TOPP-LEONE LINLEY DISTRIBUTION

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Abstract
In this paper, we introduce Topp-Leone Lindley (TL-L) distribution using the methodology of Topp-Leone generated family and some of the properties are studied. The probability density function (Pdf) is found to be decreasing, right skewed unimodal, leftt skewed unimodal or approximately symmetric for some fixed parameter values, while the hazard function is increasing or decreasing for some fixed parameter values. The moments, moment generating function, quantile function and Renyi entropy are obtained. The maximum likelihood estimation method is used to estimate the parameters of the new distribution. Finally, a real lifetime dataset is used to illustrate the usefulness of TL-L distribution.

Keywords: *J-shaped distribution; Topp-Loene Lindley distribution; maximum likelihood; cumulative hazard function; unit interval.*

1.0 Introduction

The Lindley distribution as a model for analyzing lifetime data was first introduced by Lindley (1958). The Lindley distribution have attracted the interest of many researchers because of its wide range of applications and flexibility in various area such as medicine, engineering, biological science, management, public health, etc. A random variable, *X* is said to be distributed according to the Lindley distribution if the cumulative distribution (CDF) is given by

$$F(x) = 1 - \left(1 + \frac{\theta x}{\theta + 1}\right) e^{-\theta x}, \qquad x > 0,\ \theta > 0 \tag{1.1}$$

where θ is a rate parameter. It follows that the probability density function (Pdf) corresponding to (1.1) is given as

$$f(x) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}, \qquad x > 0,\ \theta > 0 \tag{1.2}$$

It is a well-known fact that the most popularly used distribution to model continuous variables with a unit interval is the Beta distribution. Many researchers have proposed an alternative to the Beta distribution which include the Kumaraswamy distribution presented by Kumaraswamy (1980) in the context of hydrology studies. Jones (2009), pointed out that Kumaraswamy distribution has the properties of the Beta distribution and that the advantage of Kumaraswamy over Beta distribution is that it has a closed form cumulative distribution function (CDF). Nadarajah and Kotz (2003), disclosed the importance of another distribution defined on bounded support called the Topp-Leone (TL) distribution. This distribution was first introduced by Topp and Leone (1955) and refered to as the J-shape distribution. The Topp-Leone distribution have been studied by different authors: Ghitany et al. (2005) presented some reliability measures for the Topp-Leone distribution, Genc (2012) introduced Moments of order statistics of Topp-Leone distribution, Ahmad et al (2015) introduced exponentiated Topp-Leone distribution and David (2017) proposed an improved estimation for the Topp-Leone Distribution, amongst others.

In the literature, attention has been directed towards developing new families of probability distribution so that more flexibility could be achieved in real data analysis. In line with this objective, Eugene et al. (2002) developed the beta class distribution, Zografos and Balakrishnan (2009) developed the gamma-G distribution family, Alzaatreh et al. (2013) proposed the T-X distribution family etc. Recently, Topp-Leone generated family of distributions was proposed by Al-Shomrani et al. (2016). With the Topp-Leone generated family, Winai (2016) presented the Topp-Leone Gumbel distribution, Hesham and Soha (2017) introduced the Topp-Leone Bur-XII distribution.

In this paper, we are motivated to introduce a new distribution called “Topp-Leone Lindley Distribution” and study some of the statistical properties. The method of maximum likelihood estimate was used to estimate parameters of the distribution. Finally, a real life data set was used to illustrate the usefulness of the new distribution.

2.0 Topp-Leone Lindley (Tl-L) Distribution

In this section we introduce the Topp-Leone Lindley (TL-L) distribution using the methodology of Topp-Leone generated family of distributions was introduced by Al-Shomrani et al. (2016).

Let Y be a random variable from Topp-Leone distribution with parameter α denoted by $Y \sim TL(\alpha)$. The pdf and CDF are respectively given as:

$$G_{TL}(y) = y^{\alpha}(2-y)^{\alpha}, \qquad 0 < t < 1,\ \alpha > 0 \tag{2.1}$$

and

$$g_{TL}(y) = 2\alpha y^{\alpha-1}(1-y)(2-y)^{\alpha-1}, \qquad 0 < t < 1,\ \alpha > 0 \tag{2.2}$$

If T is a continuous random variable with CDF $F(t)$ then the Topp-Leone generated (TL-G) family of distribution, $G(t)$ has its CDF defined as:

$$G(t) = [F(t)]^{\alpha}[2-F(t)]^{\alpha} = \left[1-\left(\bar{F}(t)\right)^2\right]^{\alpha}, \qquad 0 < t < 1,\ \alpha > 0 \tag{2.3}$$

The corresponding Pdf, $g(t)$ is written as:

$$g(t) = 2\alpha\ f(t)\bar{F}(t)[F(t)]^{\alpha-1}[2-F(t)]^{\alpha-1} = 2\alpha\ f(t)\bar{F}(t)\left[1-\left(\bar{F}(t)\right)^2\right]^{\alpha-1} \tag{2.4}$$

where $F(t)$ is the CDF of the baseline distribution, $\bar{F}(t) = 1 - F(t)$ and $f(t) = \frac{dF(t)}{dx}$ is the Pdf of the baseline distribution. By substituting (1.1) into (2.3), we obtain the CDF of the distribution called the Topp-Leone Lindley (TL-L) distribution given as

$$G(t) = \left\{1 - \left[\frac{\theta + 1 + \theta t}{\theta + 1} e^{-\theta t}\right]^2\right\}^{\alpha}, \; 0 < t < 1, \; \alpha > 0 \tag{2.5}$$

and the corresponding pdf written (2.4) is given as

$$g(t) = \frac{2\alpha\theta^2}{\theta + 1}(1 + t)\left(\frac{\theta + 1 + \theta t}{\theta + 1}\right)e^{-2\theta t}\left\{1 - \left[\frac{\theta + 1 + \theta t}{\theta + 1} e^{-\theta t}\right]^2\right\}^{\alpha - 1} \tag{2.6}$$

Figure 1 below shows that the Pdf of TL-L is decreasing, right skewed unimodal, leftt skewed unimodal or approximately symmetric for some fixed parameter values.

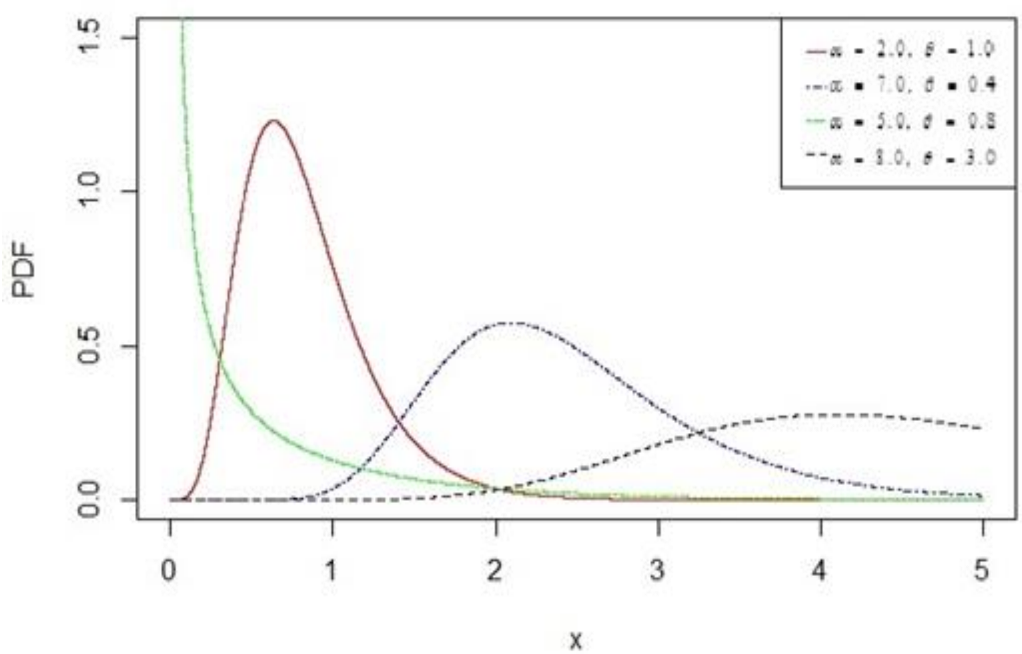

Figure 1: The Pdf of TL-L distribution for different values of the parameters

2.1 The Reiability Functions if TL-L

In this section, we consider the functions used for reliability study. These include the survival function $\overline{G}(t)$, hazard rate function $h(t)$, inverse hazard function $hr(t)$ and cumulative hazard function $H(t)$ for the TL-L distribution which are respectively given as

$$\overline{G}(t) = 1 - G(t) = 1 - \left\{1 - \left[\frac{\theta+1+\theta t}{\theta+1} e^{-\theta t}\right]^2\right\}^{\alpha}$$

(2.7)

$$h(t) = \frac{g(t)}{\overline{G}(t)} = \frac{\frac{2\alpha\theta^2}{\theta+1}(1+t)\left(\frac{\theta+1+\theta t}{\theta+1}\right)e^{-2\theta t}\left\{1 - \left[\frac{\theta+1+\theta t}{\theta+1} e^{-\theta t}\right]^2\right\}^{\alpha-1}}{1 - \left\{1 - \left[\frac{\theta+1+\theta t}{\theta+1} e^{-\theta t}\right]^2\right\}^{\alpha}}$$

(2.8)

$$hr(t) = \frac{g(t)}{G(t)} = \frac{\frac{2\alpha\theta^2}{\theta+1}(1+t)\left(\frac{\theta+1+\theta t}{\theta+1}\right)e^{-2\theta t}}{\left\{1 - \left[\frac{\theta+1+\theta t}{\theta+1} e^{-\theta t}\right]^2\right\}}$$

(2.9) $$H(t) = \ln \overline{G}(t) = \ln\left[1 - \left\{1 - \left[\frac{\theta+1+\theta t}{\theta+1} e^{-\theta t}\right]^2\right\}^{\alpha}\right] \qquad (2.10)$$

Figure 2 below show the hazard function of TL-L to be increasing or decreasing for some fixed parameter values.

2.2 The Limit of the Topp-Leone Lindley Distribution

In this section, we consider the behavior of the TL-L distribution as $t \to 0$ and $t \to \infty$ using the Pdf in (2.6) as follows:

$$\lim_{t\to 0} g(t) = \lim_{t\to 0}\left\{\frac{2\alpha\theta^2}{\theta+1}(1+t)\left(\frac{\theta+1+\theta t}{\theta+1}\right)e^{-2\theta t}\left\{1 - \left[\frac{\theta+1+\theta t}{\theta+1} e^{-\theta t}\right]^2\right\}^{\alpha-1}\right\} = 0$$

Since $\underset{t\to 0}{Lim}\left\{1-\left[\frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right]^2\right\}^{\alpha-1} = 0$

Similarly,

$$\underset{t\to\infty}{Limit}\, g(t) = \underset{t\to\infty}{Limit}\left\{\frac{2\alpha\theta^2}{\theta+1}(1+t)\left(\frac{\theta+1+\theta t}{\theta+1}\right)e^{-2\theta t}\left\{1-\left[\frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right]^2\right\}^{\alpha-1}\right\} = 0$$

Since $\underset{t\to 0}{Lim}\left\{e^{-2\theta t}\right\} = 0$

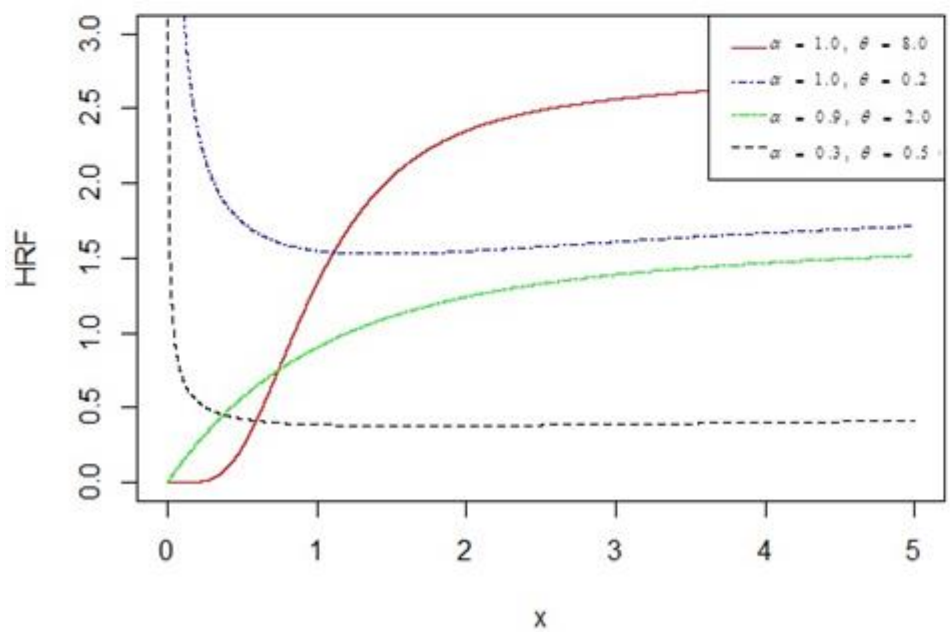

Figure 2: The HRF of TL-L distribution for different values of the parameters

2.3 Expansion of the CDF and PDF of Tl-L Distribution

Using the binomial series expansion,

$$(1+q)^a = \sum_{i=0}^{\infty}\binom{a}{i}q^i$$

The CDF of the TL-L distribution can be expanded as

$$G(t) = \left[1 - \left\{\left(1 + \frac{\theta t}{\theta + 1}\right)e^{-\theta t}\right\}^2\right]^\alpha = \sum_{i=0}^{\infty} \sum_{j=0}^{2i} \binom{\alpha}{i}\binom{2i}{j}(-1)^i \left(\frac{\theta}{\theta + 1}\right)^j t^j e^{-2i\theta t}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{2i} \Omega(i, j) t^j e^{-2i\theta t} \tag{2.11}$$

where $\Omega(i, j) = \binom{\alpha}{i}\binom{2i}{j}(-1)^i \left(\frac{\theta}{\theta + 1}\right)^j$

Similarly, the corresponding pdf can be expressed as:

$$g(t) = \frac{2\alpha\theta^2}{\theta + 1}(1 + t)\left(\frac{\theta + 1 + \theta t}{\theta + 1}\right)e^{-2\theta t}\left\{1 - \left[\frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t}\right]^2\right\}^{\alpha - 1}$$

$$= \frac{2\alpha\theta^2}{\theta + 1} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \binom{\alpha - 1}{i}\binom{2i + 1}{j}\left(\frac{\theta}{\theta + 1}\right)^j (-1)^i (1 + t) t^j e^{-2\theta[i+1]t}$$

$$= \frac{2\alpha\theta^2}{\theta + 1} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \Phi(i, j)(1 + t) t^j e^{-2\theta[i+1]t} \tag{2.12}$$

where $\Phi(i, j) = \binom{\alpha - 1}{i}\binom{2i + 1}{j}\left(\frac{\theta}{\theta + 1}\right)^j (-1)^i$

3.0 Statistical Properties of the Tl-L Distribution

3.1 Moment and Related Measure

The k^{th} raw moment of a continuous random variable Y, denoted μ_k can be defined as:

$$\mu'_k = E(Y^k) = \int_{-\infty}^{\infty} y^k f(y)dy \tag{3.1}$$

Therefore, for a continuous random variable T having the TL-L distribution, the kth raw moment becomes;

$$\mu'_k = E\left(T^k\right) = \frac{2\alpha\theta^2}{1 + \theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \binom{\alpha - 1}{i}\binom{2i + 1}{j}\left(\frac{\theta}{\theta + 1}\right)^j (-1)^i \int_0^1 t^{k+j}(1 + t)e^{-2\theta[i+1]t}dt$$

Using the expression $e^{tx} = \sum_{k=0}^{\infty} \frac{t^k x^k}{k!}$, we have

$$\mu_k' = \frac{2\alpha\theta^2}{1+\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m} \left[2\theta(i+1)\right]^m}{m!} \int_0^1 t^{k+j+m} (1+t) dt \tag{3.2}$$

Let $u = 1+t$ in the equation (3.2), we have

$$\mu_k' = \frac{2\alpha\theta^2}{1+\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m+1} \left[2\theta(i+1)\right]^m}{m!} \int_0^1 (1-u)^{k+j+m} u du$$

$$\mu_k' = \frac{2\alpha\theta^2}{1+\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m+1} \left[2\theta(i+1)\right]^m}{m!} \beta\left[k+j+m+1, 2\right]$$

$$= \frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, k) \tag{3.3}$$

Where $\beta(n,m) = \int_0^1 t^{n-1} (1-t)^{m-1} dt$ is a beta function and

$$M_1(\alpha, \theta, k) = \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m+1} \left[2\theta(i+1)\right]^m}{m!} \beta\left[k+j+m+1, 2\right]$$

From (3.3), we obtain the first four raw moment of TL-L distribution as

$$\mu_1' = \frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, 1) \qquad \mu_2' = \frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, 2)$$

$$\mu_3' = \frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, 3) \qquad \mu_4' = \frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, 4)$$

Similarly, we define the central moment of the random variable X by:

$$\mu_k = E(X-\mu)^k = E\left\{ \sum_{i=1}^{k} \binom{k}{i} X^{k-i} (-\mu)^i \right\}$$

$$= \sum_{i=1}^{k} \binom{k}{i} (-1)^i \mu_{K-1}' \mu^i \tag{3.4}$$

The mean (μ), variance (σ^2), coefficient of variation (ω), skewness (S_k) and kurtosis (K_s) for the TL-L distribution can be obtained respectively using the expression:

$$\mu = \frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, 1),$$

$$\sigma^2 = \mu - \mu^2 = \frac{2\alpha\theta^2}{1+\theta} \left\{ M_1(\alpha, \theta, 2) - \frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right\},$$

$$\omega = \frac{\sqrt{\frac{2\alpha\theta^2}{1+\theta} \left\{ M_1(\alpha, \theta, 2) - \frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right\}}}{\frac{2\alpha\theta^2}{1+\theta} M_1(\alpha, \theta, 1)},$$

$$S_k = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$

$$= \frac{\frac{2\alpha\theta^2}{\theta+1} \left\{ M_1(\alpha, \theta, 3) - 3\left[M_1(\alpha, \theta, 2) \frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right] + 2\left[\left(\frac{2\alpha\theta^2}{\theta+1} \right)^2 M_1^1(\alpha, \theta, 1) \right] \right\}}{\left\{ \frac{2\alpha\theta^2}{1+\theta} \left[M_1(\alpha, \theta, 2) - \frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right] \right\}^{\frac{3}{2}}}$$

and

$$K_s = \frac{\mu_4}{(\mu_2)^2}$$

$$= \frac{\frac{2\alpha\theta^2}{\theta+1} \left\{ M_1(\alpha, \theta, 4) - 4\left[M_1(\alpha, \theta, 3) \frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right] + 6\left[M_1(\alpha, \theta, 2) \left(\frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right) \right] - 3\left[\left(\frac{2\alpha\theta^2}{\theta+1} \right)^3 M_1^4(\alpha, \theta, 1) \right] \right\}}{\frac{2\alpha\theta^2}{1+\theta} \left\{ M_1(\alpha, \theta, 2) - \frac{2\alpha\theta^2}{\theta+1} M_1(\alpha, \theta, 1) \right\}^2}$$

3.2 Moment Generating Function

The moment generating function of TL-L distribution denoted by $M_T(z)$ can be derived as follows;

$$M_T(z) = E\left(e^{zt}\right) = \int_0^{\infty} e^{zt} g(t)dt \tag{3.5}$$

$$= \frac{2\alpha\theta^2}{1+\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j (-1)^i \int_0^1 t^j (1+t) e^{-\left[2\theta(i+1)-z\right]t} dt$$

Let $u = 1+t$

$$= \frac{2\alpha\theta^2}{1+\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m}\left[2\theta(i+1)-z\right]^m}{m!} \int_0^1 (1-u)^{j+m} udu$$

$$= \frac{2\alpha\theta^2}{1+\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m}\left[2\theta(i+1)-z\right]^m}{m!} \beta\left[k+m+1, 2\right]$$

$$= \frac{2\alpha\theta^2}{1+\theta} M_2(\alpha, \theta, z) \tag{3.6}$$

where

$$M_2(\alpha, \theta, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{2i+1} \sum_{m=0}^{\infty} \binom{\alpha-1}{i} \binom{2i+1}{j} \left(\frac{\theta}{\theta+1}\right)^j \frac{(-1)^{i+m}\left[2\theta(i+1)-z\right]^m}{m!} \beta\left[k+m+1, 2\right]$$

3.2 Quantile Function

The quantile function of a distribution with CDF $G(x)$ is defined by $q = G(Q(w))$ where $0 < w < 1$. Thus, the quantile function of the TL-L distribution is derived as follows

$$w = \left[1 - \left\{\frac{\theta + 1 + \theta Q(w)}{\theta+1} e^{-\theta Q(w)}\right\}^2\right]^{\alpha}$$

$$\left(\frac{\theta + 1 + \theta Q(w)}{\theta+1}\right) e^{-\theta Q(w)} = \sqrt{1 - w^{\frac{1}{\alpha}}}$$

$$H(w)e^{H(w)} = -(1+\theta)\left(\sqrt{1-w^{\frac{1}{\alpha}}}\right)e^{-(1+\theta}$$

$$Q(w) = -\left\{1+\frac{1}{\theta}+\frac{1}{\theta}W_{-1}\left[(1+\theta)\left(\sqrt{1-w^{\frac{1}{\alpha}}}\right)e^{-(1+\theta}\right]\right\}$$

(3.7)

Where $W_{-1}(.)$ the negative branch of the Lambert W is function, see Corless (1996) for details. Equation (3.7) can be used to generate random samples from the TL-L distribution.

3.3 Renyi Entropy

Entropy is an important concept in statistics. It measure the level of uncertainty with respect to a random variable say *T*. Renyi (1961) defined the entropy of a random variable *T* with pdf $g(t)$ as

$$J_R(s) = \frac{1}{1-s}\log\Phi(s), \qquad s>0, s\neq 1 \tag{3.8}$$

where

$$\Phi(s) = \int_0^\infty f^s(t)dt$$

Using the Pdf of TL-L distribution in (2.4), we have

$$\int_0^\infty f^s(t)dt = \left(\frac{2\alpha\theta^2}{1+\theta}\right)^s \sum_{i=0}^{\infty}\sum_{j=0}^{2si+s}\binom{s(\alpha-1)}{i}\binom{2si+s}{j}\left(\frac{\theta}{\theta+1}\right)^j(-1)^i\int_0^1 t^j(1+t)^s e^{-2\theta[i+1]st}dt$$

$$= \left(\frac{2\alpha\theta^2}{1+\theta}\right)^s \sum_{i=0}^{\infty}\sum_{j=0}^{2si+s}\sum_{m=0}^{\infty}\binom{s(\alpha-1)}{i}\binom{2si+s}{j}\left(\frac{\theta}{\theta+1}\right)^j\frac{(-1)^{i+m}\left[2\theta(i+1)s\right]^m}{m!}\int_0^1 t^{j+m}(1+t)^s dt$$

Let $u = 1+t$

$$= \left(\frac{2\alpha\theta^2}{1+\theta}\right)^s \sum_{i=0}^{\infty}\sum_{j=0}^{2si+s}\sum_{m=0}^{\infty}\binom{s(\alpha-1)}{i}\binom{2si+s}{j}\left(\frac{\theta}{\theta+1}\right)^j\frac{(-1)^{i+m}\left[2\theta(i+1)s\right]^m}{m!}\int_0^1 (1-u)^{j+m}u^s du$$

$$= \left\{ \left(\frac{2\alpha\theta^2}{1+\theta}\right)^s \sum_{i=0}^{\infty} \sum_{j=0}^{2si+s} \sum_{m=0}^{\infty} \binom{s(\alpha-1)}{i} \binom{2si+s}{j} \left(\frac{\theta}{\theta+1}\right)^j \times \frac{(-1)^{i+m}\left[2\theta(i+1)s\right]^m}{m!} \beta\left[j+m+1, s+1\right] \right\}$$

$$J_R(s) = \frac{1}{1-s}\log\left\{ \left(\frac{2\alpha\theta^2}{1+\theta}\right)^s \sum_{i=0}^{\infty} \sum_{j=0}^{2si+s} \sum_{m=0}^{\infty} \binom{s(\alpha-1)}{i} \binom{2si+s}{j} \left(\frac{\theta}{\theta+1}\right)^j \times \frac{(-1)^{i+m}\left[2\theta(i+1)s\right]^m}{m!} \beta\left[j+m+1, s+1\right] \right\}$$

(3.9)

4.0 Maximum Likelihood Estimation

Let t_i, $i = 1, 2, 3, \cdots, n$ be a random sample from the TL-L distribution, then the maximum likelihood function of (2.4) is denoted by $L(x_i, \Phi)$ is defined as:

$$L(x_i, \Phi) = \prod_{i=1}^{n} \left\{ \frac{2\alpha\theta^2}{\theta+1}(1+t_i)\left(\frac{\theta+1+\theta t_i}{\theta+1}\right)e^{-2\theta t_i}\left\{1-\left[\frac{\theta+1+\theta t_i}{\theta+1}e^{-\theta t_i}\right]^2\right\}^{\alpha-1} \right\}$$

(4.1)

and the log-likelihood function denoted by $\ell_n(x_i, \Phi)$ where $\Phi = (\alpha, \theta)^T$ is the vector parameter is obtained as

$$\ell_n(x_i, \Phi) = n\ln\left(2\alpha\theta^2\right) - n\ln(1+\theta) - 2\theta\sum_{i=1}^{n} t_i + \sum_{i=1}^{n}\ln(1+t_i) + \sum_{i=1}^{n}\ln\psi(t_i) + (\alpha-1)\sum_{i=1}^{n}\ln\xi(t_i) \tag{4.1}$$

where $\psi(t_i) = \left(\frac{\theta + 1 + \theta t}{\theta + 1}\right)$ and $\xi(t_i) = \left\{1 - \left[\frac{\theta + 1 + \theta t}{\theta + 1} e^{-\theta t}\right]^2\right\}$

The partial derivatives of $\ell_n(x_i, \Phi)$ with respect to the parameters α and θ are:

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left\{1 - \left[\frac{\theta + 1 + \theta t_i}{\theta + 1} e^{-\theta t_i}\right]^2\right\} \tag{4.2}$$

$$\frac{\partial \ell_n}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta + 1} - 2\sum_{i=1}^{n} t_i + \sum_{i=1}^{n} \frac{\psi_i'}{\psi_i} + \sum_{i=1}^{n} \frac{\xi_i'}{\xi_i} \tag{4.3}$$

MLEs ($\hat{\alpha}, \hat{\theta}$) of (α, θ), can be obtained by solving simultaneously the equations $\frac{\partial \ell_n}{\partial \alpha} = 0$ and $\frac{\partial \ell_n}{\partial \theta} = 0$ which are non-linear system. This was achieved in this work by using numerical method called Newton-Raphson iterative scheme using the R-Software package.

For the interval estimation and hypothesis testing of the parameters of the model (α, θ), we derive the 2×2 observed information matrix $J(\Phi) = \{J_{m,n}\}$ $for\ (m, n = \alpha, \theta)$ whose elements are the second derivatives of the log-likelihood function as

$$J(\Phi) = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\theta} \\ J_{\theta\alpha} & J_{\theta\theta} \end{pmatrix} \tag{4.4}$$

5.0 Real Data Application

In this section, we demonstrate the application of the TL-L distribution by considering a real life data sets presented in Table 1.

Table 1: 66 breaking stress of carbon fibers (in Gba) reported in Nicholas and Padgett (2006)

3.70,	2.74,	2.73,	2.50,	3.60,	3.11,	3.27,	2.87,	1.47,	3.11,	3.56,	4.42,
2.41,	3.19,	3.22,	1.69,	3.28,	3.09,	1.87,	3.15,	4.90,	1.57,	2.67,	
2.93,	3.22,	3.39,	2.81,	4.20,	3.33,	2.55,	3.31,	3.31,	2.85,	1.25,	4.38,
1.84,	0.39,	3.68,	2.48,	0.85,	1.61,	2.79,	4.70,	2.03,	1.89,	2.88,	2.82,
2.05,	3.65,	3.75,	2.43,	2.95,	2.97,	3.39,	2.96,	2.35,	2.55,	2.59,	2.03,
1.61,	2.12,	3.15,	1.08,	2.56,	1.80,	2.53					

In addition to TL-L distribution, we also considered the fit of the Lindley-Exponential distribution (LED), Exponential distribution (ED) and Lindley distribution (LD) with respective pdf given as follows:

$$f_{LE}(t) = \frac{\lambda\theta^2 e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{\theta-1}\left(1-\log\left(1-e^{-\lambda t}\right)\right)}{\theta+1}, \quad f_E(t) = \theta e^{-\theta t} \quad \text{and}$$

$$f_L(t) = \frac{\theta^2(1+t)e^{-\theta t}}{1+\theta}$$

The negative log likelihood $(-\ell)$, Akaike information criterion (AIC), Kolmogorov-Smirnov (KS) statistics, Cramer-von Mises (CVM) statistics and Anderson-Darling (AD) statistics for the distributions considered are presented in Table 2.

Table 2: Summary Statistics of the Dataset

Distr.	*Estimates*	ℓ	*AIC*	*KS*	*CVM*	*AD*
TL-L	$\alpha = 6.3386$ $\theta = 0.7128$	93.0464	190.093	0.1434	0.30829	1.7161
LED	$\lambda = 1.0040$ $\theta = 9.9681$	95.5855	195.1712	0.1558	0.3770	2.1146
LD	$\theta = 0.5902$	122.3841	246.7681	0.2977	2.0914	10.6922
EXP	$\lambda = 0.36237$	132.9944	267.9887	0.358114	2.871046	14.034266

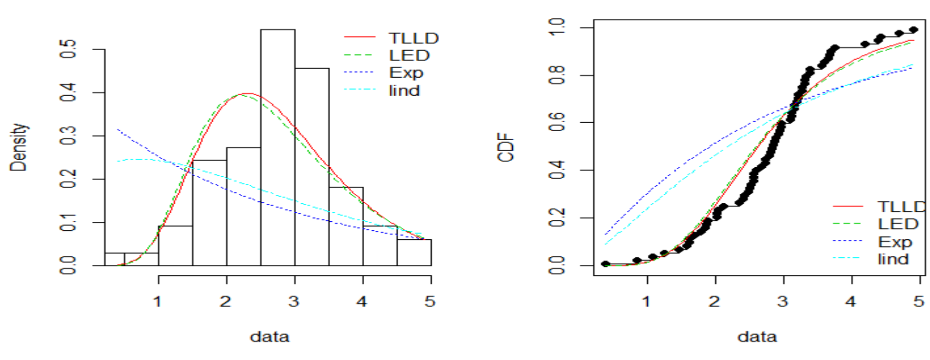

Figure 3: Fitted Pdf's and CDF's for the Concerned Data Set

Considering KS test, CVM test and AD test Table 2, it is clear to say that the TL-L distribution performed better than the other distributions considered. This is also explained in Figure 3 where the density and the cumulative distribution functions shows a better fit for the dataset.

6.0 Conclusion

This paper introduced a new distribution called "Topp-Leone Lindley (TL-L) Distribution" which is a new extension of the classical Lindley distribution for flexibility. Some of the mathematical properties such as the raw moment, moment generating function, and the renyi entropy are studied. The maximum likelihood estimation (MLE) method was used to estimate parameters of the new distribution. An application of the TL-L distribution to a real lifetime data set demonstrates that the new distribution can provide a better fit than some other distribution.

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