ISOMORPHISM OF NORMAL CLOSURE OF SUBSET OF ORDER $\mathbf{k}$ ($\mathbf{k} \in \mathbb{N}, \mathbf{k} < \infty$) IN A FINITE GROUP

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Abstract
In this paper, isomorphism of normal closure (N) of a subset H of a group G were investigated using S_n where $n = 8$. The subsets of N were shown to be abelian, cyclic and found to be subgroup of G. Finally normal closure (N) of G were shown to be isomorphic to conjugacy class $C_4 \times C_4$ which was confirmed by an existing theorem.

Keywords*: Isomorphism; Normal Closure of a Group; Finite group; Conjugacy Classes.*

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1. Introduction

Let G be a finite group of order k where $\mathrm{k} = 2^4$. A normal closure of a subset H of a group G is the intersection of all normal subgroups of G containing H. Clearly the normal closure of H is a normal subgroup generated by H. Those elements of a group that generates or forms the group by their products or product of their powers or their inverses are the generators of that group.
In what follows, the elements of G are permutation group gotten from $\mathrm{S_n}$ where $\mathrm{n} = 8$ with its operation "o" as composition of mapping. These elements are generator of the finite group which are derived by the product of the subgroup and its inverses. The elements of normal closure are closed and normal in G. Many authors have worked on normal closure of subset of a group producing sound results. Sibertin, (1980) demonstrated that if the elements $\mathrm{S} \subset \mathrm{F}$ where F is a free group are solution of an equation $w(x)$, then $w(x)$ belongs to the normal closure of finitely many short equations associated to S. Herg *et al.*, (2014) presented a research work on the normal closure of cyclic subgroup yielding positive results. McHaffey, (1965) and Coleman, (1962) worked on isomorphism of finite groups but none have been done on isomorphism of normal closure of subset H of a finite group G. This work is motivated by the earlier work done by (Aja *et al.*, 2019). The objective of this paper is to show that normal closure of subset H of a group G is isomorphic to $\mathrm{C_4} \times \mathrm{C_4}$ and not to $\mathrm{C_8} \times \mathrm{C_2}$. In section two, we outline definitions, theorems and lemma. In section three, isomorphism of normal closure is shown using permutation group and in section four we conclude our results.

2. Preliminaries

Definition 2.1 A group is an ordered pair $(\mathrm{G},*)$ where G is a set and $*$ is a binary operation on G satisfying the following axioms:
(i) $(\mathrm{a} * \mathrm{b}) * \mathrm{c} = \mathrm{a} * (\mathrm{b} * \mathrm{c})$ for all $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{G}$ that is $*$ is associative
(ii) There exist an element e in G called an identity of G, such that for all $\mathrm{a} \in \mathrm{G}$ we have $\mathrm{a} * \mathrm{e} = \mathrm{e} * \mathrm{a} = \mathrm{a}$

(iii) For each $a \in G$, there is an element a^{-1} of G, called an inverse of a such that $a * a^{-1} = a^{-1} * a = e$

Definition 2.2 The group $(G,*)$ is called abelian (or commutative) if $a * b = b * a$ for all $a, b \in G$.

Proposition 2.3 If G is a group under the operation $*$ then

(i) The identity of G is unique

(ii) For each $a \in G, a^{-1}$ is uniquely determined

(iii) $(a^{-1})^{-1} = a$ for all $a \in G$.

(iv) $(a * b)^{-1} = (b^{-1}) * (a^{-1})$

Definition 2.4 Let G be a group and $x \in G$, then the order of x is the smallest positive integer n such that $x^n = 1$. It is denoted by $|x|$. In this case, x is said to be of order n. If no positive power of x is the identity, the order of x is defined to be infinity and x is said to be of infinite order.

Definition 2.5 Let $(G,*)$ and (H, o) be groups. A map $\varphi: G \to H$ such that $\varphi(x * y) = \varphi(x) \text{ o } \varphi(y)$ for all $x, y \in G$ is called a homomorphism.

Definition 2.6 The map $\varphi: G \to H$ is called an isomorphism and G, H are said to be isomorphism or of the same isomorphism type written $G \cong H$ if

(i) φ is a homomorphism

(ii) φ is a bijection

Definition 2.7 A group action of a group G on a set A is a map from $G \times A$ to A (written as $g.a$, for all $g \in G$ and $a \in A$) satisfying the following properties

(i) $g_1.(g_2.a) = (g_1 g_2).a$ for all $g_1, g_2 \in G, a \in A$ and

(ii) $1.a = a$ or all $a \in A$

Definition 2.8 Let G be a group. The subset H of G is a subgroup of G if H is nonempty and is closed under product and inverses (that is $x, y \in H$ implies $x^{-1} \in H$ and $xy \in H$). If H is a subgroup of G we shall write $H \leq G$

Definition 2.9 Let G be a finite group consisting of n elements, a permutation group of degree n is a one to one mapping of S_n onto itself.

Proposition 2.10 If $H = \langle x \rangle$ then $|H| = |x|$. More specifically

(i) If $|H| = n < \infty$ then $x^n = 1$ and $1, x, x^2, \dots x^{n-1}$ are all the distinct elements of H and

(ii) If $|H| = \infty$ then $x^n \neq 1$ for all $n \neq 0$ and $x^a = x^b$ for all $a \neq b$ in $\mathbb{Z}$.

Proof: Let $|x| = n$ and $n < \infty$ then the elements $1, x, x^2, \dots, x^{n-1}$ are distinct because if $x^a = x^b$ with $0 \leq a < b < n$, then $x^{b-a} = x^0 = 1$ contrary to n being the smallest positive power of x giving the identity. Thus H has at least n elements. Now if x^t is any power of x, using division algorithm we write $t = nq + k$ where $0 \leq k < n$, so $x^t = x^{nq+k} = (x^n)^q x^k = 1^q x^k = x^k \in \{1, x, x^2, \dots, x^{n-1}\}$

Proof: Suppose $|x| = \infty$ then no positive power of x is the identity. If $x^a = x^b$ for some a and b with say $a < b$, then $x^{b-a} = 1$, a contradiction. Distinct powers of x are distinct element of H so $|H| = \infty$

Proposition 2.11 Let G be an arbitrary group $x \in G$ and let $m, n \in \mathbb{Z}$. If $x^n = 1$ and $x^m = 1$ then $x^d = 1$ where $d = (m, n)$. In particular if $x^m = 1$ for some $m \in \mathbb{Z}$, then $|x|$ divides m.

Theorem 2.12 Any two cyclic groups of the same order are isomorphic. More specifically, if $n \in \mathbb{Z}^+$ and $\langle x \rangle$ and $\langle y \rangle$ are both cyclic groups of order n, then the map

$\varphi: \langle x \rangle \to \langle y \rangle$

$x^k \mapsto y^k$

Is well defined and is an isomorphism.

Proof: Suppose $\langle x\rangle$ and $\langle y\rangle$ are both cyclic groups of order n. Let $\varphi:\langle x\rangle \to \langle y\rangle$ be defined $\varphi(x^k) = y^k$, we prove that φ is well defined that is if $x^r = x^s$ then $(x^r) = \varphi(x^s)$. $x^r = x^s \Rightarrow \frac{x^r}{x^s} = 1 \Rightarrow x^{r-s} = 1$. By Proposition 2.10 $n|r-s$. Write $r = tn + s$ so $\varphi(x^r) = \varphi(x^{tn+s}) = y^{tn+s} = (y^n)^t y^s = y^s = \varphi(x^s)$.

Hence φ is well defined.

Let $x^a, x^b \in \langle x\rangle$ then $\varphi(x^a) = y^a$ and $\varphi(x^b) = y^b$. $\varphi(x^a x^b) = y^a y^b = \varphi(x^a)\varphi(x^b)$. Hence φ is homomorphism. Since the element y^k of $\langle y\rangle$ is the image of x^k under φ, then φ is surjective. Since both groups have the same finite order, any surjection from one to the other is a bijection. Hence φ is an isomorphism. See (Dummit and Foote, 2004).

Definition 2.13 Let G be a group, then two elements a and b of G are said to be conjugate in G if there is some $g \in G$ such that $b = gag^{-1}$(that is if and only if they are in the same orbit of G acting on itself by conjugation). The orbit of G acting on itself by conjugation are called the conjugacy classes of G.

Definition 2.14 Let G be a finite group, a conjugacy class $C(G)$ is a non-empty subset of G such that the following holds

(i) Given any $x, y \in C$ there exist $g \in G$ such that $gxg^{-1} = y$

(ii) If $x \in C$ and $g \in G$ then $gxg^{-1} \in C$. In other words it is closed under the action of group on itself.

3. Results and Discussion

Let G be a finite group and H a subset of G. The normal closure N of H is defined by $N = \{g^{-1}xg: x \in H, g \in G\}$. The elements of N are defined as follows:

$N = \{1, x, x^2, x^3, x^y, (x^y)^2, (x^y)^3, xx^y, x(x^y)^2, x(x^y)^3, x^2(x^y), x^2(x^y)^2, x^2(x^y)^3, x^3(x^y), x^3(x^y)^2, x^3(x^y)^3\}$. Thus N has sixteen elements. These elements are displayed below:

$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$ $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$

$x^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$ $x^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 5 & 6 & 7 & 8 \end{pmatrix}$

$x^y = y^{-1}xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$ $(x^y)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 5 & 8 & 7 \end{pmatrix}$

$(x^y)^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 8 & 7 & 5 & 6 \end{pmatrix}$ $xx^y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

$x(x^y)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 6 & 5 & 8 & 7 \end{pmatrix}$, $x(x^y)^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 8 & 7 & 5 & 6 \end{pmatrix}$

$\mathbf{x^2(x^y)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 7 & 8 & 6 & 5 \end{pmatrix}$, $x^2(x^y)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix}$

$x^2(x^y)^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 8 & 7 & 5 & 6 \end{pmatrix}$ $x^3x^y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 7 & 8 & 6 & 5 \end{pmatrix}$

$x^3(x^y)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix}$ $x^3(x^y)^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix}$

Having listed the elements of N, we now want to show that N is abelian

Given $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$

$y^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 3 & 4 & 1 & 2 \end{pmatrix}$

$y^{-1}xy =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$

Thus $y^{-1}xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$

Since x commute with all its powers, we will consider two elements x and x^y of N

$xx^y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

$x^y x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

Since $xx^y = x^y x$, it follows that N is abelian.

Now given that $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$

$x^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}$

$x^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix} =$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 5 & 6 & 7 & 8 \end{pmatrix}$

$x^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = e \qquad (1)$$

From (1) x has order 4 since 4 is the smallest positive integer such that $x^4 = e$. Therefore a subset say $W = \{1, x, x^2, x^3\}$ of N is cyclic. This subset W is a subgroup of G which implies that W is a cyclic subgroup of G.

Again, consider $x^y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$

$(x^y)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 5 & 8 & 7 \end{pmatrix}$

$(x^y)^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 5 & 8 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 8 & 7 & 5 & 6 \end{pmatrix}$

$(x^y)^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 8 & 7 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 7 & 8 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = e$ (2)

Consider also $xx^y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

$(xx^y)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix}$

$(xx^y)^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 8 & 7 & 5 & 6 \end{pmatrix}$

$(xx^y)^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 8 & 7 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 1 & 2 & 7 & 8 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = e$ (3)

From (2) x^y has order 4 because 4 is the smallest positive integer such that $(x^y)^4 = e$. Similarly from (3) xx^y has order 4 since 4 is the smallest positive integer such that $(xx^y)^4 = e$. Therefore two subsets say $Y = \{1, x^y, (x^y)^2, (x^y)^3\}$ and $B = \{1, xx^y, (xx^y)^2, (xx^y)^3\}$ of N is cyclic. Y and B are subgroups of G which implies that they are cyclic subgroup of G.

Theorem 3.1 (Fundamental theorem of finitely generated abelian group): This state that every finitely generated abelian group G is isomorphic to a direct product of cyclic groups.

4. Conclusion

From our results, the normal closure of a subset of a finite group have shown to be isomorphic to conjugacy classes. This is achieved by showing that the subset of N is abelian and cyclic. Permutation group was also utilized to showcase the cyclic nature of N. Then using theorem 3.1 it follows that N is isomorphic to $C_4 \times C_4$ since $\langle x \rangle$ is isomorphic to C_4 and $\langle x^y \rangle$ is isomorphism to C_4 and N has no order greater than $2^2 = 4$.

References

Aja, R.O., Obasi, U.E. and E.O. Eze (2019), The Number of Conjugacy Classes and Irreducible Characters in a Finite Group. Earthline Journal of Mathematical Sciences 2(1): 181-190. https://doi.org/10.34198/ejms.2119.181190

Coleman, D.B. (1962), Finite groups with Isomorphic Group Algebras. Transactions of the American Mathematical Society 105(1):1-8.

Dummit, D.S. and Foote, R.M. (2004). Abstract Algebra Third Edition. John Wiley & Sons, Inc.

Herg L.V, Weizhou and Xiuyun G. (2014). Finite Groups with Small Normal Closure of Cyclic Subgroups. Journal of Communication in Algebra 49844996, https://doi.org/10.1080/00927872.2013.829080

McHaffey, R. (1965), Isomorphism of Finite Abelian Groups. Journal of American Mathematical Monthly 72(1):48-50, DOI: 10.2307/2313001, https://www.jstor.org/stable/2313001

Sibertin-Blanc, C. (1980), Normal closure of one variable equations in free groups. Proceedings of The American Mathematical Society 80(1): 1-5