THE VOLUME OF A RECTANGULAR FRUSTUM: AN ALTERNATIVE METHOD

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Abstract
This paper presents an improvement on an early presentations on volumes of Frustums, a useful tool in a construction outfit. As a follow-up to an earlier presentation that considered Square and Conical Frustums, this paper considers the Volumes of a Rectangular Frustum as related to a Square Frustum, noting how they could depend on each other for estimation. It was noted that earlier papers discussed how the Egyptians obtained the volume of the pyramid as one-third the height multiplied by the sum of the two different areas A_1 (from a large pyramid) and A_2 (a smaller pyramid), added to the square-root of the product of the two areas {i.e. $V = \frac{1}{3}h\left[A_1 + A_2 + \sqrt{A_1 A_2}\right]$}. It was also noted that the usual method of estimating the volume of such dissected solid figures was by first calculating the volume of the original big pyramidal container, chopping off the top part as needed, calculating the volume of the chopped-off pyramid, and then subtracting the chopped-off volume from the original big pyramid. Such cumbersome calculation having been reduced by Olagunju (2011), to obtain a proven formula for a Square Frustum as $V_{SF} = \frac{1}{6}h(D^2 + Dd + d^2)$, and in furtherance to this, Olagunju (2016) showed that the volume of a square frustum could be estimated using the known volume of a conical frustum (and vice versa). This has now been extended to obtain the volume of a Rectangular Frustum as $V_{RF} = \frac{h}{3\sqrt{5}}(D^2 + Dd + d^2)$. Where V_{RF} = Volume of Rectangular Frustum, D = Diagonal of the large Rectangular base, d = diagonal of the small Rectangular top, and h = the height of the Rectangular Frustum.

Keywords: *Volume, Pyramids, Frustum, Rectangular, Square, Diagonals.*

1.0 Background

Scientific knowledge is expected to be progressive in order to bring about some improvement in application for the purpose of sustaining progressive development. This explains why man looks for new ways of doing things every new day. Thus, the essence of education, especially Mathematics, is to find a way of improving on earlier findings. Sometimes, this could be achieved by establishing the relationship between existing models. This accounts for why the established formula for the Volume of Square Frustums is considered here in order to generate a formula for the Rectangular Frustum.

2.0 Significance of the Study

This work seeks to obtain a formula for estimating the volume of a Rectangular based Frustum, and illustrate same with a view of comparing the results of the new formula with the old existing method. This will help teachers and students of Mathematics as well as construction firms to gain some time in their effort in estimating such.

3.0 Some Clarifications

3.1 Pyramids and their Classification

According to Hart (2005), as reported in Olagunju (2011, 2016), a Pyramid is a Polyhedron having one polygonal face (called 'base') and all other faces as Triangles, meeting at the Vertex (called 'Apex'). A special kind of Pyramid whose base is circular and all slant-edge lines meet at the vertex is referred to as a Cone or a Conical Pyramid. When a part of a Pyramid is chopped off, a plane parallel to the base is created and it becomes a Truncated Pyramid, usually referred to as a Frustum. Pyramids are classified by their dimensions. While a *Regular pyramid* is one with a base with regular polygon (e.g. Square-Based, Rectangular-Based), a *Right pyramid* is one whose apex is joined to the center of the base by a perpendicular line. Another type with one single cross-sectional shape having lengths scaling linearly with its height is referred to as an *Arbitrary pyramid*.

3.2 Frustums and their Classification

A Frustum is a Truncated Pyramid in which one part has been chopped-off to a given height. At the point of cut, the top platform is usually parallel to the base platform. Frustums are classified using the shape of their bases and tops which are usually the same. A frustum is said to be *Rectangular* when the Base and Top are both Rectangular, or *Square* when the Base and Topare both in the form of Squares (usually, the area of one end-face is smaller than the other), and *Conical* when the Base and Top are circular (usually, the radius of one end-face is smaller than the other).

4.0 Brief Information on Volumes of Allied Shapes

4.1 Volumes of Pyramids and Pyramidal Frustums

As noted by Olagunju (2011, 2016), Harris and Stocker (1998) showed that the Volume of a pyramid is given as one-third of the product of the base-area and the perpendicular height.

i.e. Pyramidal Volume = $\frac{1}{3}$(base-area x $\perp$ height).

Thus, the volume of the Truncated Pyramid (i.e. Pyramidal Frustum) is given as the difference between the complete pyramid and the chopped small pyramid.

i.e. $V_{PF} = V_{BP} - V_{SP}$ **where**

V_{PF} = Volume of Pyramidal Frustum,

V_{BP} = Volume of Complete Big Pyramid,

V_{SP} = Volume of the Small chopped pyramid

4.2 Volume of a Square Frustum

Consider the work of Olagunju (2016) on the Square Pyramidal Frustum, where $l = b$ (square base.) and base area = l x b,

Since $V_P = \frac{1}{3} l \text{ x } b \text{ x } h = \frac{1}{3} l^2 h = \frac{1}{3} b^2 h$

Then, we obtain the volume of a Truncated Square Pyramid thus:

If the volume of a Big Square-based Pyramid is $V_{BP} = \frac{1}{3} L^2 H$

And if the volume of the chopped-off small Pyramid is $V_{SP} = \frac{1}{3} l^2 x$

Then, $$V_{SF} = V_{BP} - V_{SP} \Rightarrow V_{PF} = \frac{1}{3} L^2 H - \frac{1}{3} l^2 x = \frac{1}{3}(L^2 H - l^2 h) \qquad \textbf{(4.1)}$$

Where: l = length of Small-Square-Top (Base of chopped-off top pyramid),
x = height of Small Pyramid
L = Length of Big-Square-Base Pyramid,
H = Height of Big Pyramid [(H = x + h), h = height of Frustum],.

V_{SF} = Volume of Square Frustum

V_{BP} = Volume of Big Pyramid

V_{SP} = Volume of Small Pyramid

This leads to the fact that if the base and top diagonals of a Square Frustum are given as D and d respectively, and if the height is h, then, the Volume of the Square frustum designated as V_{SF} is obtained as $$V_{SF} = \frac{h}{6}\left[D^2 + Dd + d^2 \right] \qquad \textbf{(4.2)}$$

Where: d = Diagonal of the Small-Square-Top,
D = Diagonal of the Big-Square Base
L = Length of Big-Square-Base Pyramid,
h = Height of Square Frustum

V_{SF} = Volume of Square Frustum

5.0 Formula for the Volume of a Rectangular Frustum

5.1 Important Information concerning the Rectangular-Based Model

5.11 Since the original Pyramid has a Rectangular base, then, the top chopped-off Small Pyramid also has a Rectangular base.

5.12 If the ratio of the rectangular base-length to the base-breadth is 2:1 (i.e. **2L:L**), then, the top rectangular chopped-off pyramid, top-length and top-breadth will also be of ratio 2:1 (i.e. ***2l:l***).

5.13 The ratio of the height of the top chopped-off pyramid to the height of the original big pyramid equals the ratio of the diagonal of the top chopped-off pyramid to the Diagonal of the original big pyramid ***x : H = d : D***

5.14 Since the Big Rectangular Base-Length and the Small Rectangular Top-length are in the ratio L : *l* , and this affects their diagonals, then, $\mathbf{D^2 : d^2 = L :} \boldsymbol{l}$

5.2 Developing Formula for the Volume of a Rectangular Frustum

As against method of subtracting the volume of a small chop-off rectangular pyramid from the complete big rectangular pyramid as earlier mentioned, we proceed to obtain a formula for the volume of a rectangular frustum as follows, considering Figure 5.1 below:

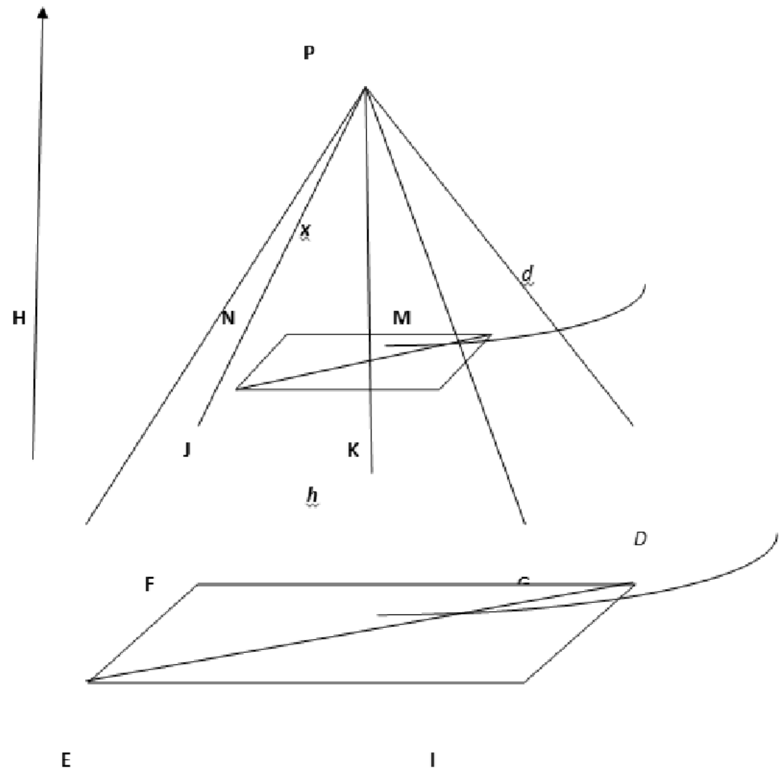

Figure 5.2

Given application of the listed dimensions below to Fig. 5.2 above,
x *= height of Small Pyramid,*
h *= height of Frustum,*
H *= Height of Large Pyramid, (height of small pyramid + height of Frustum)*
Base-Length = ***2L***
Base-breadth = ***L***
Top-Length = ***2l***
Top-breadth = ***l***

We note if the base and top diagonals of the pyramidal frustum be ***D*** and ***d*** respectively.

Since its height is ***h***, and the Volume is designated as V_{RF} **,**

Then, by the old method, the Volume of Rectangular Frustum will be the difference between the Volume of large Rectangular Pyramid and Volume of chopped Rectangular Pyramid.

We note here that $H = h + x$ (Heights of the pyramids) **(5.21)**

By Pythagoras applied to rect. JKMN and rect. EFGI above, From the Small Top-Rectangle, $d = \sqrt{5}l$ and $d^2 = 5l^2$ **(5.22)**

From the Big Base-Rectangle, $D = \sqrt{5}L$ and $D^2 = 5L^2$ **(5.23)**

But Base-Area of the Big Rectangular Pyramid = $2L$ x $L = 2L^2$

While the Volume of the Big Rectangular Pyramid = $2L^2H = 2L^2(x+h)$

Also, Base- Area of the chopped Small Rectangular Pyramid = $2l$ x $l = 2l^2$

While the Volume of the chopped Small Rectangular Pyramid = $2l^2x$

Substituting these in the old technique whereby

$$V_{PF} = \tfrac{1}{3}L^2H - \tfrac{1}{3}l^2x = \tfrac{1}{3}(L^2H - l^2h) \quad \textbf{(4.2)}$$

We have equation (4.2) becoming

$$V_{RF} = \tfrac{1}{3}\left[2L^2(x+h) - 2l^2x\right]$$

$$\Rightarrow \quad V_{RF} = \tfrac{2}{3}\left[L^2(x+h) - l^2x\right]$$

But equations (5.22) and (5.23) above, $d = \sqrt{5}l$ and $D = \sqrt{5}L$

Thus, $V_{RF} = \tfrac{2}{3}\left[\left(\tfrac{D}{\sqrt{5}}\right)^2(x+h) - \left(\tfrac{d}{\sqrt{5}}\right)^2 x\right]$

$$\Rightarrow \quad V_{RF} = \tfrac{2}{3}\tfrac{1}{5}\left[D^2(x+h) - d^2x\right]$$

$$\Rightarrow \quad V_{RF} = \tfrac{2}{15}\left[D^2h + D^2x - d^2x\right]$$

$$\Rightarrow \quad V_{RF} = \tfrac{2}{15}\left[D^2h + x\left(D^2 - d^2\right)\right]$$

$$\Rightarrow \quad V_{RF} = \tfrac{2}{15}\left[D^2h + x\left(D-d\right)\left(D+d\right)\right] \quad \textbf{(5.2)}$$

Considering similar ΔPJM and ΔPEG

We note that $\dfrac{x}{d} = \dfrac{x+h}{D}$

Implying that $x(D-d) = dh$ **(5.3)**

Putting (5.3) into (5.2), we have

$$V_{RF} = \tfrac{2}{15}\left[D^2h + dh(D+d)\right]$$

Hence $V_{RF} = \tfrac{2}{15}\left[D^2h + Ddh + d^2h\right]$

$$\Rightarrow \quad V_{RF} = \tfrac{2h}{15}\left[D^2 + Dd + d^2\right] \quad \textbf{(5.4)}$$

Equation (5.4) determines the Volume of the said Rectangular Frustum

6.0 Illustrations – of Rectangular Frustum (Old and New Methods)

For the purpose of illustration, we shall consider few examples below, calculating the volumes using the method of subtracting smaller chopped pyramids from the original big pyramids, in comparison with the use of the formula just derived. For this purpose, we shall designate the volume and height of the original big rectangular pyramid by $\boldsymbol{V_{BRP}}$ and ($\boldsymbol{x + h}$ i.e. H) respectively, volume and height of the small chopped-off rectangular pyramid by $\boldsymbol{V_{SRP}}$ and $\boldsymbol{x}$ respectively, and the volume and height of the Rectangular Frustum by $\boldsymbol{V_{RF}}$ and $\boldsymbol{h}$ respectively.

6.1 Illustration I

When x = 5, h = 5 , l = 4 and L = 8.
From equation (5.21), ***H : (x + h)***.

Solution: $V_{SP} = \tfrac{2}{3}xl^2 = \tfrac{2}{3}(5)4^2 = \dfrac{160}{3}$

$V_{BP} = \tfrac{2}{3}(h+x)L^2 = \tfrac{2}{3}(5+5)8^2 = \dfrac{1280}{3}$

Thus, $V_{RF} = V_{BP} - V_{SP} = \dfrac{1280}{3} - \dfrac{160}{3} = \dfrac{1120}{3}$ unit^3 (6.1.1)

Now, by the new formula, from equations (5.22) and (5.23),
We have $D^2 = 5L^2$, $d^2 = 5l^2$ and $Dd = 5Ll$
Also, from equation (5.4),
We have $V_{RF} = \tfrac{2h}{15}\left[D^2 + Dd + d^2\right]$
Now, h = 5, $d^2 = 5l^2 = 5(4)^2 = 80$, $D^2 = 5L^2 = 5(8)^2 = 320$ and
$Dd = 5Ll = 5(8)(4) = 160$
Hence, $V_{RF} = \tfrac{2h}{15}\left[D^2 + Dd + d^2\right]$ becomes

$$V_{RF} = \tfrac{10}{15}\left[320 + 160 + 80\right] = \frac{1120}{3} \text{ unit}^3 \qquad (6.1.2)$$

It is noted that results in (6.1.1) and (6.1.2) confirm the authenticity of the formula.

6.2 Illustration II

When x = 7, h = 7 , l = 11 and L = 22.
From equation (5.21), ***H : (x + h)***.

Solution: $V_{SP} = \tfrac{2}{3}xl^2 = \tfrac{2}{3}(7)11^2 = \dfrac{1694}{3}$

$V_{BP} = \tfrac{2}{3}(h+x)L^2 = \tfrac{2}{3}(7+7)22^2 = \dfrac{13552}{3}$

Thus, $V_{RF} = V_{BP} - V_{SP} = \dfrac{13552}{3} - \dfrac{1694}{3} = \dfrac{11858}{3}$ unit^3 (6.2.1)

Now, by the new formula, from equations (5.22) and (5.23),
We have $D^2 = 5L^2$, $d^2 = 5l^2$ and $Dd = 5Ll$
Also, from equation (5.4),
We have $V_{RF} = \tfrac{2h}{15}\left[D^2 + Dd + d^2\right]$

Now, h = 7, $d^2 = 5l^2 = 5(11)^2 = 605$, $D^2 = 5L^2 = 5(22)^2 = 2420$ and $Dd = 5Ll = 5(22)(11) = 1210$

Hence, $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ becomes

$$V_{RF} = \tfrac{14}{15}\left[2420 + 1210 + 605\right] = \frac{11858}{3} \text{ unit}^3 \quad (6.2.2)$$

It is noted that results in (6.2.1) and (6.2.2) confirm the authenticity of the formula.

6.3 Illustration III

When x = 0.8, h = 0.8 , l = 0.3 and L = 0.6.
From equation (5.21), ***H : (x + h)***.

Solution: $V_{SP} = \tfrac{2}{3}xl^2 = \tfrac{2}{3}(0.8)(0.3)^2 = \dfrac{0.144}{3}$

$$V_{BP} = \tfrac{2}{3}(h + x)L^2 = \tfrac{2}{3}(0.8 + 0.8)(0.6)^2 = \frac{1.152}{3}$$

Thus, $V_{RF} = V_{BP} - V_{SP} = \dfrac{1.152}{3} - \dfrac{0.144}{3} = \dfrac{1.008}{3}$ unit^3 (6.3.1)

Now, by the new formula, from equations (5.22) and (5.23),
We have $D^2 = 5L^2$, $d^2 = 5l^2$ and $Dd = 5Ll$
Also, from equation (5.4),
We have $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$

So, h = 0.8, $d^2 = 5l^2 = 5(0.3)^2 = 0.45$, $D^2 = 5L^2 = 5(0.6)^2 = 1.80$ and $Dd = 5Ll = 5(0.6)(0.3) = 0.90$

Hence, $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ becomes

$$V_{RF} = \tfrac{1.6}{15}\left[1.80 + 0.90 + 0.45\right] = \frac{1.008}{3} \text{ unit}^3 \quad (6.3.2)$$

It is noted that results in (6.3.1) and (6.3.2) confirm the authenticity of the formula.

6.4 Illustration IV

When x = 7, h = 14 , l = 4 and L = 12.
From equation (5.21), ***H : (x + h)***.

Solution: $V_{SP} = \tfrac{2}{3}xl^2 = \tfrac{2}{3}(7)4^2 = \dfrac{224}{3}$

$$V_{BP} = \tfrac{2}{3}(h + x)L^2 = \tfrac{2}{3}(7 + 14)12^2 = \frac{6048}{3}$$

Thus, $V_{RF} = V_{BP} - V_{SP} = \frac{6048}{3} - \frac{224}{3} = \frac{5824}{3}$ unit^3 (6.4.1)

Now, by the new formula, from equations (5.22) and (5.23),
We have $D^2 = 5L^2$, $d^2 = 5l^2$ and $Dd = 5Ll$
Also, from equation (5.4),
We have $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$
Now, h = 14, $d^2 = 5l^2 = 5(4)^2 = 80$, $D^2 = 5L^2 = 5(12)^2 = 720$ and
$Dd = 5Ll = 5(12)(4) = 240$
Hence, $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ becomes

$$V_{RF} = \tfrac{28}{15}\left[720 + 240 + 80\right] = \frac{5824}{3} \text{ unit}^3 \quad (6.4.2)$$

It is noted that results in (6.4.1) and (6.4.2) confirm the authenticity of the formula.

6.5 Illustration V

When x = 0.9, h = 2.7 , *l* = 0.6 and L = 2.4
From equation (5.21), ***H : (x + h)***.

Solution: $V_{SP} = \tfrac{2}{3}xl^2 = \tfrac{2}{3}(0.9)(0.6)^2 = \frac{0.648}{3}$

$V_{BP} = \tfrac{2}{3}(h+x)L^2 = \tfrac{2}{3}(0.9+2.7)(2.4)^2 = \frac{41.472}{3}$

Thus, $V_{RF} = V_{BP} - V_{SP} = \frac{41.472}{3} - \frac{0.648}{3} = \frac{40.824}{3}$ unit^3 (6.5.1)

Now, by the new formula, from equations (5.22) and (5.23),
We have $D^2 = 5L^2$, $d^2 = 5l^2$ and $Dd = 5Ll$
Also, from equation (5.4),
We have $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$
So, h = 2.7, $d^2 = 5l^2 = 5(0.6)^2 = 1.80$, $D^2 = 5L^2 = 5(2.4)^2 = 28.80$ and
$Dd = 5Ll = 5(2.4)(0.6) = 7.20$
Hence, $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ becomes

$$V_{RF} = \tfrac{5.4}{15}\left[28.80 + 7.20 + 1.80\right] = \frac{40.824}{3} \text{ unit}^3 \quad (6.5.2)$$

It is noted that results in (6.5.1) and (6.5.2) confirm the authenticity of the formula.

6.6 Illustration VI

When x = 6.8, h = 10.2 , *l* = 3.0 and L = 7.5
From equation (5.21), ***H : (x + h)***.

Solution: $V_{SP} = \frac{2}{3}xl^2 = \frac{2}{3}(6.8)(3.0)^2 = \frac{122.4}{3}$

$V_{BP} = \frac{2}{3}(h+x)L^2 = \frac{2}{3}(10.2+6.8)(7.5)^2 = \frac{1912.5}{3}$

Thus, $V_{RF} = V_{BP} - V_{SP} = \frac{1912.5}{3} - \frac{122.4}{3} = \frac{1790.1}{3}$ unit^3

(6.6.1)

Now, by the new formula, from equations (5.22) and (5.23),

We have $D^2 = 5L^2$, $d^2 = 5l^2$ and $Dd = 5Ll$

Also, from equation (5.4),

We have $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$

So, h = 10.2, $d^2 = 5l^2 = 5(3)^2 = 45$, $D^2 = 5L^2 = 5(7.5)^2 = 281.25$ and

$Dd = 5Ll = 5(7.5)(3) = 112.5$

Hence, $V_{RF} = \frac{2h}{15}\left[D^2 + Dd + d^2\right]$ becomes

$$V_{RF} = \frac{20.4}{15}\left[281.25 + 112.5 + 45\right] = \frac{1790.1}{3} \text{ unit}^3 \qquad (6.6.2)$$

It is noted that results in (6.6.1) and (6.6.2) confirm the authenticity of the formula.

Conclusion and Recommendation

Having confirmed the usefulness and effectiveness of this formula, it has therefore been proven that this new method can now be used for the calculation of the volume of a Rectangular Frustum as it removes the cumbersomeness involved in the use of the old method, which involved the process of having to calculate two different volumes and then carrying out some subtraction.

Precaution

Since every model has its own precaution(s), it should be noted that this fails when the conditions are not properly applied.

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