}+B'_0\frac{du'}{dy'}=0 \quad (7)$$

$$\frac{1}{\rho C_p}\frac{d}{dy'}(k^*\frac{dT'}{dy'})=0 \quad (8)$$

Where Eq. (6) represents the velocity field equation in which the variable viscosity is a factor in the first term. The viscosity is taken as an inverse function of temperature following (Singh and Agarwal, 2013). That is

$$\frac{1}{\mu^*}=\frac{1}{\mu_0}\left(1+a\left(T'-T_0\right)\right) \quad (9a)$$

Eq. (9a) is equivalent to Eq. (9b) below

$$\mu^*=\mu_0(1-a(T'\Delta T)). \quad (9b)$$

Eq. (7) is the expression for the induced magnetic field and Eq. (8) is the expression for the temperature field with the variable thermal conductivity taken following (Hazarika and Konch, 1986), as

$$\frac{1}{k^*}=\frac{1}{k_0}\left(1+b\left(T'-T_0\right)\right). \quad (10a)$$

This is similar to Eq. (10b) below

$$k^*=k_0(1-b(T'\Delta T)). \quad (10b)$$

The symbols a, b and T_0 are constants whose values depend on the standard state and thermal properties (that is υ-kinematic viscosity and k-thermal conductivity) of the fluid. Also, for liquids the value of a is greater than zero (a > 0) and for gases its value is less than zero (a < 0); μ_0 and k_0 are the dynamic viscosity and thermal conductivity of the fluid far away from the wall (ambient fluid).

The boundary conditions are given as

$$u'=0, B'_{x'}=0, \frac{dT'}{dy'}=-\frac{q}{k^*}, at, y'=0 \quad (11)$$

With

$$u'=0, \frac{dB'_{x'}}{dy'}=0, T'=T'_0, at, y'=h \quad (12)$$

4. Solution Method

Applying the non-dimensional parameters represented by Eq. (13) below.

$$y=\frac{y'}{h}, u=\frac{\nu_0 u'}{g\beta h^2\Delta T'}, B=\frac{\nu'}{g\beta h^2\Delta T'}\sqrt{\frac{\mu_e}{\rho}}B'_{x'},$$

$$T=\frac{T'-T'_0}{\Delta T^{'}}, \Delta T'=\frac{hq}{k}, \tag{13}$$

$$Pm=\nu_0\sigma\mu_e, Ha=\frac{B_0 h}{\nu_0}\sqrt{\frac{\mu_e}{\rho}}, \lambda=\mathrm{a}\Delta\mathrm{T},$$

$$\varepsilon=b\Delta T.$$

The governing equations are obtained in non-dimensional form as the set of nonlinear second order ordinary differential Eqs. (14) - (16) below.

$$\frac{d^2u}{dy^2}-\lambda(1+\lambda T)\frac{dT}{dy}\frac{du}{dy}+Ha\frac{dB}{dy}+(1+\lambda T)T=0 \tag{14}$$

$$\frac{d^2B}{dy^2}+PmHa\frac{du}{dy}=0 \tag{15}$$

$$\frac{d^2T}{dy^2}-\varepsilon(1+\varepsilon T)\left(\frac{dT}{dy}\right)^2=0 \tag{16}$$

While the boundary conditions are transformed to Eqs. (17)-(18).

$$u=0, B=0, \frac{dT}{dy}=-1 at, y=0 \tag{17}$$

$$u=0, \frac{dB}{dy}=0, T=0 at, y=1 \tag{18}$$

The dimensionless parameters are defined as stated below:

$Ha=\frac{B_0 h}{\nu_0}\sqrt{\frac{\mu_e}{\rho}}$ Is the Hartmann number, $Pm=\nu_0\sigma\mu_e$ the magnetic Prandtl number, $\lambda=\mathrm{a}\Delta\mathrm{T}$ is the viscosity parameter and $\varepsilon=b\Delta T$ the thermal conductivity parameter.

The Eqs. (14)-(18) are solved by using the Differential Transform Method (DTM) (Zhou, 1986) to obtain the numerical solution with the help of the symbolic software called (Maple, 2016). Applying DTM to the Eqs. (14) - (18) we obtain the iterative relations (19) to (22) below.

$$U(K+2)=\frac{K}{(K+2)!}.\begin{pmatrix} \lambda.\sum_{r=0}^{K}(r+1).(K-r+1) \\ .U(r+1).T(K-r+1) \\ +\lambda^2.\sum_{r=0}^{K}\sum_{t=0}^{r}(t+1).(r-t+1) \\ .U(t+1).T(r-t+1).T(K-r) \\ -Ha.(K+1).B(K+1) \\ -T(K)-\lambda.\sum_{r=0}^{K}T(r).T(K-r) \end{pmatrix} \quad (19)$$

$$B(K+2)=\frac{K}{(K+2)!}.\left(-Pm.Ha.(K+1).U(K+1)\right) \quad (20)$$

$$T(K+2)=\frac{K}{(K+2)!}.\begin{pmatrix} \varepsilon.\sum_{r=0}^{K}(r+1).(K-r+1). \\ T(r+1).T(K-r+1). \\ \varepsilon^2.\sum_{r=0}^{K}\sum_{t=0}^{r}(t+1).(r-t+1).T(t+1). \\ T(r-t+1).T(K-r) \end{pmatrix} \quad (21)$$

Also, the boundary conditions give the expressions below

$$U(0)=0;U(1)=a;T(1)=-1;T(0)=c;B(0)=0;B(1)=d \quad (22)$$

Where $U(K)$, B (K) and T (K) are the differential transforms of U(y), B(y) and T(y) respectively. The unknown constants a, c and d from the boundary conditions are determined for various fluid situations using the software called (Maple, 2016). Hence, the polynomials (23)-(29) obtained by using the DTM are the expressions for the Velocity field, the induced Magnetic field, the Temperature field, the Induced current density, the skin friction and the Nusselt number respectively.

$$\begin{aligned} U(y) &= a.y - 1/2.(1/2\lambda^2.a.c + 1/2\lambda.c^2 + 1/2.Ha.d + 1/2\lambda.a + 1/2.c).y^2 + 1/6.(\lambda..\varepsilon^2.a.c \\ &+ \lambda..\varepsilon.a + \lambda^3.a.c + \lambda^2.c^2 + \lambda.Ha.d + \lambda^2.a + \lambda.c + a..\lambda^2 + \lambda^2..\varepsilon^2.a.c^2 + \lambda^2..\varepsilon..a.c + \lambda^4.a.c^2 \\ &+ \lambda^3.c^3 + \lambda^2.Ha.d.c + \lambda^3.a.c + \lambda^2.c^2 + Ha^2.Pm.a + 1 + 2.\lambda.c).y^3 + \ldots \end{aligned} \quad (23)$$

$$\begin{aligned} B(y) &= d.y - 1/2.a.Pm.Ha.y^2 - 1/3.Pm.Ha.(-1/2\lambda^2.a.c - 1/2.\lambda.c^2 - 1/2.Ha.d \\ &- 1/2.\lambda.a - 1/2.c).y^3 - 1/4.Pm.Ha.[1/6\lambda.a..\varepsilon.(\varepsilon.c+1)(1+\lambda.c) \\ &+ 1/3\lambda^2.c.(\lambda.a + c)(1+1/2\lambda.c) + 1/6.Ha\lambda.d.(1+\lambda.c) + 1/3\lambda^2.a + 1/2\lambda.c \\ &+ 1/6.(Ha)^2.Pm.a + 1/6].y^4 + \ldots \end{aligned} \quad (24)$$

$$\mathrm{T(y)} = \mathrm{c} - \mathrm{y} + \varepsilon/2.(\varepsilon.\mathrm{c}+1).\,\mathrm{y}^2 - \varepsilon^2/6.(3+4.\varepsilon.\mathrm{c}+2.\varepsilon^2.\mathrm{c}).\,\mathrm{y}^3$$
$$+1/6.(1/2.(1+\varepsilon^2.\mathrm{c}).(\varepsilon^2.\mathrm{c}+\varepsilon)^2 - \mathrm{c}^3\varepsilon^6 + \mathrm{c}^2\varepsilon^5 + 1/4.(11+15\mathrm{c}).\varepsilon^4 \qquad (25)$$
$$+(11/4)\varepsilon^3)\,\mathrm{y}^4 + \dots$$

$$\mathrm{J} = \mathrm{d} - \mathrm{a.Pm.Ha.y} - \mathrm{Pm.Ha.}(-(1/2)\lambda^2.\mathrm{a.c} - (1/2)\lambda.\mathrm{c}^2 - (1/2).\mathrm{Ha.d} - (1/2)\lambda.\mathrm{a} - (1/2).\mathrm{c}).\mathrm{y}^2$$
$$-\mathrm{Pm.Ha.}((1/6)\lambda.(2\mathrm{a}.((1/2)\varepsilon^2.\mathrm{c}+(1/2)\varepsilon)+\lambda^2.2.\mathrm{a.c}+\lambda.\mathrm{c}^2+\mathrm{Ha.d}+\lambda.\mathrm{a}+\mathrm{c})+(1/6).\lambda^2\dots$$
$$+(1/2).\varepsilon).\mathrm{c} - (2.(-(1/2)\lambda^2.\mathrm{a.c} - (1/2)\lambda.\mathrm{c}^2 - (1/2).\mathrm{Ha.d} - (1/2)\lambda.\mathrm{a} - (1/2).\mathrm{c}))\mathrm{c})$$
$$+(1/6).\mathrm{Ha}^2.\mathrm{Pm.a} + 1/6 + (1/3)\lambda.\mathrm{c}).\,\mathrm{y}^3 + \dots \qquad (26)$$

$$\tau_0 = (1-\lambda T)\left(\frac{dU}{dy}\right)_{y=0} = (-\mathrm{c}.\lambda+1).\mathrm{a} \qquad (27)$$

$$\tau_1 = -(1-\lambda T)\left(\frac{dU}{dy}\right)_{y=1} = (\mathrm{c}-1+(1/2).\varepsilon^2.\mathrm{c}$$
$$+(1/2)\varepsilon+(1/6)\varepsilon.(-2.\mathrm{c}\varepsilon^2-2.\varepsilon)+(1/6).\mathrm{Ha}^2.Pm.\lambda.c^2+\dots \qquad (28)$$

$$\mathrm{Nu} = (1-\varepsilon T)\left(\frac{dT}{dy}\right)_{y=1} = \mathrm{c}.\varepsilon - 1 \qquad (29)$$

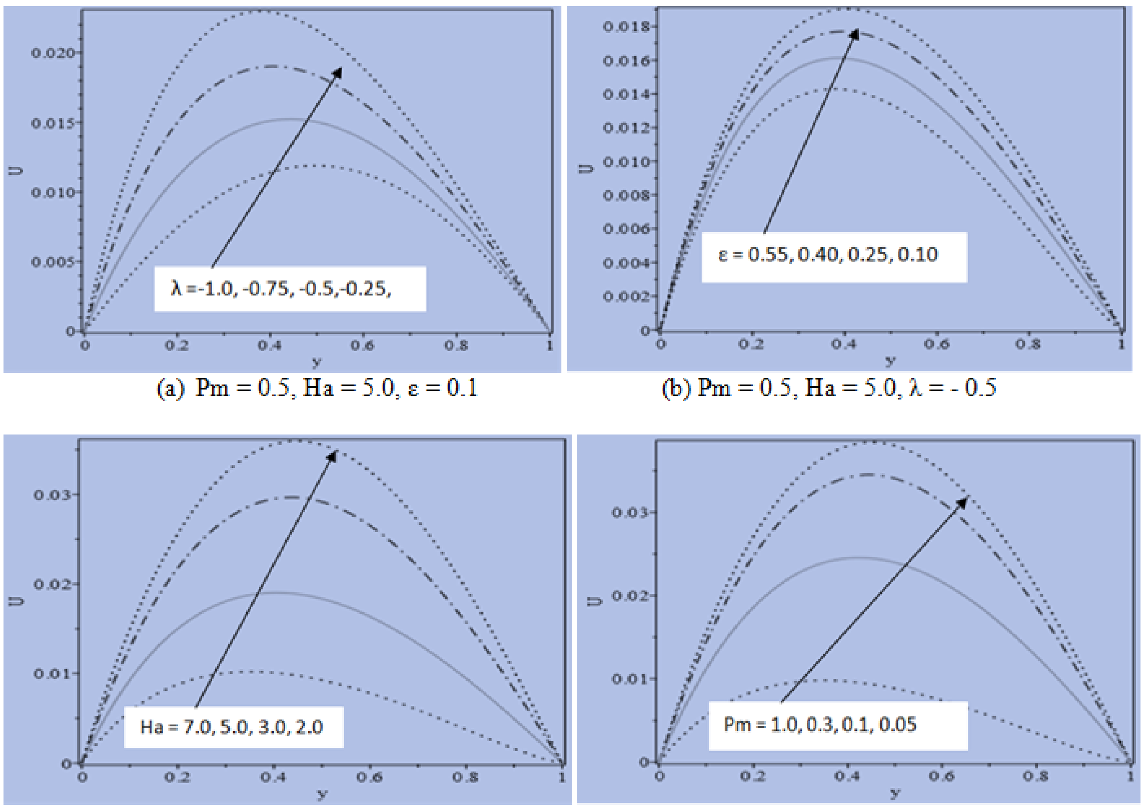
The skin friction (τ) in a streamline flow is the gradient of the fluid velocity at the walls. While the ratio of convection heat transfer to the conduction heat transfer call the Nusselt number (Nu) is the dimensional temperature gradient at the surface of the fluid. It measures the convection heat transfer at the surface of the fluid.

Considering fluid flow with constant viscosity and thermal conductivity as in the case of (Sarveshanand and Singh, 2015) the viscosity and conductivity parameters λ and ε respectively are taken to be zero in the present problem. The results are then compared with that of (Sarveshanand and Singh, 2015) when the suction velocity is neglected. The comparison which is presented in Table 4 below shows an excellent agreement up to an appreciable degree of accuracy.

5. Results and Discussion

The system of differential Eqs. (14)-(18) including the boundary conditions which governs the natural convection flow with induced magnetic field, variable viscosity and variable thermal conductivity are solved by an efficient quasi-analytic technique called the Differential Transform Method proposed by (Zhou, 1986). The method is programmed and implemented by using computer software called (Maple, 2016). The convergence of DTM as a solution technique has been established in the work of (Oke, 2017). Hence, this gives the justification for the use of the method in the present problem. The problem is described by the variable viscosity parameter (λ), the variable thermal conductivity parameter (ε), the Hartmann number (Ha) and the magnetic

Prandtl number (Pm). To achieve the objectives of this discussion, the values of the parameters have been chosen as follows: $\lambda = -0.5$, $\varepsilon = 0.1$, $Pm = 0.5$, $Ha = 5.0$. It is necessary to note that negative values of λ signify that viscosity grows with increasing temperature. While positive values of it signifies a situation where the fluid viscosity decreases when temperature increases. Hence, the results are shown graphically by plotting the values of the dimensionless velocity U, the dimensionless temperature T and the dimensionless Induced magnetic field B from Figures 2-9. Each representing the results of the variation of a particular parameter involved in the problem. The velocity profiles are plotted in Fig. 2. The temperature profile is shown in Figure 3 and the Induced magnetic profiles in Figure 4.



(a) $Pm = 0.5$, $Ha = 5.0$, $\varepsilon = 0.1$ (b) $Pm = 0.5$, $Ha = 5.0$, $\lambda = -0.5$

(c) $Pm = 0.5$, $\varepsilon = 0.1$, $\lambda = -0.5$ (d) $Ha = 0.5$, $\varepsilon = 0.1$, $\lambda = -0.5$

Figure 2: Velocity profile varying the parameters (a) λ (b) ε (c) Ha (d) Pm

Varying the viscosity parameter λ as displayed in Fig. 2(a), we see that as λ increases, the velocity profile increases. This is true because increasing the variable viscosity parameter λ means that the fluid viscosity is decreasing (Eq. 9). This decreases the boundary layer thickness and invariably increases the velocity. This is in agreement with established literature. In Fig. 2(b), the thermal conductivity parameter ε is varied and it is discovered that as ε decreases, the velocity increases. This is physically true since a decrease in ε increases the fluid's thermal conductivity and also increases the thermal boundary layer and consequently an increased velocity is achieved. Also, Fig. 2(c) illustrates the velocity profile resulting from the varying the Hartmann number (Ha). It shows that the decrease in Hartmann number causes the velocity profile to increases. This happens because increasing Ha increases the magnetic field which causes a force known as the Lorentz force that works against the motion of the fluid. Fig. 2(d) shows the velocity profile when we vary the magnetic Prandt number (Pm). It shows that as Pm decreases the velocity profile increases. Usually a decrease Pm reduces the induced magnetic field leading to lesser development of electric current and reduction in the thickness of the momentum boundary layer. (Raju C, Sandeep N, and Saleem S, 2016). This results into an increase of the flow velocity. We can also see that the Temperature profile decreases as the thermal conductivity parameter (ε) increases in Fig. 3. This is the case because it is an inverse linear function of temperature.

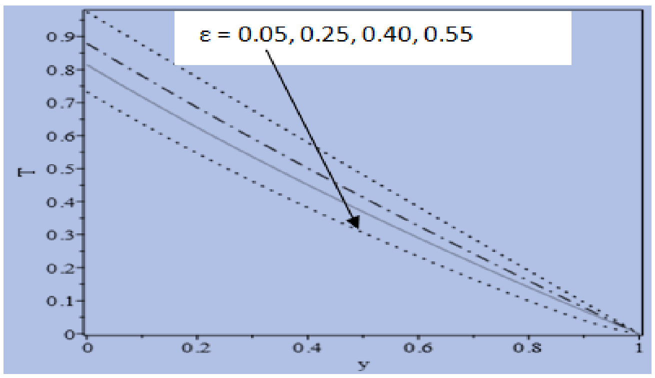
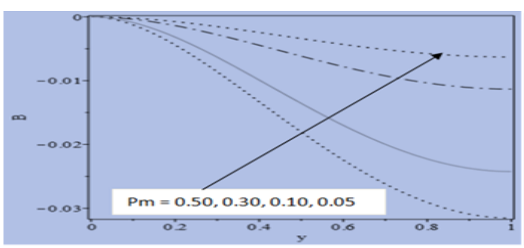
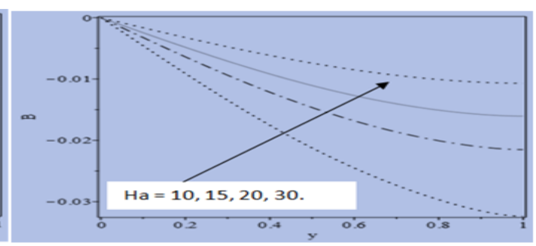


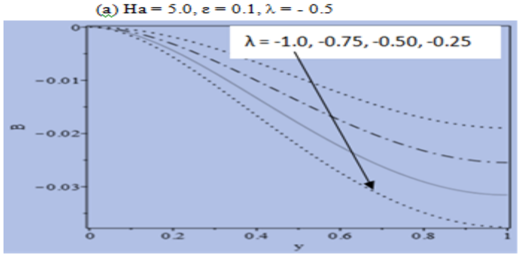
Figure 3: Temperature profile varying thermal conductivity parameter
Ha = 5.0, Pm = 0.5, λ = - 0.5



(a) Ha = 5.0, ε = 0.1, λ = - 0.5



(b) Pm = 0.5, ε = 0.1, λ = - 0.5

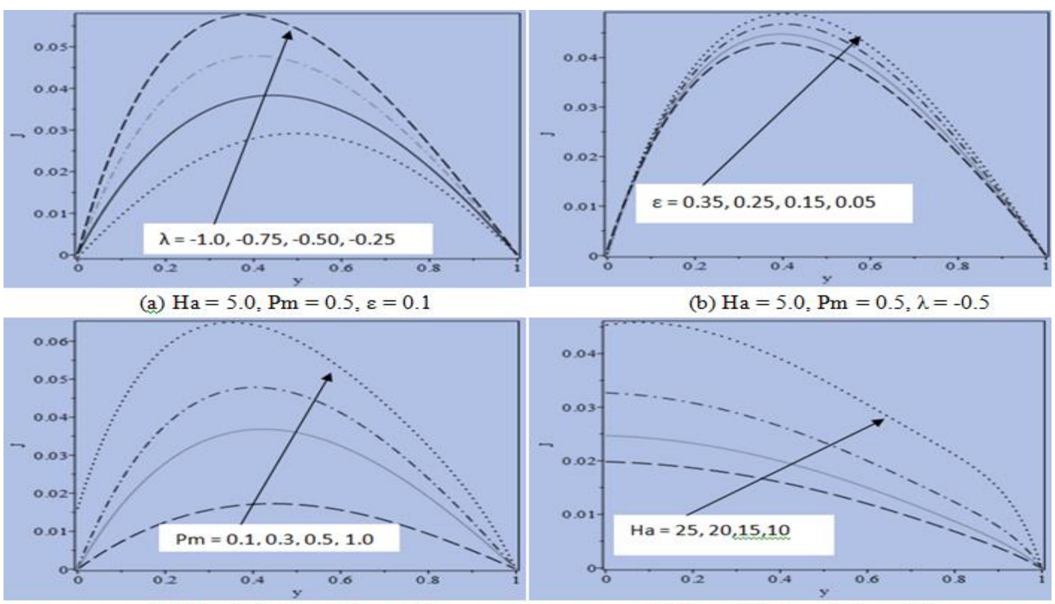


(c) Pm = 0.5, ε = 0.1, Ha = 5.0

Figure 4: Induced magnetic field profile varying (a) Pm (b) Ha, (c) λ

Figure 4(a) shows the induced magnetic field (B) profile decreasing as the magnetic Prandt number (Pm) increases. This occurs because the magnetic diffusion rate is dominant over the viscous diffusion rate (since Pm < 1). For the Hartmann number (Ha), the B profile increases as Ha increases shown in Fig. 4(b). An enhancement of Ha results in the increase in strength of the applied magnetic field and as expected this increases the strength of B too. Finally Fig. 4(c) shows the B profile decreasing as the viscosity parameter λ increases. This is expected since increase in viscosity slows down the velocity of flow of the fluid which invariably reduces the induced magnetic field. It is also in agreement with the work of (Sarveshanand and Singh, 2015).

Figure 4 shows the effect of the viscosity parameter (λ), thermal conductivity parameter (ε), Hartmann number (Ha) and Pm respectively on the induced current density (J). In Fig. 4(a) it is observed that for increasing values of the viscosity parameter, J increases. As λ increases the viscosity of the fluid decreases so that the velocity of the flow increases, hence the induced current increases. Whereas increasing the values of the thermal conductivity parameter, J decreases as shown in Fig. 4(b). This is because increased thermal conductivity reduces the flow velocity. Meanwhile, the flow velocity of a fluid which conducts electricity when induced magnetic field is present has direct relation to induced current density. Fig. 4(c) shows increasing induced current density as Pm increases. Increasing Pm enhances the strength of the induced magnetic field through the boundary layer which helps in increasing the induced current density. Also, the induced current density decreases for increasing values of Ha as shown in Fig. 4(d). This happens because increase in Hartmann number reduces the value of B which in turns leads to decrease of the induced current density.



(a) Ha = 5.0, Pm = 0.5, ε = 0.1 (b) Ha = 5.0, Pm = 0.5, λ = -0.5

(c) Ha = 5.0, λ = - 0.5, ε = 0.1 (d) Pm = 0.5, λ = - 0.5, ε = 0.1

Figure 5: Induced current density profile varying (a) λ (b) ε (c) Pm (d) Ha

Table 1 below show the effects of the governing parameters on the coefficient of skin-friction (τ) for each of the two plates. Skin friction represents the shearing stress on the vertical plates. Also, Table 2 represents the coefficient of heat transfer. The Table 2(a) shows the impact of the viscosity parameter (λ) and the thermal conductivity parameter (ε) on the skin friction around the two plates. It can be seen that decreasing (λ) which means increasing the viscosity of the fluid results in decrease of the skin friction on both plates. As ε increases, the thermal conductivity decreases. The result is a reduction in the skin friction as is shown in the table. Table 2(b) shows the effects of Ha and Pm on the skin friction. By increasing Ha we observe a decrease of the skin friction on both plates. Likewise, increase in Pm number results in reduced skin friction on both plates. Table 2 shows that increasing the thermal conductivity parameter (ε) results in the increase of the Nusselt number. Table 3 illustrates the result of comparing the present work with that of (Sarveshenand and Singh, 2015) where the effect of suction is neglected and viscosity and thermal conductivity are taken to be constants. The table shows that the results agree to a very appreciable degree.

Table 2: The effect of viscosity and thermal conductivity parameters on Skin friction

Pm = 0.5, Ha = 5.0, ε = 0.1			Pm = 0.5, Ha = 5.0, λ = -0.5		
λ	τ_0	τ_1	ε	τ_0	τ_1
-0.25	0.1873328767	0.05480475148	0.06	0.1710445466	0.05226560801
-0.50	0.1681933866	0.05076471212	0.18	0.1623129241	0.04776437190
-0.75	0.1337382533	0.04606872428	0.30	0.1529528984	0.04275929348
-1.0	0.08527815485	0.0429504871	0.42	0.1418072596	0.03449723841

Table 3: The effect of Hartmann number and the Magnetic Prandtl number on Skin friction

ε = 0.1, Pm = 0.5, λ = - 0.5			ε = 0.1, Ha = 5.0, λ = -0.5		
Ha	τ_0	τ_1	Pm	τ_0	τ_1
			0.05		
2.0	0.2529463058	0.1063653069	0.10	0.2645022722	0.1145415174
3.0	0.2226774848	0.0855253785	0.30	0.2459785919	0.1014893390
5.0	0.1681933866	0.0507647123	1.0	0.1971261089	0.0687520409
7.0	0.1108471274	0.0096407477		0.1076783003	0.0069932707

Table 4: The effect of thermal conductivity parameters on heat transfer coefficient (Nu)

Pm = 0.5, Ha = 5.0, λ = -0.5	
ε	Nu
0.06	0.9417954488
0.18	0.8358788851
0.30	0.7428475456
0.42	0.6607550459

Table 5: The comparison of the present work with that of Sarveshanand and Singh, 2015

	Sarveshanand and Singh $V = 0,\ y = 0.5$		Present work $\lambda = \varepsilon = 0,\ y = 0.5$	
Pm	U	B	U	B
0.25	0.0376366	0.0188846	0.0376366	0.0188846
0.5	0.0267296	0.0282822	0.0267296	0.0282822
1.5	0.0120880	0.0438716	0.0120880	0.0438716
2.0	0.0094176	0.0475406	0.0094176	0.0475406

CONCLUSION

The combined effect of viscosity and thermal conductivity (both of which are temperature dependent) on a steady two dimensional, magneto-hydrodynamic, natural convection boundary layer flow of an incompressible fluid which is viscous and electrically conducting is presented in this paper. The effects of all the parameters involve in the problem are tested on U, B, and T; J and the skin friction. The results have been shown in graphs and tables and discussed extensively. From the investigation, it is concluded that the influence of temperature dependence in viscosity and conductivity is so important on the flow pattern and thermodynamics such that neglecting this condition of the two important fluid properties makes the results obtained in the convection problem to be either under determined or over determined. In addition, the study also concluded that:

(1) Increasing the viscosity of the fluid, the Magnetic Prandtl number, the Hartmann number and reducing the conductivity of the fluid retards the velocity profile of the flow.

(2) The temperature profile decreases due to increases in the thermal conductivity parameter.

(3) The induced Magnetic profile is enhanced by increasing the values of the Hartmann number; it is however decreased by increasing the values of the Magnetic Prandtl number and decreasing the viscosity of the fluid.

(4) When the viscosity of the fluid and the Hartmann number are increased, the induced current density reduces on both plates. But it decreases on both plates when magnetic Prandtl number is increased and the conductivity of the fluid is reduced.

(5) Increasing the viscosity of the fluid by reducing the value of the viscosity measuring parameter λ, and decreasing the thermal conductivity by increasing the thermal conductivity parameter ε, is to reduce the skin friction in the two plates.

(6) Increasing the Hartmann number and the magnetic Prandtl number reduces the skin friction on both plates.

(7) As the thermal conductivity reduces by increasing the thermal conductivity parameter ε, the convection heat exchange in the fluid increases.

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Declaration of Conflicting Interests
The Authors declare that there is no conflict of interest.

NOMENCLATURE	
b channel width(gab between the plates)	
$B'_{x'}$ Induced Magnetic field in the x' direction	μ^* Variable viscosity of the fluid
B_0 constant magnetic flux density	β Coefficient if thermal expansion
g gravitational acceleration	ρ Density of the fluid
Ha Hartmann number	v Fluid kinematic viscosity
J Induced current density	σ Electrical conductivity of the fluid
Nu Nusselt number	μ_e Magnetic permeability
T Temperature of the fluid	κ^* Variable thermal conductivity of the fluid
u' Velocity of the fluid along x' axis	
v' Velocity of fluid along the y' axis	
T_0 Reference temperature	
Pm magnetic Prandtl number	

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