MODELING A LIFETIME PROCESS WITH GOMPERTZ EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract
In this article, a new lifetime model called the Gompertz Exponential (Gom-E) distribution is proposed and studied. Some mathematical properties of the proposed distribution such as the cumulative distribution, hazard, survival, quantile, moment, generating functions and order statistics were derived. The estimated parameter values were obtained using the method of maximum likelihood estimator. A simulation was carried out to examine the flexibility of the proposed model. A real life application was also performed to further examine the applicability and the flexibility of the proposed model. The results show that the Gom-E model performs better than some existing models.

***Keywords**: Exponential distribution, Gompertz distribution, Maximum likelihood, Moment generating function, Quantile Function and Order statistics.*

1. Introduction

The study of new families of probability models by distribution theory researchers is motivated by the need to improve the flexibility of existing model. This involves compounding two or more distributions of either same distribution or different distributions (Alshawarbeh *et al.* 2013).

The exponential distribution plays a greater role in modeling memoryless processes in lifetime processes. However, of most important, the exponential distribution has peculiarity of cases with constant failure rate. Hence, the need to improve the exponential distribution to be able to model cases with non-constant failure rate.

In a bid to improve the flexibility of the exponential distribution, many researchers have proposed different versions of the exponential distribution. For example, Gupta and Kundu (1999) proposed the generalized exponential distribution. The failure time data was modeled using the Lehman alternative in Gupta, Gupta and Gupta (1998). Oguntunde *et al.* (2014a) proposed the Kumaraswamy inverse exponential distribution. Oguntunde *et al.* (2014b) proposed the exponentiated generalized inverted exponential distribution. Oguntunde and Adejumo (2014c) proposed the transmuted inverse exponential distribution. Anake et al. (2015) proposed the Fractional Beta exponential distribution. Abouammoh and Alshingiti (2009) proposed the generalized inverted distribution. Dey *et al.* (2017) proposed the generalized exponential distribution. Olapade (2014) proposed the extended generalized exponential distribution. Eghwerido *et al.* (2019) proposed the extended new generalized exponential distribution. Efe-Eyefia *et al.* (2020) proposed the Weibull alpha power inverted exponential distribution. Eghwerido *et al.* (2020a) proposed the Gompertz alpha power inverted exponential distribution. Nadarajah *et al.* (2014) proposed the truncated exponential skew symmetric distribution. Unal et al. (2018) proposed the alpha power inverted exponential distribution.

Rastogi and Oguntunde (2018) examined the performance of the Kumaraswamy inverse exponential distribution.

Let X be the random sample from the exponential distribution. Then, Gupta and Kundu (1999) defined the probability density function (pdf) of the exponential distribution as

$$g(x;\lambda)=\lambda\exp(-\lambda x),\ x>0,\ \lambda>0. \tag{1}$$

where λ is the scale parameter.

The corresponding cumulative distribution function (cdf) is given as

$$G(x;\lambda)=1-\exp(-\lambda x), \qquad x>0, \lambda>0. \tag{2}$$

The Gompertz-G family of distribution was proposed in Alizadeh et al. (2017). The probability density of the Gompertz-G family is given

$$f(x)=\varphi g(x)[1-G(x)]^{-\beta-1}\exp\left\{\left(\frac{\varphi}{\beta}\right)\left\{1-[1-G(x)]^{-\beta}\right\}\right\} \qquad \varphi>0, \beta>0 \tag{3}$$

where $g(x)$ and $G(x)$ are the baseline pdf and cdf.

The cumulative distribution function that corresponds to the pdf is given as:

$$F(x)=1-\exp\left\{\left(\frac{\varphi}{\beta}\right)\left\{1-[1-G(x)]^{-\beta}\right\}\right\}; \qquad \varphi>0, \beta>0 \tag{4}$$

where φ and β are extra shape parameters added to make the distribution more flexible.

This study is motivated as a result of inability to model real life scenarios with non-monotonically processes using the exponential distribution. Thus, the Gompertz distribution with a non-monotonically distributed is compounded with exponential distribution to introducing flexibility and non-monotonicity.

This study aim at proposing a class of the exponential distribution called Gompertz exponential distribution using the Gompertz-G family characterization.

2. The Gompertz Exponential Distribution (Gom-E)

Let $x_1, x_2, ..., x_n$ be a random sample from the Gom-E distribution, then the pdf of the Gom-E is given as

$$f_{GomE}(x)=\varphi\lambda\exp(-\lambda x)\exp\left\{(\beta+1)\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp[\beta\lambda x])\right\} \tag{5}$$

The cumulative distribution function of the Gom-E distribution is given as

$$F_{GomE}(x)=1-\exp\left\{\left(\frac{\varphi}{\beta}\right)\{1-\exp[\beta\lambda x]\}\right\} \tag{6}$$

Figure 1 shows the plot for different values of parameter for the Gom-E distribution. The plot shows that the distribution is skewed to the left and could be decreasing.

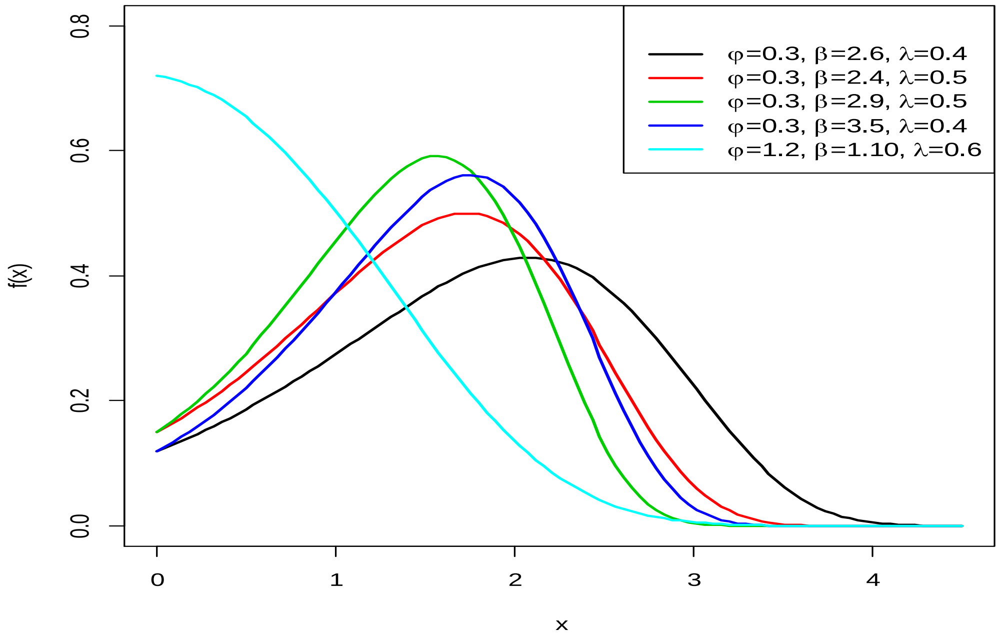

Figure 1: The Plot of the density function of the Gom-E distribution

2.1 The Survival Function

The Survival Function is also known as the reliability function (Eghwerido *et al*. 2020b) is defined as:

$$S(x) = 1 - F(x) = \exp\left\{\left(\frac{\varphi}{\beta}\right)\left[1 - \exp(\beta\lambda x)\right]\right\} \tag{7}$$

2.2 Hazard Rate Function

The Hazard rate Function (hrf) is the ratio of the pdf and the survival function (Eghwerido et al. 2020b). The hazard rate function is given as:

$$hrf(x) = \frac{f(x)}{S(x)} = \varphi\lambda \exp(-\lambda x)\exp((\beta+1)\lambda x) \tag{8}$$

Figure 2 shows the plot of the Hazard function for the Gom-E distribution. Figure 2 shows that the hazard rate function of the Gom-E distribution is bathtub and increasing.

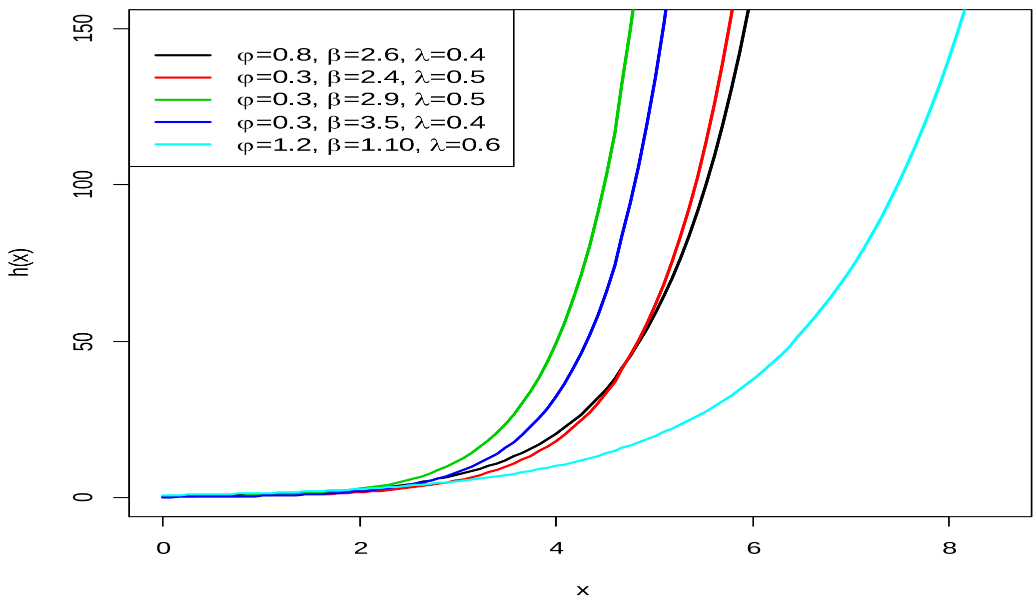

Figure 2: The Plot of the hrf of the Gom-E distribution

3. The Mixture Representation

In this section, we express the Gom-E distribution in power series form in terms of the exponential distribution. However, the last quantity in Equation (5) can be expressed as

$$\exp\left\{(\beta+1)\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp(\beta\lambda x))\right\}=\sum_{k=0}^{\infty}\frac{\left[(\beta+1)\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp(\beta\lambda x))\right]^{k}}{k!} \tag{9}$$

Substituting (11) into (5), we have

$$f_{GomE}(x)=\varphi\lambda\exp(-\lambda x)\sum_{k=0}^{\infty}\frac{\left[(\beta+1)\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp(\beta\lambda x))\right]^{k}}{k!} \tag{10}$$

More so, the binomial expansion in Equation (10) can be expressed as

$$\left[(\beta+1)\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp(\beta\lambda x))\right]^{k}=\sum_{j=0}^{k}\binom{k}{j}(\beta+1)^{k-j}\lambda^{k-j}x^{k-j}\varphi^{j}\beta^{-j}(1-\exp[\beta\lambda x])^{j} \tag{11}$$

Also the last quantity in Equation (11) can be expressed as

$$\left(1-\exp[\beta\lambda x]\right)^{j}=\sum_{i=0}^{j}(-1)^{i}\binom{j}{i}\exp[i\beta\lambda x] \tag{12}$$

Thus, the pdf of the Gom-E distribution can be expressed in power series as

$$f_{Gom-E}(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}V_{k,j,i}(x)\exp[-(1-i\beta)\lambda x] \tag{13}$$

where

$$V_{k,j,i}(x)=\binom{k}{j}\binom{j}{i}\frac{(-1)^{i}(\beta+1)^{k-j}\lambda^{k-j+1}x^{k-j}\varphi^{j+1}\beta^{-j}}{k!}$$

The corresponding power series cdf is given as

$$F_{Gom-E}(x)=1-\sum_{k=0}^{\infty}\sum_{j=0}^{k}V_{k,j}^{*}(x)\exp-[-j\beta\lambda x] \tag{14}$$

where

$$V_{k,j}^{*}(x)=\binom{k}{j}\frac{(-1)^{j}\varphi^{k}\beta^{-k}}{k!}$$

4. Some Statistical Properties of the Gom-E Distribution

This section deals with some mathematical or statistical properties of the new distribution. These properties include the quantile and random number generation, moment, moment generating function, order statistics and Renyi and δ-entropies.

4.1 The Cumulative Hazard Rate Function (Chrf)

The Cumulative hazard rate function (Chrf) is defined as the negative logarithm of the survival function (Oguntunde et al. 2014a)

$$Chrf(x)=-\log(S(x))=\left(\frac{\varphi}{\beta}\right)(\exp(\beta\lambda x)-1) \tag{15}$$

4.2 The Reversed Hazard Rate Function (Rhrf)

The Reversed hazard rate function (Rhrf) is the ratio of the pdf and the cdf.

$$Rhrf(x)=\frac{\varphi\lambda\exp\left\{\beta\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp(\beta\lambda x))\right\}}{1-\exp\left\{\beta\lambda x+\left(\frac{\varphi}{\beta}\right)(1-\exp(\beta\lambda x))\right\}} \tag{16}$$

4.3 The Quantile Function and random number generation

Let X be a random variable such that X is Gom-E distributed for $u\in(0,1)$. Then, the quantile function is obtained by inverting the cdf of the Gom-E distribution as

$$Q(u) = x = \beta^{-1}\lambda^{-1}\log\left(1-\left(\frac{\varphi}{\beta}\right)^{-1}\log(1-u)\right) \tag{17}$$

where u is the Quantile parameter. The Equation (17) can be used to generate random numbers for the Gom-E model.

The median of the Gom-E distribution is obtained by setting $u = 0.5$. Thus, the median is given as

$$Q(0.5) = x_{0.5} = \beta^{-1}\lambda^{-1}\log\left(1-\left(\frac{\varphi}{\beta}\right)^{-1}\log(0.5)\right) \tag{18}$$

More so, the first quantile is obtained from Equation (16) as

$$Q(0.25) = x_{0.25} = \beta^{-1}\lambda^{-1}\log\left(1-\left(\frac{\varphi}{\beta}\right)^{-1}\log(0.75)\right) \tag{19}$$

and the corresponding third quantile is given as

$$Q(0.75) = x_{0.75} = \beta^{-1}\lambda^{-1}\log\left(1-\left(\frac{\varphi}{\beta}\right)^{-1}\log(0.25)\right) \tag{20}$$

4.4 Moment

The moment of the Gom-E distribution for the random variable X is given as

$$M_x(r) = E(x^r) = \int_0^\infty x^r f_{GomE}(x)dx \tag{21}$$

Thus,

$$M_x(r) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} V_{k,j,i}(x)\int_0^\infty x^{k-j}\exp[-(1-i\beta)\lambda x]dx$$

$$M_x(r) = E(x^r) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j} v_{k,j,i}(x)\Gamma(k-j) \tag{22}$$

where

$$v_{k,j,i}(x) = \binom{k}{j}\binom{j}{i}\frac{(-1)^i(\beta+1)^{k-j}(1-i\beta)^{j-k-1}\varphi^{j+1}\beta^{-j}}{k!} \tag{23}$$

4.5 Moment Generating Function (mgf)

The moment generating function of the Gom-E distribution for the random variable X is given as

$$M_x(t) = E(e^{tx}) = \int_0^\infty \exp(tx) f_{Gom-E}(x)dx \tag{24}$$

$$M_x(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\int_0^\infty \exp(tx)V_{k,j,i}(x)dx$$

$$M_x(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\binom{j}{i}\frac{(-1)^i(\beta+1)^{k-j}\lambda^{k-j+1}\varphi^{j+1}\beta^{-j}}{k!}\int_0^\infty x^{k-j}\exp(tx)\exp[-(1-i\beta)\lambda x]dx \tag{25}$$

The function $\exp(tx)$ can be expressed as $\exp(tx) = \sum_{p=0}^{\infty}\frac{x^p t^p}{p!}$, then, the Equation (25) becomes

$$M_x(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{p=0}^{\infty}\binom{k}{j}\binom{j}{i}\frac{(-1)^i(\beta+1)^{k-j}\lambda^{k-j+1}\varphi^{j+1}t^p\beta^{-j}}{k!\,p!}\int_0^\infty x^{k+p-j}\exp[-(1-i\beta)\lambda x]dx \tag{26}$$

After integration and some algebraic simplifications, we have the moment generating function given as

$$M_x(t) = E(e^{tx}) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{p=0}^{\infty} e_{k,j,i,p}(x)\Gamma(k+p-j) \tag{27}$$

where

$$e_{k,j,i,p}(x) = \binom{k}{j}\binom{j}{i}\frac{(-1)^i(\beta+1)^{k-j}\lambda^{-p}(1-i\beta)^{j-k-p-1}\varphi^{j+1}t^p\beta^{-j}}{k!\,p!}$$

4.6 Order Statistics

The order statistics of the Gom-E density function is given as

$$f_{i:n}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1} f(x)[1-F(x)]^{n-k} \tag{28}$$

On substituting, we have

$$f_{i:n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)\sum_{p=0}^{n-k}\binom{n-k}{p}(-1)^p F(x)^{k+p-1} \tag{29}$$

On simplifying, we have

$$f_{i:n}(x) = \frac{n!}{(k-1)!(n-k)!}\sum_{p=0}^{n-k}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\sum_{r=0}^{\infty}\sum_{h=0}^{i+r} S_{p,m,j,i,r,h}(x)\exp[h\beta\lambda x] \tag{30}$$

where

$$S_{p,m,j,i,r,h}(x) = \binom{n-k}{p}\binom{k+p-1}{m}\binom{j}{i}\binom{i+r}{h}\frac{(-1)^{p+m+h}(\beta+1)^{j-i}m^{r}x^{j-i}\lambda^{j-i+1}\varphi^{i+r+1}\beta^{-r-i}}{j!r!}$$

4.6 Renyi and δ-entropies

The Renyi entropy of a random variable X represents a measure of variation of the uncertainty. However, It can be defined as

$$I_{\delta}(x) = \frac{1}{1-\delta}\log\int_{-\infty}^{\infty} f^{\delta}(x)dx, \quad \delta > 0 \text{ and } \delta \neq 1. \tag{31}$$

$$f^{\delta}(x) = \varphi^{\delta}\lambda^{\delta}\exp\{-\delta\beta\lambda x\}\exp\left\{(\beta+1)\delta\lambda x + \frac{\varphi\delta}{\beta}(1-\exp[\beta\lambda x])\right\} \tag{32}$$

After some algebraic simplification, we then have

$$I_{\delta}(x) = \frac{1}{1-\delta}\log\left[\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}\binom{k}{j}\binom{j}{m}\frac{\delta^{k}(\beta+1)^{k-j}\lambda^{\delta-1}\varphi^{\delta+j}\beta^{-k-1}}{k!}\Gamma(k-j)\right] \tag{33}$$

4.7 Maximum Likelihood Estimator

Let $x_1, x_2, ..., x_n$ be a random sample of size n from the Gom-E distribution with parameters $(\varphi, \lambda, \beta)$. Then log-likelihood function $L(\varphi, \lambda, \beta)$ can be expressed as

$$\log\prod_{i=0}^{n}(f(x)) = \ell = n\log\varphi + n\log\lambda + \sum_{i=1}^{n}\beta\lambda x_i + \frac{n\varphi}{\beta} - \sum_{i=1}^{n}\left(\frac{\varphi}{\beta}\right)\exp[\beta\lambda x_i] \tag{34}$$

Taking the partial derivative of the estimated parameters and equating to zero, we have.

$$\frac{\partial\ell}{\partial\varphi} = \frac{n}{\varphi} + \frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{1}{\beta}\right)\exp[\beta\lambda x] = 0 \tag{35}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \beta\sum_{i=0}^{n}x_i - \sum_{i=1}^{n}\varphi x_i\exp[\beta\lambda x_i] = 0 \tag{36}$$

$$\frac{\partial\ell}{\partial\beta} = \lambda\sum_{i=1}^{n}x_i - \frac{n\varphi}{\beta^2} - \sum_{i=1}^{n}\frac{\varphi}{\beta^2}\exp[\beta\lambda x_i] - \sum_{i=1}^{n}\frac{\lambda x_i\varphi}{\beta}\exp[\beta\lambda x_i] = 0. \tag{37}$$

However,

$$\frac{\partial\ell}{\partial\varphi} = 0, \; \frac{\partial\ell}{\partial\lambda} = 0, \text{ and } \frac{\partial\ell}{\partial\beta} = 0.$$

The Equations (35), (36) and (37) are nonlinear. Thus, they can be solved using the Newton-Raphson iterative numerical techniques in Maple and R.

Also, we obtain the 3×3 observed information matrix through

$$\begin{pmatrix} \hat{\varphi} \\ \hat{\lambda} \\ \hat{\beta} \end{pmatrix} \sim \left[\begin{pmatrix} \varphi \\ \lambda \\ \beta \end{pmatrix}, \begin{pmatrix} V_{\varphi\varphi} & V_{\lambda\varphi} & V_{\beta\varphi} \\ V_{\varphi\lambda} & V_{\lambda\lambda} & V_{\beta\lambda} \\ V_{\varphi\beta} & V_{\lambda\beta} & V_{\beta\beta} \end{pmatrix} \right] \tag{49}$$

where $V_{(.)}$ is the variance-covariance matrix. Thus, the Fisher information matrix $\hat{\varphi} = \varphi, \hat{\lambda} = \lambda$ and $\hat{\beta} = \beta$.

$$V^{-1} = -E \begin{bmatrix} \frac{\partial^2 \ln \ell}{\partial \varphi^2} & \frac{\partial^2 \ln \ell}{\partial \varphi \partial \lambda} & \frac{\partial^2 \ln \ell}{\partial \varphi \partial \beta} \\ \frac{\partial^2 \ln \ell}{\partial \lambda \partial \varphi} & \frac{\partial^2 \ln \ell}{\partial \lambda^2} & \frac{\partial^2 \ln \ell}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \ln \ell}{\partial \beta \partial \varphi} & \frac{\partial^2 \ln \ell}{\partial \beta \partial \lambda} & \frac{\partial^2 \ln \ell}{\partial \beta^2} \end{bmatrix} \tag{50}$$

Solving the inverse dispersion matrix yields the asymptotic variance and covariances for the ML estimators of φ, λ and β. The approximate 100(1- ϕ) % confidence intervals for (φ, λ and β) are determined respectively as $\hat{\varphi} \pm Z_{\frac{\phi}{z}} \frac{\sigma_{\varphi\varphi}}{\sqrt{n}}$, $\hat{\lambda} \pm Z_{\frac{\phi}{z}} \frac{\sigma_{\lambda\lambda}}{\sqrt{n}}$ and $\hat{\delta} \pm Z_{\frac{\phi}{z}} \frac{\sigma_{\delta\delta}}{\sqrt{n}}$ where Z_{ϕ} is the upper $100\phi_{th}$ the percentile of the standard normal distribution.

5. Simulation Study

Simulation is conducted to investigate the behaviour and performance of the Gom-E distribution for different sample size. The simulations are as follow:

- Data are generated from the random variable $x = \beta^{-1}\lambda^{-1} \log\left(1 - \left(\frac{\varphi}{\beta}\right)^{-1} \log(1-u)\right)$,

where $u \sim (0,1)$.

- The parameters value of the Gom-E distribution are set at $\varphi = 2.5, \beta = 0.6, \, and \, \lambda = 5.7$.
- The sample sizes of the Gom-E distribution are taken as $n = 10, 50, 100, 150, 250 \, and \, 350$.
- The Gom-E distribution sample size is replicated 1000 times.

The average estimates (AEs), biases, variance, root means squared errors (RMSEs) and means squared errors (MSEs) are evaluated using the Monte Carlo study. The results are given in Table 1.

Table 1: Simulation results for the mean estimates, biases, variance, and MSEs of $\hat{\varphi}, \hat{\beta}$, and $\hat{\lambda}$.

n	Parameter	AE	Bias	Variance	MSE
10	φ	2.1734	-0.3266	1.1817	1.2884
	λ	5.9206	0.2206	0.4335	0.4822
	β	1.7980	1.1980	3.7434	5.1786
50	φ	2.4041	-0.0959	0.2740	0.2832
	λ	5.8660	0.1660	0.0723	0.0999
	β	0.7859	0.1859	0.3321	0.3666
100	φ	2.4031	-0.0969	0.1406	0.1500
	λ	5.8880	0.1880	0.0397	0.0751
	β	0.6769	0.0769	0.1490	0.1550
150	φ	2.3991	-0.1009	0.0915	0.1017
	λ	5.9069	0.2069	0.0298	0.0727
	β	0.6389	0.0389	0.0879	0.0894

Table 1 continues

n	Parameter	AE	Bias	Variance	MSE
250	φ	2.3977	-0.1023	0.0515	0.0620
	λ	5.9032	0.2032	0.0199	0.0612
	β	0.6218	0.0218	0.0524	0.0528
350	φ	2.4101	-0.0899	0.0368	0.0449
	λ	5.8880	0.1880	0.0168	0.0521
	β	0.6082	0.0082	0.0352	0.0352

In Table 1, the values indicate that the MSEs of the MLEs of the Gom-E parameters converges to zero as the size of the sample increases as expected in first order asymptotic theory. As the sample size increases, the estimated mean tends to the true parameter values. The bias of the shape parameter is negative as sample size increases.

6. Application of real life data

A survival times of 121 patients data with breast cancer obtained in Muhammad et al. (2015) are applied to the proposed model to examined the performance of the model based on its statistic. Several criteria were used to determine the distribution for the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC). The proposed model is compared with extended generalized exponential (EGE) distribution, Kumaraswamy exponential (KE) distribution, Frechet exponential (FE) distribution, alpha power inverted exponential (APIE) distribution, extended exponential (EXE) distribution, exponentiated extended generalized exponential (EEXGE) distribution. The observations are as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0,126.0, 127.0, 129.0, 129.0, 139.0, 154.0

The descriptive statistics of the dataset are showed in Table 1. Table 2 is the measure of comparison for the various distributions under consideration. Table 3 shows the test statistic values. Table 4 shows the estimated parameters value for the real life application.

Table 2: Descriptive Statistics for the breast cancer patients

Skewness	1st Q	Mean	Median	Variance	Max	Min	Kurtosis	3rd Q
-0.05882353	17.50	46.33	40.00	1244.464	154.00	0.30	1.489412	60.00

Table 3: Statistical values of the 121 patients with breast cancer

Distribution	$-\ell$	AIC	BIC	CAIC	HQIC
Gom-E	579.5154	1165.031	1173.418	1165.236	1168.437
EGE	1160.116	1166.116	1174.504	1166.321	1169.523
EEXGE	1160.187	1168.187	1179.37	1168.532	1172.729
KE	583.0251	1172.05	1180.438	1172.255	1175.457
EXE	585.1277	1174.255	1179.847	1174.357	1176.526
FE	1283.621	1289.621	1298.008	1289.826	1293.027
APIE	619.1023	1242.205	1247.796	1242.306	1244.476

Table 4: Estimated values of parameters for breast cancer

Distribution	λ	α	β	φ
Gom-E	0.1144370		0.0787309	0.1301529
	(0.04)		(0.02)	(0.012)
EGE	0.2221620	1.5158268		0.1248924
	(0.05)	(0.01)		(0.055)
EEXGE	0.1355167	1.5168561	0.4126839	0.4960374
	(0.051)	(0.01)	(0.4)	(0.011)
KE	0.23656929	1.47487385	0.09393062	
	(0.2)	(0.5)	(0.1)	
EXE	0.01307601			0.65053794
	(0.099)			(1.68)
FE	0.002425186	0.045548469	0.638469830	
	(0.158)	(0.25)	(0.11)	
APIE	4.080036	126.929575		
	(1.0)	(20)		

The performance of a model is determined by the value that corresponds to the lowest Akaike Information Criteria (AIC) value is regarded as the best model. However, in the real life case considered, the Gom-E distribution has the lowest AIC value of 1165.031. Hence, it is considered as the best model for this data.

7. Conclusion

In this paper, we study a three-parameter distribution called the Gompertz exponential (Gom-E) distribution. Some mathematical properties of the Gom-E distribution have been derived and studied. The density function of the order statistics is obtained as a mixture of Gompertz exponential densities. The parameter estimation is obtained by maximum. The shape of the distribution could be inverted bathtub or decreasing (depending on the value of the parameters). An application to a real life data shows that the Gom-E distribution competes favourably with the some class of the exponential distributions like EGE, EEXGE, KE, EXE, FE and APIE.

Conflicts of Interest

The Authors declare that there are no conflicts of interest.

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