ON MATHEMATICAL MODEL FOR ZIKA VIRUS DISEASE CONTROL WITH WOLBACHIA-INFECTED MOSQUITOES

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Abstract
In this paper, a mathematical model for controlling the spread of zika virus disease with wobachia-infected aedes aegypti mosquitoes was developed. The model consists a system of 14 non-linear ordinary differential equations. These equations were used to describe the transmission dynamics of zika virus disease in human and aedes aegypti populations, in the presence of wolbachia-infected mosquitoes used for control. Approximate analytical solution to the model was obtained through homotopy perturbation method, and was simulated at the baseline parameter values. Graphically, it was seen that the population of infected humans and the population of wolbachia-free mosquitoes diminished, while the population of wolbachia-infected mosquitoes remain on the increase as time was increased. This result showed that zika virus disease can be eradicated by introducing reasonable number of wolbachia-infected mosquitoes in the zika endemic area.

Keywords: *Zika, aedes aegypti, wolbachia, homotopy, microcephaly.*

Introduction

Zika virus disease or zika as it is generally called is a mosquito-borne, Flavivirus disease (Gao *et al.*, 2016). The disease is named after Zika forest in Uganda, where the virus (Zika virus) was first isolated from a rhesus monkey, in 1947 (Dick *et al.*, 1952). Zika virus is primarily transmitted to humans through the bites of infected female aedes aegypti mosquitoes (Gao *et al*., 2016). Apart from transmission through mosquito bites, there are evidences that zika virus can equally be transmitted through sex, blood transfusion and mother-to-fetus (Musso *et al*., 2014).

Zika virus disease has been confirmed to be associated with microcephaly in infants born to mothers who were infected with the virus during pregnancy (Mlakar *et al*., 2016; Cauchemez *et al*., 2016). Microcephaly is a congenital disorder in which brain does not develop properly. Babies born with this condition have a characteristic small head with circumference equal or less than 32 cm, which is below the standard size recognized by the World Health Organization (WHO)(CDC, 2016). In the recent outbreak of zika in Brazil, there have been over 5000 confirmed cases of microcephaly (Gao *et al*., 2016). In addition to microcephaly, zika has been linked to an apparent increased risk of the neurological disorder, Guillain-Barre Syndrome (GBS) (Cao-Lormeau *et al.*,2016). These associations of zika with microcephaly and GBS prompted the WHO's declaration of zika virus disease a Public Health Emergency of International Concern in February 2015 (WHO, 2016). The first reported incidence of zika virus disease in man was in Nigeria in 1954, during investigation in Afikpo, Eastern Nigeria, of an outbreak of jaundice suspected of being yellow fever, (Macnamara, 1954). In 2007, zika outbreak occurred in Yap Island, Federated state of Micronesia, in the North pacific (Hayes,

2009; Duffy *et al.*, 2009). This was followed by a severe outbreak in French Polynesia between 2013 and 2014, with over 30,000 reported cases (Cao-Lormeau *et al.,* 2013). From French Polynesia, the disease spread to New Caledonia, The Cook Islands and Eastern Islands (Musso *et al*., 2014). This recent and still-ravaging outbreak of zika began in April 2015, in Brazil (Gatherer and Kohl, 2016), and has spread to many other countries in south and central Americas and the Caribbean, with over 140,000 suspected and confirmed cases by the end of February 2016. In the United States, zika has been detected in Florida, New York City, Texas, and other places (CDC, 2016; Meaney-Delman, 2016; Fauci *et al.,* 2016).

Only recently, the literature has being flooded with mathematical models on zika virus dynamics and its control. This is because zika virus disease was not regarded serious ailment until its association with some congenital and neurological complications was conformed. There are models that focus on understanding the transmission dynamics of the disease. The models in this category can be found in Moreno *et al.,* (2016), Perkins *et al.,* (2016), Dantas *et al.,* (2017), Kurchaski *et al.,* (2016) and others. There are also models which incorporate sexual transmission in the dynamics of the disease. These include models in Shah *et al* (2017), Gao *et al.,* (2017) and Towers *et al.,* (2016). As a way of controlling the spread of zika virus disease, Wang *et al.,* (2017), proposes the use of wolbachia-infected aedes aegypti mosquitoes to stop the spread of zika virus disease. This method has been effectively applied in controlling similar mosquito borne avivirus diseases such as dengue and West Nile, for example, Britton *et al*., (2013) and Ndii *et al.,* (2012). Our model in this work is also based on this innovative mosquito control method.

The rest of this work is organized as follows: In section 2, we present the idea behind using Wolbachia-infected mosquitoes as agent for controlling the spread of zika virus disease. This is followed by section 3, where we presented our model with its basic assumptions. Section 4 is where we discussed and apply the homotopy perturbation method to obtain the approximate analytical solution to the model. In section 5, we performed numerical experiment on the solution and discussed the result.

Using Wolbachia to Fight Zika Virus Disease

Wolbachia is a common and widespread group of bacteria naturally found in reproductive tissues of some arthropods. They are transmitted maternally through the cytoplasm of eggs of their hosts. They have various mechanisms such as cytoplasmic incompatibility (CI), parthenogenesis, and feminization through which they manipulate the reproduction process of their hosts to their advantage (Werren, 1997). Some strains of wolbachia have the ability to shorten the life span of the host insect, while others reduce the ability of the host insects to transmit disease infections to humans. Scientists therefore see these as ways to limit disease transmission by these wolbachia-infected insects such as mosquitoes (Scott, 2015).

Even though aedes aegypti is not a natural host to wolbachia, it can be manually infected with it in the laboratory (Hughes and Britton, 2013). These wolbachia-infected mosquitoes are then released in the zika-endemic area to mate with the wolbachia-free ones. In using wolbachia to fight zika, it is believed that two mechanisms are involved: (1) the presence of wolbachia in the infected mosquitoes increases the incubation period (or reduces the incubation rate) of the virus in these mosquitoes. Since the adult life of the mosquito is short (about 14 days), most of the mosquitoes carrying the virus die before they become infectious. Hence, the infected

wolbachia-carrier mosquitoes may not transmit the virus to humans through their bites. (2) wolbachia induces cytoplasmic incompatibility (CI), which helps its host mosquitoes to invade the population of wolbachia-free aedes aegypti mosquitoes in the wild (Koiller *et al.*, 2014). CI is a biological phenomenon that prevents the development of embryos (non-hatching of eggs) when wolbachia-carrier male mosquitoes mate with wolbachia-free female mosquitoes or mate with female mosquitoes that carry a different wolbachia strain (Mains *et al.*, 2013). The effect of cytoplasmic incompatibility on the mosquitoes' reproduction process is summarized below.

- wolbachia-free male with wolbachia-free female produces wolbachia-free eggs.
- wolbachia-free male with wolbachia-carrier female produces wolbachia-carrier eggs.
- wolbachia-carrier male with wolbachia-free female to produce non-viable eggs.
- wolbachia-carrier male with wolbachia-carrier female produce wolbachia-carrier eggs.

Consequently, wolbachia-carrier female aedes aegypti will have the advantage of producing more offspring than the wolbachia-free female mosquitoes (Hankock *et al.*, 2011). Therefore, vast number of aedes aegypti mosquitoes will eventually carry wolbachia and will be less capable to transmit zika virus to humans. In this work, we assume that the wolbachia strain, *wMel,* which will reduce zika virus transmission from mosquito to humans is used.

The Mathematical Model

Human beings may contract zika virus when bitten by infectious female aedes aegypti mosquitoes or when infected human passes the virus to uninfected human through unsafe sex, unsafe blood transfusion or perinatal transmission from mother to child. On the other hand, transmission of zika virus from human to mosquito occurs when an adult, uninfected female aedes aegypti mosquito bites human to suck blood. If the human is already infected with the virus, he may pass it to the mosquito. The mosquito once infected remains so and continues to infect humans throughout its life time. The model we propose is made up of three major populations; human population, adult female wolbachia-free aedes aegypti population and adult female wolbachia-carrier aedes aegypti population used as control. We use a set of non-linear ordinary differential equations to model the dynamics of zika virus in these populations.

Zika Dynamics in Human Population

The total human population, $N_H(t)$, at any time, t, is divided into 8 compartments or classes, namely, the susceptible class, $S_H(t)$; the latent or the exposed class $E_H(t)$, the symptomatically infectious class; $I_{Hs}(t)$, the asymptomatically infectious class $I_{Has}(t)$, the treatment class, $I_T(t)$; the non-treatment class, $I_{NT}(t)$;the partially recovered class, $R_{H1}(t)$ and the totally recovered class, $R_{H2}(t)$.

Individuals in the human population are recruited into the susceptible class either through migration into the zika- endemic area at the rate, Π_H, or through birth of zika virus-free offspring at the rate μ_H. The susceptible class acquires zika virus either through infectious wolbachia-free mosquito bites, with probability,α_{MH}, or through humans in the infectious classes; treatment class, non-treatment class and partially recovered class, with probability, α_{HH}, to move to the exposed class. The susceptible humans may also contract the virus through the bites of wolbachia-carrier mosquitoes with a very negligible probability of infection, $\alpha_{MwH} \ll \alpha_{MH}$. The exposed class becomes either asymptomatically or symptomatically infectious, in the proportions v and $(1-v)$, respectively at the incubation rate β_H. Due to transmission of zika

virus from infected pregnant mothers to their offspring, we assume that the proportion δ, of the young ones does not carry the virus, hence, it is transferred to the susceptible class, $S_H(t)$, while the infected proportion $1-\delta$, moves to the exposed class, $E_H(t)$. The proportion, ω of the symptomatically infectious class receives treatment at the rate τ, to be in the treatment class, $I_T(t)$, whereas $(1-\omega)$ receives no treatment, and stays in the non-treatment class, $I_{NT}(t)$. The treatment and the non-treatment classes recover partially at the rates, v_1, and v_2, respectively, to move to the partially recovered class, $R_{H1}(t)$. After some period of time, the partially recovered and the asymptomatically infectious classes recover fully at the rates γ_1 and γ_2 respectively, to be in the totally recovered class, $R_{H2}(t)$. An individual remains in the totally recovered class, and cannot be re-infected with the virus by any means until natural death occurs.

All the classes in the human population benefits from the natural mortality rate σ_{H}, whereas only the diseased classes are affected by zika-induced death at the rate σ'_{H}, which is negligible.

Zika Dynamics in the Aedes aegypti Populations

The population of wolbachia-free adult female aedes aegypti mosquitoes is grouped into 3 classes namely; the susceptible mosquitoes, $S_M(t)$,(those that can contract the virus by biting infectious humans); the exposed mosquitoes, $E_M(t)$, (those that have contracted Zika virus but not infectious), and the infectious mosquitoes, $I_M(t)$,(those that have contracted the virus and are infectious). The wolbachia-free male mosquitoes mate with their female counterparts in the wolbachia-free and wolbachia-carrier mosquito populations to produce wolbachia-free and wolbachia-carrier offspring, respectively. The female wolbachia-free mosquitoes join the susceptible class through migration at the rate Π_M, or through oviposition at the rate μ_M. To model the effect of cytoplasmic incompatibility, we assume that the proportion q, of the eggs produced by the female wolbachia-free mosquitoes are viable, while $(1-q)$ are non-viable. Susceptible mosquitoes contract zika virus when they bite humans in the infectious classes at the biting rate b_1, with probability of infection, α_{HM}, and move to the exposed class. After some period of time (incubation period) in the exposed class, the mosquitoes become infectious and move to the infectious class at the rate, β_M. The mosquitoes remain infectious throughout their lifetime until they die naturally at the rate, σ_M.

Similarly, the female adult wolbachia-carrier aedes aegypti mosquitoes are grouped in the same manner, with the following compartments, the susceptible wolbachia-carrier class; $S_{Mw}(t)$, the exposed wolbachia-carrier class; $E_{Mw}(t)$, and the infectious wolbachia-carrier class; $I_{Mw}(t)$.

The dynamics of zika virus disease in the wolbachia-carrier mosquito population is similar to that of the wolbachia-free mosquitoes, except at the infectious stage where the probability of the wolbachia-carrier mosquitoes to transmit the virus to the susceptible humans is negligible.

The assumptions above and the flow diagram (Figure 1) lead to the following system of ordinary differential equations as our model for the transmission and control of zika virus disease.

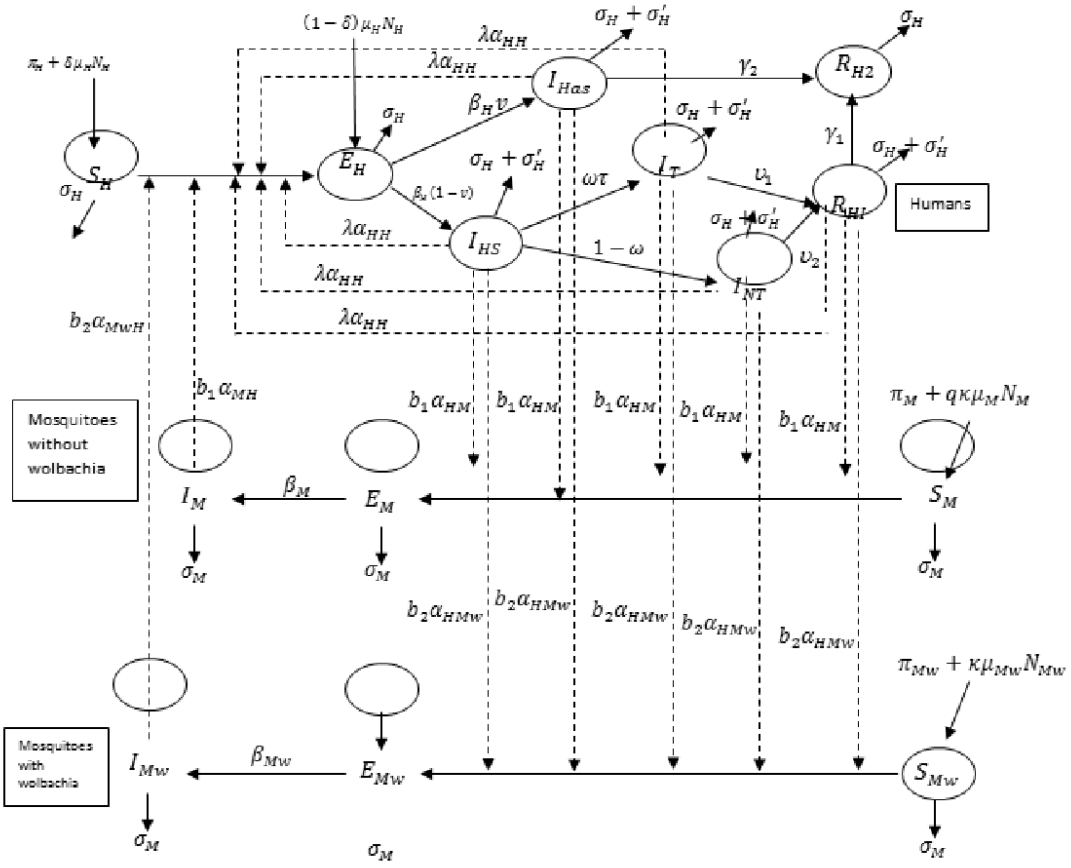

Figure 1: Flow Diagram for the Disease Transmission and Control

$$\frac{dS_H(t)}{dt} = \prod_H + \delta_{\mu H} N_H - (\mathrm{b}_1 \alpha_{MH} \frac{I_M}{N_M} + b_2 \alpha_{MwH} \frac{I_{Mw}}{N_{Mw}} + \lambda \alpha_{HH} F(N_H) + \sigma_H) S_H$$

$$\frac{dE_H(\mathrm{t})}{dt} = (1-\delta)\mu_H N_H + (b_1 \alpha_{MH} \frac{I_M}{N_M} + b_2 \alpha_{MwH} \frac{I_{Mw}}{N_{Mw}} + \lambda \alpha_{HH} F(N_H)) S_H - (\beta_H + \sigma_H) E_H$$

$$\frac{dI_{H\alpha s}(t)}{dt} = \nu \beta_H E_H - (\gamma_2 + \sigma_H + \sigma'_H) I_{H\alpha s},$$

$$\frac{dI_{Hs}(t)}{dt} = (1-\nu)\beta_H E_H - (\tau\omega + 1 - \omega + \sigma_H + \sigma'_H) I_{Hs}, \quad (1)$$

$$\frac{dI_T(t)}{dt} = \tau\omega I_{Hs} - (\nu_1 + \sigma_H + \sigma'_H) I_T,$$

$$\frac{dI_{NT}(t)}{dt} = (1-w)I_{Hs} - (v_2 + \sigma_H + \sigma'_H)I_{NT},$$
$$\frac{dR_{H1}(t)}{dt} = v_1 I_T + v_2 I_{NT} - (\gamma_1 + \sigma_H + \sigma'_H)R_{H1},$$
$$\frac{dR_{H2}(t)}{dt} = \gamma_1 R_{H1} + \gamma_2 I_{H\alpha s} - \sigma_H R_{H2},$$
$$\frac{dS_M(t)}{dt} = \prod_M + q\kappa\mu_M N_M - b_1\alpha_{HM} S_M(t)F(N_H) - \sigma_M S_M(t),$$
$$\frac{dE_M(t)}{dt} = b_1\alpha_{HM} S_M(t)F(N_H) - (\beta_M + \sigma_M)E_M(t),$$
$$\frac{dI_M(\mathrm{t})}{dt} = \beta_M E_M(\mathrm{t}) - \sigma_M I_M(t),$$
$$\frac{dS_{Mw}(t)}{dt} = \prod_{Mw} + \kappa\mu_{Mw} N_{Mw} - b_2\alpha_{HMw} S_{Mw}(t)F(N_H) - \sigma_M S_{Mw}(t),$$
$$\frac{dE_{Mw}(t)}{dt} = b_2\alpha_{HMw} S_{Mw}(t)F(N_H) - (\beta_{Mw} + \sigma_M)E_{Mw}(t),$$
$$\frac{dI_{Mw}(t)}{dt} = \beta_{Mw} E_{Mw}(t) - \sigma_M I_{Mw}(t),$$

Where $F(N_H) = \left(\frac{I_{H\alpha s} + I_{Hs} + I_T + I_{NT} + R_{H1}}{N_H} \right)$. The initial conditions are $S_H(0) = S_H^0$, $E_H(0) = E_H^0$, $I_{H\alpha s}(0) = I_{H\alpha s}^0$, $I_{Hs}(0) = I_{Hs}^0$, $I_T(0) = I_T^0$, $I_{NT}(0) = I_{NT}^0$, $R_{H1}(0) = R_{H1}^0$, $R_{H2}(0) = R_{H2}^0$, $S_M(0) = S_M^0$, $E_M(0) = E_M^0$, $I_M(0) = I_M^0$, $S_{Mw}(0) = S_{Mw}^0$, $E_{Mw}(0) = E_{Mw}^0$, $I_{Mw}(0) = I_{Mw}^0$, which we assume to be all non-negative quantities. The total human population N_H, the total wolbachia-free mosquito's population N_M and the total wolbachia-infected mosquito population N_{Mw} satisfy the differential equations

$$\frac{dN_H}{dt} = \prod_H + (\mu_H - \sigma_H)N_H - \sigma'_H N'_H \qquad (2)$$

$$\frac{dN_M}{dt} = \prod_M + (q\kappa\mu_M - \sigma_M)N_M \qquad (3)$$

$$\frac{dN_{Mw}}{dt} = \prod_{Mw} + (\kappa\mu_{Mw} - \sigma_M)N_{Mw} \qquad (4)$$

respectively.

The domain of existence of the solution to the system can be described as

$D = D_1 \bigcup D_2 \bigcup D_3$

Where

$$D_1 = \{(S_H, E_M, I_{H\alpha s}, I_{Hs}, I_{NT}, I_T, R_{H1}, R_{H2}) \in \mathbb{R}_+^8 \mid S_H + E_M + I_{\alpha s} + I_s + I_{NT} + I_T + R_{H1} + R_{H2} \le N_H\}$$

$$D_2 = \{(S_M, E_M, I_M) \in \mathbb{R}_+^3 \mid S_M + E_M + I_M \le N_M\} \quad (5)$$

$$D_3 = \{(S_{Mw}, E_{Mw}, I_{Mw}) \in \mathbb{R}_+^3 \mid S_{Mw} + E_{Mw} + I_{Mw} \le N_{Mw}\}$$

Theorem 1: *Given that the initial solution X_0 is positive in D, the system of equation (1) possesses a unique positive solution X(t) that remains in D at all times.*

The existence and uniqueness of the solution are guaranteed since the right hand side of (1) has continuous partial derivatives with respect to each population class. Also it is easy to show through differential inequality that the individual solutions are all positive at all times. For example, using the first equation in (1), we have

$$\frac{dS_H(t)}{dt} = \prod_H + \delta\mu_H N_H - b_1\alpha_{MH} S_H \frac{I_M}{N_M} - b_2\alpha_{MwH} \frac{I_{Mw}}{N_{Mw}} S_H - \lambda\alpha_{HH} S_H F(N_H) - \sigma_H S_H,$$

$$> -\left(b_1\alpha_{MH} \frac{I_M}{N_M} + b_2\alpha_{MwH} \frac{I_{Mw}}{N_{Mw}} + \lambda\alpha_{HH} F(N_H) + \sigma_H \right) S_H$$

$$\Rightarrow \frac{dS_H(t)}{S_H(t)} > -\left(b_1\alpha_{MH} \frac{I_M}{N_M} + b_2\alpha_{MwH} \frac{I_{Mw}}{N_{Mw}} + \lambda\alpha_{HH} F(N_H) + \sigma_H \right)$$

$$\Rightarrow S_H(t) > S_H^0 e^{-\int^t \left(b_1\alpha_{MH} \frac{I_M(s)}{N_M(s)} + b_2\alpha_{MwH(s)} \frac{I_{Mw}}{N_{Mw}} + \lambda\alpha_{HH} F(N_H) + \sigma_H \right) ds} > 0$$

The positivity of other solutions can be shown in similar manner. Therefore, the solution is positive in $D \; \forall \; t > 0$.

Next, we show that D is positive invariant with respect to the flow of the model system. We can show this using the total populations in (2), (3) and (4).
From (2), we have that

$$\frac{dN_H}{dt} = \prod_H + (\mu_H - \sigma_H) N_H - \sigma'_H N'_H$$

$$\le \prod_H + (\mu_H - \sigma_H) N_H$$

$$\Rightarrow N_H(t) \le \frac{\prod_H}{\sigma_H - \mu_H} + N_H^0 e^{-(\sigma_H - \mu_H)t}$$

Therefore, $\limsup_{t\to\infty} N_H(t) \le \frac{\prod_H}{\sigma_H - \mu_H}$. This shows the total human population is bounded.

Hence, all the solutions in the human population remain in $D \; \forall \; t > 0$
Similarly, in the wolbacha-free and wolbachia-infected mosquito populations, we have

$\limsup_{t\to\infty} N_M(t) \le \frac{\prod_M}{\sigma_M - qk\mu_M}$ and $\limsup_{t\to\infty} N_{Mw}(t) \le \frac{\prod_{Mw}}{\sigma_M - k\mu_{Mw}}$ respectively. Hence, all the solutions for the mosquito populations remain in the domain $D\ \forall\ t > 0$. This shows that D is positive invariant.

Approximate Analytical Solution

We seek an approximate analytical solution to the system of nonlinear equation using the homotopy perturbation method (HPM), introduced by Jihuan He in 1998 (Biazar and Aminikhan, 2009). This method is a combination of homotopy, and the traditional perturbation method in which the solution to the system of differential equation is obtained as an infinite series in the independent variable, t. This method has been employed to solve a variety of linear and nonlinear differential equations arising in science and engineering. In the area of mathematical epidemiology, the method has being effectively used to find approximate analytical solution to the system of nonlinear differential equations that governs the spread of the diseases, see (Ebenezer *et al.*,2016), (Khan *et al.*, 2013), (Adamu *et al.*, 2017) and others.

The main idea of this method is the introduction of a parameter, $p \in [0,1]$, called homotopy parameter, such that starting from $p = 0$, the system undergoes a sequence of changes or deformations until $p = 1$. Essentially, at $p = 0$, the system is in its simplest form, and admits a much simple solution. As p increases to 1 the equation deforms gradually to the original equation, and the approximate analytical solution is obtained.

To apply HPM, consider the non-linear differential equation

$$A(u) - f(t) = 0, \quad t \in \Omega \tag{6}$$

with boundary condition $B\left(u, \frac{\partial u}{\partial n}\right) = 0$

where A is the general nonlinear differential operator, B is the boundary operator, $f(t)$ is a known analytic function. The idea is to split the differential operator A into linear, L and non-linear, N,

parts, so that (6) becomes

$$L(u) + N(u) - f(t) = 0, \quad t \in \Omega \tag{7}$$

Then, the homotopy, $v(t,p)$: $\Omega \times [0,1] \to \mathbb{R}$, is constructed which satisfies the equation

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[(v) - f(t)] = 0 \tag{8}$$

which when rearranged, gives the simpler form

$$H(v,p) = L(v) - [1-p]L(u_0) + p[N(v) - f(t)] = 0 \tag{9}$$

where $p \in [0,1]$ is the embedding parameter and u_0 is an initial approximate solution to the original differential equation (6). From (8) and using $p \in [0,1]$, we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{10}$$

$$H(v,1) = A(v) - f(t) = 0 \tag{11}$$

It is assumed that the solution to (9) can be written as a Maclaurin's series in p as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{12}$$

which on setting $p = 1$ we get the approximate solution to the system of differential equation (6).

Obtaining Approximate Analytical Solution to the Model Equations

We start by constructing a homotopy for the system that satisfies (1), with the assumption that the initial approximate solution, u_0, to the system is zero. Hence, (9) gives

$$\frac{dS_H(t)}{dt} = p\left(\prod_H + \delta\mu_H N_H - \left(b_1\alpha_{MH}\frac{I_M}{N_M} + b_2\alpha_{MwH}\frac{I_{Mw}}{N_{Mw}} + \lambda\alpha_{HH}F(N_H) + \sigma_H\right)S_H\right),$$

$$\frac{dE_H(t)}{dt} = p\left((1-\delta)\mu_H N_H + \left(b_1\alpha_{MH}\frac{I_M}{N_M} + b_2\alpha_{MwH}\frac{I_{Mw}}{N_{Mw}} + \lambda\alpha_{HH}F(N_H)\right)S_H - (\beta_H + \sigma_H)E_H\right),$$

$$\frac{dI_{H\alpha s}(t)}{dt} = p(\nu\beta_H E_H - (\gamma_2 + \sigma_H + \sigma'_H)I_{H\alpha s}),$$

$$\frac{dI_{Hs}(t)}{dt} = p((1-\nu)\beta_H E_H - (\tau\omega + 1 - \omega + \sigma_H + \sigma'_H)I_{Hs}),$$

$$\frac{dI_T(t)}{dt} = p(\tau\omega I_{Hs} - (\nu_1 + \sigma_H + \sigma'_H)I_T),$$

$$\frac{dI_{NT}(t)}{dt} = p((1-\omega)I_{Hs} - (\nu_2 + \sigma_H + \sigma'_H)I_{NT}),$$

$$\frac{dR_{H1}(t)}{dt} = p(\nu_1 I_T + \nu_2 I_{NT} - (\gamma_1 + \sigma_H + \sigma'_H)R_{H1}),$$

$$\frac{dR_{H2}(t)}{dt} = p(\gamma_1 R_{H1} + \gamma_2 I_{H\alpha s} - \sigma_H R_{H2}),$$

$$\frac{dS_M(t)}{dt} = p(\prod_M + q\kappa\mu_M N_M - b_1\alpha_{HM}S_M(t)F(N_H) - \sigma_H S_M(\mathrm{t})), \quad (13)$$

$$\frac{dE_M(t)}{dt} = p(b_1\alpha_{HM}S_M(t)F(N_H) - (\beta_M + \sigma_H)E_M(\mathrm{t})),$$

$$\frac{dI_M(t)}{dt} = p(\beta_M E_M(t) - \sigma_M I_M(t)),$$

$$\frac{dS_{Mw}(t)}{dt} = p(\prod_{Mw} + \kappa\mu_{Mw}N_{Mw} - b_2\alpha_{HMw}S_{Mw}(t)F(N_H) - \sigma_M S_{Mw}(\mathrm{t})),$$

$$\frac{dE_{Mw}(t)}{dt} = p(b_2\alpha_{HMw}S_{Mw}(t)F(N_H) - (\beta_{Mw} + \sigma_M)E_{Mw}(\mathrm{t})),$$

$$\frac{dI_{Mw}(t)}{dt} = p(\beta_{Mw}E_{Mw}(t) - \sigma_M I_{Mw}(t)),$$

$$\frac{dN_H(t)}{dt} = p(\prod_H + (\mu_H - \sigma_H)N_H - \sigma'_H N'_H),$$
$$\frac{dN_M(t)}{dt} = p(\prod_M + (q\kappa\mu_M - \sigma_M)N_M),$$
$$\frac{dN_{Mw}(t)}{dt} = p(\prod_{Mw} + (\kappa\mu_{Mw} - \sigma_{Mw})N_{Mw}),$$

Using two-term approximations for each class, we write the solution to (13) in the form

$$S_H(t) = S_{1,0} + pS_{1,1} + p^2 S_{1,2},\ E_H(t) = E_{1,0} + pE_{1,1} + p^2 E_{1,2},$$
$$I_{H\alpha s}(t) = I_{H1,0} + pI_{H1,1} + p^2 I_{H1,2},\ I_{Hs}(t) = I_{H2,0} + pI_{H2,1} + p^2 I_{H2,2},$$
$$I_T(t) = I_{1,0} + pI_{1,1} + p^2 I_{1,2},\ I_{NT}(t) = I_{2,0} + pI_{2,1} + p^2 I_{2,2},$$
$$R_{H1}(t) = R_{1,0} + pR_{1,1} + p^2 R_{1,2},\ R_{H2}(t) = R_{2,0} + pR_{2,1} + p^2 R_{2,2},$$
$$S_M(t) = S_{2,0} + pS_{2,1} + p^2 S_{2,2},\ E_M(t) = E_{2,0} + pE_{2,1} + p^2 E_{2,2}, \quad (14)$$
$$I_M(t) = I_{3,0} + pI_{3,1} + p^2 I_{3,2},\ S_{Mw}(t) = S_{3,0} + pS_{3,1} + p^2 S_{3,2},$$
$$E_{Mw}(t) = E_{3,0} + pE_{3,1} + p^2 E_{3,2},\ I_{Mw}(t) = I_{4,0} + pI_{4,1} + p^2 I_{4,2},$$
$$N_H(t) = N_{1,0} + pN_{1,1} + p^2 N_{1,2},\ N_M(t) = N_{2,0} + pN_{2,1} + p^2 N_{2,2},$$
$$N_{Mw}(t) = N_{3,0} + pN_{3,1} + p^2 N_{3,2},$$

where the coefficients of p^0, p^1 and p^2 in each equation are functions of t to be determined. By substituting (14) into (13) and comparing the coefficients of p^0, p^1 and p^2, we get the following system of ordinary differential equations for each power of p.

For p^0, we have

$$S'_{1,0}(t) = 0, E'_{1,0}(t) = 0, I'_{H1,0}(t) = 0, I'_{H2,0}(t) = 0, I'_{1,0}(t) = 0, I'_{2,0}(t) = 0, R'_{1,0}(t) = 0,$$
$$R'_{2,0}(t) = 0, S'_{2,0}(t) = 0, E'_{2,0}(t) = 0, I'_{3,0}(t) = 0, S'_{3,0}(t) = 0, E'_{3,0}(t) = 0,$$
(15)
$$I'_{4,0}(t) = 0, N'_{1,0}(t) = 0, N'_{2,0}(t) = 0, N'_{3,0}(t) = 0,$$

with the initial conditions,

$$S_{1,0}(0) = S_H^0, E_{1,0}(0) = E_H^0, I_{H1,0}(0) = I_{H\alpha s}^0, I_{H2,0}(0) = I_{Hs}^0, I_{1,0}(0) = I_T^0, I_{2,0}(0) = I_{NT}^0,$$
$$R_{1,0}(0) = R_{H1}^0, R_{2,0}(0) = R_{H2}^0, S_{2,0}(0) = S_M^0, E_{2,0}(0) = E_M^0, I_{3,0}(0) = I_M^0, S_{3,0}(0) = S_{Mw}^0,$$

$$E_{3,0}(0) = E_{Mw}^0, I_{4,0}(0) = I_{Mw}^0, N_{1,0}(0) = N_H^0, N_{2,0}(0) = N_M^0, N_{3,0}(0) = N_{Mw}^0,$$

Solving (15) and applying the initial conditions, we have

$$S_{1,0}(t) = S_H^0, E_{1,0}(t) = E_H^0, I_{H1,0}(t) = I_{H\alpha s}^0, I_{H2,0}(t) = I_{Hs}^0, I_{1,0}(t) = I_T^0, I_{2,0}(t) = I_{NT}^0,$$

$$R_{1,0}(t)=R_{H1}^{0}, R_{2,0}(t)=R_{H2}^{0}, S_{2,0}(t)=S_{M}^{0}, E_{2,0}(t)=E_{M}^{0}, I_{3,0}(t)=I_{M}^{0}, S_{3,0}(t)=S_{Mw}^{0},$$

(16)

$$E_{3,0}(t)=E_{Mw}^{0}, I_{4,0}(t)=I_{Mw}^{0}, N_{1,0}(t)=N_{H}^{0}, N_{2,0}(t)=N_{M}^{0}, N_{3,0}(t)=N_{Mw}^{0},$$

For p^1, we have the system

$$S'_{1,1}(t)=\prod_{H}+\delta\mu_{H}N_{1,0}-\left(b_1\alpha_{MH}\frac{I_{3,0}}{N_{2,0}}+b_2\alpha_{MwH}\frac{I_{4,0}}{N_{3,0}}+\lambda\alpha_{HH}\frac{n'_{10}}{N_{1,0}}+\sigma_{H}\right)S_{1,0}$$

$$E'_{1,1}(t)=(1-\delta)\mu_{H}N_{1,0}+\left(b_1\alpha_{MH}\frac{I_{3,0}}{N_{2,0}}+b_2\alpha_{MwH}\frac{I_{4,0}}{N_{3,0}}+\lambda\alpha_{HH}\frac{n'_{10}}{N_{1,0}}\right)S_{1,0}-(\beta_{H}+\sigma_{H})E_{1,0}$$

$$I'_{H1,1}(t)=\nu\beta_{H}E_{1,0}-(\gamma_2+\sigma_{H}+\sigma'_{H})I_{H1,0}$$

$$I'_{H2,1}(t)=(1-\nu)\beta_{H}E_{1,0}-(\tau\omega+1-\omega+\sigma_{H}+\sigma'_{H})I_{H2,0}$$

$$I'_{1,1}(t)=\tau\omega I_{H2,0}-(\nu_1+\sigma_{H}+\sigma'_{H})I_{1,0}$$

$$I'_{2,1}(t)=(1-\omega)I_{H2,0}-(\nu_2+\sigma_{H}+\sigma'_{H})I_{2,0}$$

$$R'_{1,1}(t)=\nu_1I_{1,0}+\nu_2I_{2,0}-(\gamma_1+\sigma_{H}+\sigma'_{H})R_{1,0}$$

$$R'_{2,1}(t)=\gamma_1R_{1,0}+\gamma_2I_{H1,0}-\sigma_{H}R_{2,0}$$

$$S'_{2,1}(t)=\prod_{M}+qk\mu_{M}N_{2,0}-\left(b_1\alpha_{HM}\frac{n'_{10}}{N_{1,0}}+\sigma_{M}\right)S_{2,0}$$

$$E'_{2,1}(t)=b_1\alpha_{HM}\frac{n'_{10}}{N_{1,0}}S_{2,0}-(\beta_{M}+\sigma_{M})E_{2,0}$$

$$I'_{3,1}=\beta_{M}E_{2,0}-\sigma_{M}I_{3,0}$$

$$S'_{3,1}(t)=\prod_{Mw}+k\mu_{Mw}N_{3,0}-\left(b_2\alpha_{HM}\frac{n'_{10}}{N_{1,0}}+\sigma_{M}\right)S_{3,0} \quad (17)$$

$$E'_{3,1}(t)=b_2\alpha_{HMw}\frac{n'_{10}}{N_{1,0}}S_{3,0}-(\beta_{Mw}+\sigma_{M})E_{3,0}$$

$$I'_{4,1}=\beta_{Mw}E_{3,0}-\sigma_{M}I_{4,0}$$

$$N'_{1,1}(t)=\prod_{H}+(\mu_{H}-\sigma_{H})N_{1,0}-\sigma'_{H}N'_{1,0}$$

$$N'_{2,1}(t)=\prod_{M}+(qk\mu_{M}-\sigma_{M})N_{2,0}$$

$$N'_{3,1}(t)=\prod_{Mw}+(k\mu_{Mw}-\sigma_{M})N_{3,0}$$

with the initial conditions

$$S_{1,1}(0)=0, E_{1,1}(0)=0, I_{H1,1}(0)=0, I_{H2,1}(0)=0, I_{1,1}(0)=0, I_{2,1}(0)=0,$$

$R_{1,1}(0)=0, R_{2,1}(0)=0, S_{2,1}(0)=0, E_{2,1}(0)=0, I_{3,1}(0)=0, S_{3,1}(0)=0,$

$E_{3,1}(0)=0, I_{4,1}(0)=0, N_{1,1}(0)=0, N_{2,1}(0)=0, N_{3,1}(0)=0,$

The solution to (17) is

$$S_{1,1}(t)=\left(\prod_H+\delta\mu_H N_H^0-\left(b_1\alpha_{MH}\frac{I_M^0}{N_M^0}+b_2\alpha_{MwH}\frac{I_{Mw}^0}{N_{Mw}^0}+\lambda\alpha_{HH}\frac{n_{10}'}{N_H^0}+\sigma_H\right)S_H^0\right)t$$

$$E_{1,1}(t)=\left((1-\delta)\mu_H N_H^0+\left(b_1\alpha_{MH}\frac{I_M^0}{N_M^0}+b_2\alpha_{MwH}\frac{I_{Mw}^0}{N_{Mw}^0}+\lambda\alpha_{HH}\frac{n_{10}'}{N_H^0}\right)S_H^0-(\beta_H+\sigma_H)E_H^0\right)t$$

$$I_{H1,1}(t)=(v\beta_H E_H^0-(\gamma_2+\sigma_H+\sigma_H')I_{Hs}^0)t$$

$$I_{H2,1}(t)=((1-v)\beta_H E_H^0-(\tau\omega+1-\omega+\sigma_H+\sigma_H')I_{Hs}^0)t$$

$$I_{1,1}(t)=(\tau\omega I_{Hs}^0-(v_1+\sigma_H+\sigma_H')I_T^0)t$$

$$I_{2,1}(t)=((1-\omega)I_{Hs}^0-(v_2+\sigma_H+\sigma_H')I_T^0)t$$

$$R_{1,1}(t)=(v_1 I_T^0+v_2 I_{NT}^0-(\gamma_1+\sigma_H+\sigma_H')R_{H1}^0)t$$

$$R_{2,1}(t)=(\gamma_1 R_{H1}^0+\gamma_2 I_{H\alpha s}^0-\sigma_H R_{H2}^0)t$$

$$S_{2,1}(t)=\left(\prod_M+qk\mu_M N_M^0-\left(b_1\alpha_{MH}\frac{n_{10}'}{N_H^0}+\sigma_M\right)S_M^0\right)t \quad (18)$$

$$E_{2,1}(t)=\left(b_1\alpha_{HM}\frac{n_{10}'}{N_H^0}S_M^0-(\beta_M+\sigma_M)E_M^0\right)t$$

$$I_{3,1}(t)=(\beta_M E_M^0--+\sigma_M I_M^0)t$$

$$S_{3,1}(t)=\left(\prod_{Mw}+k\mu_{Mw}N_{Mw}^0-\left(b_1\alpha_{HM}\frac{n_{10}'}{N_H^0}+\sigma_M\right)S_{Mw}^0\right)t$$

$$E_{3,1}(t)=\left(b_2\alpha_{HMw}\frac{n_{10}'}{N_H^0}S_{Mw}^0-(\beta_{Mw}+\sigma_M)E_{Mw}^0\right)t$$

$$I_{4,1}(t)=(\beta_{Mw}E_{Mw}^0--+\sigma_M I_{Mw}^0)t$$

$$N_{1,1}(t)=(\prod_H+(\mu_H-\sigma_H)N_H^0-\sigma_H' n_{10}')t$$

$$N_{2,1}(t)=(\prod_M+(qk\mu_M-\sigma_M)N_M^0)t$$

$$N_{3,1}(t)=(\prod_{Mw}+(k\mu_{Mw}-\sigma_M)N_{Mw}^0)t$$

For the coefficients of p^2, we solve the system

$$S'_{1,2}(t) = \delta\mu_H N_{1,1} - \left(b_1\alpha_{MH}\frac{S_{1,0}I_{3,1}+S_{1,1}I_{3,0}}{N_{2,1}} + b_2\alpha_{MwH}\frac{S_{1,0}I_{4,1}+S_{1,1}I_{4,0}}{N_{3,1}} \right)$$
$$-\lambda\alpha_{HH}\frac{S_{1,0}(I_{H1,1}+I_{H2,1}+I_{1,1}+I_{2,1}+R_{1,1}}{N_{1,1}} - \lambda\alpha_{HH}\frac{S_{1,1}(I_{H1,0}+I_{H2,0}+I_{1,0}+I_{2,0}+R_{1,0}}{N_{1,1}} - \sigma_H S_{1,1}$$
$$E'_{1,2}(t) = (1-\delta)\mu_H N_{1,1} - \left(b_1\alpha_{MH}\frac{S_{1,0}I_{3,1}+S_{1,1}I_{3,0}}{N_{2,1}} + b_2\alpha_{MwH}\frac{S_{1,0}I_{4,1}+S_{1,1}I_{4,0}}{N_{3,1}} \right)$$
$$-\lambda\alpha_{HH}\frac{S_{1,0}(I_{H1,1}+I_{H2,1}+I_{1,1}+I_{2,1}+R_{1,1}}{N_{1,1}} - \lambda\alpha_{HH}\frac{S_{1,1}(I_{H1,0}+I_{H2,0}+I_{1,0}+I_{2,0}+R_{1,0}}{N_{1,1}} - (\beta_H+\sigma_H)E_{1,1}$$

$$I'_{H1,2}(t) = \nu\beta_H E_{1,1} - (\gamma_2+\sigma_H+\sigma'_H)I_{H1,1}$$
$$I'_{H2,2}(t) = (1-\nu)\beta_H E_{1,1} - (\tau\omega+1-\omega+\sigma_H+\sigma'_H)I_{H2,1}$$
$$I'_{1,2}(t) = \tau\omega I_{H2,1} - (\nu_1+\sigma_H+\sigma'_H)I_{1,1}$$
$$I'_{2,2}(t) = (1-\omega)I_{H2,1} - (\nu_2+\sigma_H+\sigma'_H)I_{2,1} \quad (19)$$
$$R'_{1,2}(t) = \nu_1 I_{1,1} + \nu_2 I_{2,1} - (\gamma_1+\sigma_H+\sigma'_H)R_{1,1}$$
$$R_{2,2}(t) = \gamma_1 R_{1,1} + \gamma_2 I_{H1,1} - \sigma_H R_{2,1}$$
$$S'_{2,2}(t) = qk\mu_M N_{2,1} - b_1\alpha_{HM}\frac{S_{2,0}(I_{H1,1}+I_{H2,1}+I_{1,1}+I_{2,1}+R_{1,1})}{N_{1,1}} - b_1\alpha_{HM}\frac{S_{2,1}(I_{H1,0}+I_{H2,0}+I_{1,0}+I_{2,0}+R_{1,0})}{N_{1,1}} - \sigma_M S_{1,1}$$
$$E'_{2,2}(t) = b_1\alpha_{HM}\frac{S_{2,0}(I_{H1,1}+I_{H2,1}+I_{1,1}+I_{2,1}+R_{1,1})}{N_{1,1}} - b_1\alpha_{HM}\frac{S_{2,1}(I_{H1,0}+I_{H2,0}+I_{1,0}+I_{2,0}+R_{1,0})}{N_{1,1}} - (\beta_M+\sigma_M)E_{2,1}$$
$$I'_{3,3}(t) = \beta_M E_{2,1} - \sigma_M I_{3,1}$$
$$S'_{3,2}(t) = k\mu_{Mw}N_{3,1} - b_2\alpha_{HMw}\frac{S_{3,0}(I_{H1,1}+I_{H2,1}+I_{1,1}+I_{2,1}+R_{1,1})}{N_{1,1}} - b_2\alpha_{HMw}\frac{S_{3,1}(I_{H1,0}+I_{H2,0}+I_{1,0}+I_{2,0}+R_{1,1})}{N_{1,1}} - \sigma_M S_{3,1}$$
$$E'_{3,2}(t) = b_2\alpha_{HMw}\frac{S_{3,0}(I_{H1,1}+I_{H2,1}+I_{1,1}+I_{2,1}+R_{1,1})}{N_{1,1}} - b_2\alpha_{HM}\frac{S_{3,1}(I_{H1,0}+I_{H2,0}+I_{1,0}+I_{2,0}+R_{1,1})}{N_{1,1}} - (\beta_{Mw}+\sigma_M)E_{3,1}$$
$$I'_{4,2}(t) = \beta_{Mw}E_{3,1} - \sigma_M I_{4,1}$$
$$N'_{1,2}(t) = (\mu_H - \sigma_H)N_{1,1} - \sigma'_H N_{1,1}$$
$$N'_{2,2}(t) = (qk\mu_M - \sigma_M)N_{2,1}$$
$$N'_{3,2}(t) = (k\mu_{Mw} - \sigma_M)N_{3,1}$$

with the initial conditions

$$S_{1,2}(0)=0, E_{1,2}(0)=0, I_{H1,2}(0)=0, I_{H2,2}(0)=0, I_{1,2}(0)=0, I_{2,2}(0)=0,$$
$$R_{1,2}(0)=0, R_{2,2}(0)=0, S_{2,2}(0)=0, E_{2,2}(0)=0, I_{3,2}(0)=0, S_{3,2}(0)=0,$$
$$E_{3,2}(0)=0, I_{4,2}(0)=0, N_{1,2}(0)=0, N_{2,2}(0)=0, N_{3,2}(0)=0,$$

The solution to (19) is

$$S_{1,2}(t)=-\left(b_1\alpha_{MH}\frac{S_H^0 i_{31}+s_{11}I_M^0}{n_{21}}+b_2\alpha_{MwH}\frac{S_H^0 i_{41}+s_{11}I_{Mw}^0}{n_{31}}\right)t-\lambda\alpha_{HH}\frac{S_H^0 n'_{11}+s_{11}n'_{10}}{n_{11}}t+(\delta\mu_H n_{11}-\sigma_H s_{11})\frac{t^2}{2}$$

$$E_{1,2}(t)=-\left(b_1\alpha_{MH}\frac{S_H^0 i_{31}+s_{11}I_M^0}{n_{21}}+b_2\alpha_{MwH}\frac{S_H^0 i_{41}+s_{11}I_{Mw}^0}{n_{31}}\right)t+\lambda\alpha_{HH}\frac{S_H^0 n'_{11}+s_{11}n'_{10}}{n_{11}}t+(1-\delta)\mu_H n_{11}-(\beta_H+\sigma_H)e_{11})\frac{t^2}{2}$$

$$I_{H1,2}(t)=(\nu\beta_H e_{11}-(\gamma_2+\sigma_H+\sigma'_H)i_{11}^*)\frac{t^2}{2}$$

$$I_{H2,2}(t)=((1-\nu)\beta_H e_{11}-(\tau\omega+1-\omega+\sigma_H+\sigma'_H)i_{21}^*)\frac{t^2}{2}$$

$$I_{1,2}(t)=(\tau\omega i_{21}^*-(\nu_1+\sigma_H+\sigma'_H)i_{11})\frac{t^2}{2}$$

$$I_{2,2}(t)=(1-\omega)i_{21}^*-(\nu_2+\sigma_H+\sigma'_H)i_{21})\frac{t^2}{2}$$

$$R_{1,2}(t)=(\nu_1 i_{11}+\nu_2 i_{21}-(\gamma_1+\sigma_H+\sigma'_H)i_{11}^*)\frac{t^2}{2}$$

$$R_{2,2}(t)=(\gamma_1 r_{11}+\gamma_2 i_{11}^*-\sigma_H r_{21})\frac{t^2}{2}$$

$$S_{2,2}(t)=-b_1\alpha_{HM}\frac{S_M^0 n'_{11}+s_{21}n'_{10}}{n_{11}}t+(qk\delta\mu_M n_{21}-\sigma_M s_{21})\frac{t^2}{2} \quad (20)$$

$$E_{2,2}(t)=-b_1\alpha_{HM}\frac{S_M^0 n'_{11}+s_{21}n'_{10}}{n_{11}}t-(\beta_M+\sigma_M)e_{21}\frac{t^2}{2}$$

$$I_{3,2}(t)=(\beta_M e_{21}-\sigma_M i_{31})\frac{t^2}{2}$$

$$S_{3,2}(t)=-b_2\alpha_{HMw}\frac{S_{Mw}^0 n'_{11}+s_{31}n'_{10}}{n_{11}}t+(k\delta\mu_{Mw}n_{31}-\sigma_M s_{31})\frac{t^2}{2}$$

$$E_{3,2}(t)=-b_2\alpha_{HMw}\frac{S_{Mw}^0 n'_{11}+s_{31}n'_{10}}{n_{11}}t-(\beta_{Mw}+\sigma_M)e_{31}\frac{t^2}{2}$$

$$I_{4,2}(t)=(\beta_{Mw}e_{31}-\sigma_M i_{41})\frac{t^2}{2}$$

$$N_{1,2}(t) = ((\mu_H - \sigma_H)n_{11} - \sigma'_H n'_{11})\frac{t^2}{2}$$

$$N_{2,2}(t) = ((qk\mu_M - \sigma_M)n_{21})\frac{t^2}{2}$$

$$N_{3,2}(t) = ((k\mu_{Mw} - \sigma_M)n_{31})\frac{t^2}{2}$$

Where we have used

$$s_{11} = \prod_H + \delta\mu_H N_H^0 - \left(b_1\alpha_{MH}\frac{I_M^0}{N_M^0} + b_2\alpha_{MwH}\frac{I_{Mw}^0}{N_{Mw}^0} + \lambda\alpha_{HH}\frac{n_0^*}{N_H^0} + \sigma_H \right) S_H^0,$$

$$i_{31} = \beta_M E_M^0 - \sigma_M I_M^0,$$

$$i_{41} = \beta_{Mw}E_{Mw}^0 - \sigma_M I_{Mw}^0,\ n'_{10} = I_{H\alpha s}^0 + I_{Hs}^0 + I_T^0 + I_{NT}^0 + R_{H1}^0,$$

$$n_{11} = \prod_H + (\mu_H - \sigma_H)N_H^0 - \sigma'_H n'_{10},$$

$$n_{21} = \prod_M + (k\mu_M - \sigma_M)N_M^0,\ n_{31} = \prod_{Mw} + (k\mu_{Mw} - \sigma_M)N_{Mw}^0,$$

$$i_{11}^* = \nu\beta_H E_H^0 - (\gamma_2 + \sigma_H + \sigma'_H)I_{H\alpha s}^0,$$

$$e_{11} = (1-\delta)\mu_H N_H^0 + \left(b_1\alpha_{MH}\frac{I_M^0}{N_M^0} + b_2\alpha_{MwH}\frac{I_{Mw}^0}{N_{Mw}^0} + \lambda\alpha_{HH}\frac{n_{10}^*}{N_H^0} \right) S_H^0 - (\beta_H + \sigma_H)E_H^0,$$

$$e_{21} = b_1\alpha_{HM}\frac{n_0^*}{N_H^0}S_M^0 - (\beta_M + \sigma_M)E_M^0,\ e_{31} = b_2\alpha_{HMw}\frac{n_0^*}{N_H^0}S_{Mw}^0 - (\beta_{Mw} + \sigma_M)E_{Mw}^0,$$

$$i_{21} = (1-w)I_{Hs}^0 - (\nu_2 + \sigma_H + \sigma'_H)I_{NT}^0,\ i_{21}^* = (1-\nu)\beta_H E_H^0 - (\tau w + 1 - w + \sigma_H + \sigma'_H)I_{Hs}^0,$$

$$i_{11} = \tau w I_{Hs}^0 - (\nu_1 + \sigma_H + \sigma'_H)I_T^0,\ r_{11} = \nu_1 I_T^0 + \nu_2 I_{NT}^0 - (\gamma_1 + \sigma_H + \sigma'_H)R_{H1}^0,$$

$$r_{21} = \gamma_1 R_{H1}^0 + \gamma_2 I_{H\alpha s}^0 - \sigma_H R_{H2}^0,$$

$$s_{21} = \prod_M + qk\mu_M N_M^0 - \left(b_1\alpha_{HM}\frac{n_0^*}{N_H^0} + \sigma_M \right) S_M^0,$$

$$s_{31} = \prod_{Mw} + k\mu_{Mw}N_{Mw}^0 - \left(b_1\alpha_{HM}\frac{n_0^*}{N_H^0} + \sigma_M \right) S_M^0,$$

Substituting the solutions (16), (18) and (20) into (14) and allowing $p \to 1$, we obtain the approximate solution to the model equation.

Table 1: Parameter values used in this model

Parameter	Value	Source	Parameter	Value	Source
$\prod_H$	100	Assumed	σ'_H	0.001	Assumed
μ_H	0.002	Assumed	$\prod_M$	50	Assumed
α_{HM}	0.75	(Gao *et al*., 2016)	$\prod_{Mw}$	50	Assumed
α_{HMw}	0.5	Assumed	μ_M	0.6	Assumed
δ	0.75	Assumed	μ_{Mw}	0.8	Assumed
β_H	$\frac{1}{3}$	(Kucharski *et al*., 2016)	k	0.45	(Ndii *et al*., 2012)
λ	0.02	(Wang *et al*., 2017)	b_1	0.05	Assumed
α_{HH}	0.8	(Gao *et al*., 2016)	b_2	0.7	Assumed
v	0.8	(Gao *et al*., 2016)	β_M	1/9	(Dantas *et al*., 2017)
τ	0.5	Assumed	β_{Mw}	1/18	Assumed
w	0.85	Assumed	α_{MH}	0.75	Assumed
v_1	0.28	Assumed	α_{MwH}	0.001	Assumed
v_2	0.5	Assumed	γ_1	0.5	Assumed
γ_2	0.25	Assumed	σ_H	0.005	Assumed
σ_M	0.15	(Ndii *et al*., 2012)	q	0.17	Assumed

Numerical Experiment

We simulate the approximate analytical solution obtained in section 4. The simulations are done with MATLAB software using the parameter values in Table 1, and the initial solutions

$S_H^0 = 1000,\ E_H^0 = 16,\ I_{H\alpha s}^0 = 20,\ I_{Hs}^0 = 10,\ I_{NT}^0 = 10,$

$R_{H1}^0 = 0,\ R_{H2}^0 = 10,\ S_M^0 = 980,\ E_M^0 = 90,$

The results of the numerical experiment are shown in the figures below. In Figure 2, we see that over time, the population of the wolbachia-carrier mosquitoes overtakes that of the wolbachia-free mosquitoes. This the resultant effect of cytoplasmic incompatibility (C.I). The effect of CI is further seen in Figure 3, where the population of the susceptible wolbachia-free mosquitoes increases until it reaches equilibrium. At this equilibrium, the wolbachia-carrier mosquitoes have established in the zika-endemic area, and subsequently prevent further increase in the production of new ones. It is easy to show that this population will start to decrease as time increases beyond this period. The effect of this bio-control method is more pronounced in Figure 4, where the population of infectious wolbachia-free mosquitoes decreases sharply after some period of time. The effect of the reduction in the population of the infectious wolbachia-free mosquitoes is shown in Figure 5, figure 6 and Figure 7. In these figures, we notice drastic

reduction in the populations of the exposed humans, the asymptomatically-infectious humans and the symptomatically-infectious humans. With these reductions in the populations of the infectious humans and wolbachia-free aedes aegypti mosquitoes, zika virus disease will not spread in the population, hence the disease is controlled.

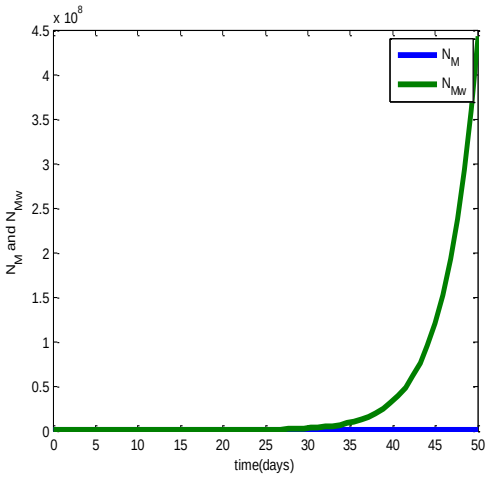

Figure 2: Total Population of Mosquitoes

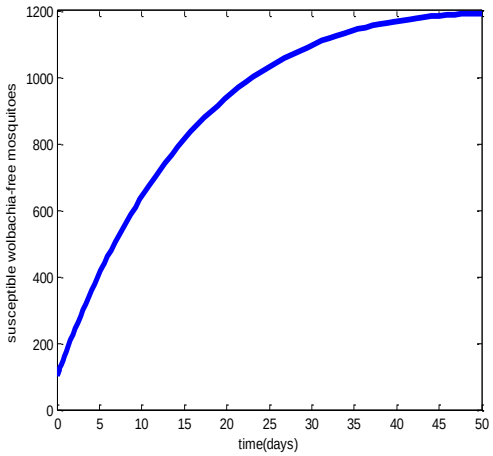

Figure 3: Susceptible wolbachia-free mosquitoes

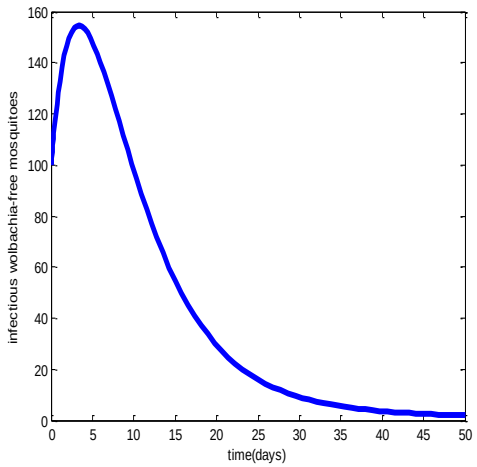

Figure 4: Infectious wolbachia-free mosquitoes

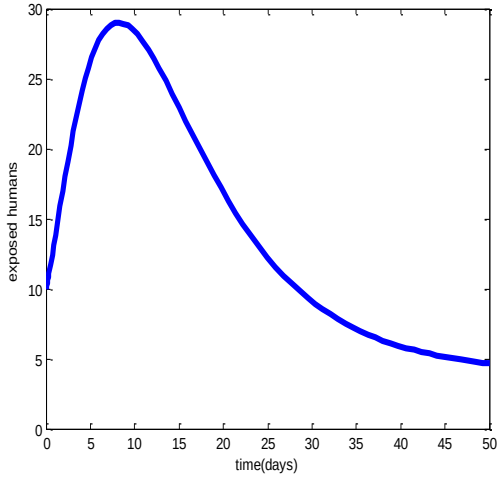

Figure 5: Exposed Humans

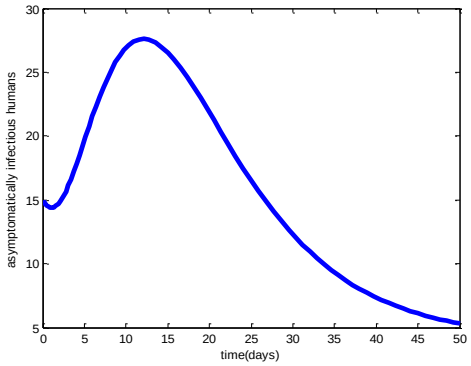

Figure 6: Asymptomatically Infectious Humans

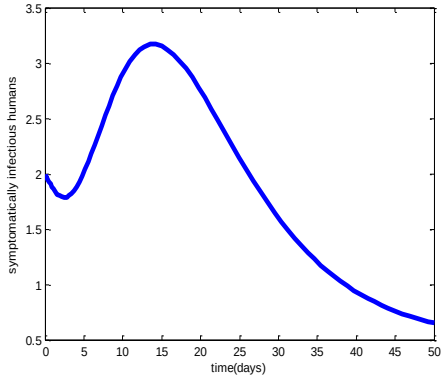

Figure 7: Symptomatically Infectious humans

Conclusion

In this work, we have presented a mathematical model for using wolbachia-carrier aedes aegypti mosquitoes to stop the spread of zika virus disease. We have demonstrated the effectiveness of using this mosquito control method by showing that in the long run, the wild aedes aegypti mosquitoes responsible for the spread of the disease is gradually replaced by the wolbachia-carrier mosquitoes which are not harmful, thereby reducing the number of people that will be exposed to the virus.

We conclude that using wolbachia-infected mosquitoes as a control for zika virus disease is an efficient and effective method for stopping the spread of this disease. This bio-control method can also be applied in controlling other mosquito-borne diseases such as malaria, West Nile, yellow fever and others, since other control methods have proved ineffective.

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