THE REVERSE ORDER RELATIONSHIP BETWEEN THE SKEW-SYMMETRIC GAME AND LINEAR PROGRAMMING PROBLEM

Okafor, Folakemi M.[1] Adiri, Emmanuel Iyasere[2]
[1]*National Mathematical Centre, Abuja, Nigeria*
[2]*Mathematics Department, College of Education, Warri, Delta State, Nigeria*
Email: folayanfe@gmail.com1

Abstract
This paper considered the conversion of Linear Programming Problem (LPP) to skew-symmetric game. In literature, a lot of research has been carried out on the conversion of a game problem to Linear Programming problem. However, it has been observed that the reverse direction of this relationship has not been investigated, which is the primary aim of this work. The researchers considered the conversion of LPP to skew-symmetric game through the formation of a super LPP which resulted in a skew- symmetric matrix game.

Keywords: *Linear programming problems, skew-symmetric game, sparsity, Super Linear Programming Problem.*

1.0 Introduction

Game Theory is the mathematical modeling of players, strategies and payoffs. (Leigh and William, 2016). Symmetric games have been studied since the early days of game theory. Gale and Stewart (1951), The established definition states that a game is symmetric if the payoff functions of all players are identical and symmetric in the other players' actions i.e players cannot, or needed not, distinguish between the other players. Strategic games may exhibit symmetries in a variety of ways. A characteristic feature, enabling the compact representation of games even when the number of players is unbound. Felix *et al* (2009).

Kim *et al* (1997), presented a pair of symmetric variational problems, also Kim and Lee (1999) and Kim and Lee (1998) extended symmetric duality theorems for multi objective variational problems. Okafor *et al* (2018) examined the relationship that exists between linear programming problem and games theory by developing a model and comparing it with that of Dorfman, etal (2012)

2.0 Symmetric and Skew-Symmetric Matrix

A square matrix A is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j, where a_{ij} is an element present at $(i,j)^{th}$ position (ith row and jth column in matrix A) and a_{ji} is an element present at $(j,i)^{th}$ position (jth row and ith column in matrix A).

In other words we can say that matrix A is said to be symmetric if transpose of matrix A is equal to matrix A itself i.e ($A^T = A$).

A square matrix A is said to be skew-symmetric if $a_{ij} = - a_{ji}$ for all i and j.

In other words, we can say that matrix A is said to be skew-symmetric if transpose of matrix A is equal to negative of matrix A (i.e. $A^T = - A$).

It is worth noting that all the main diagonal elements in skew-symmetric matrix are zero. For example;

$$A = \begin{bmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}_1$$

It is a skew-symmetric matrix because a_{ij} = - a_{ji} for all i and j

2.1 Sparsity in LPP (Ekoko, 2004)

An LP problem is said to be sparse if it contains very few non-zero elements in its matrix coefficients. Sparsity is one of the major characteristics of practical problems which can be exploited in seeking computer solution to models of such problems. In nearly all practical mathematical programming problems, a typical variable occurs in not more than about six constraints. This is true whether the constraints are linear or nonlinear and whether the variables are continuous or discrete.

2.2 Conversion of LPP to Skew Symmetric Game

Both symmetric and skew-symmetric games are subspaces of finite games. Using the Euclidean space structure of finite games, we observed that the skew-symmetric games form an orthogonal complement of symmetric games.

Consider the following linear programming problems:

$$\left.\begin{array}{l} \text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{Subject to} \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ -------------- \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ x_1 \ge 0, x_2 \ge 0, \cdots, x_n \ge 0 \end{array}\right\} \quad (2.1)$$

The dual of the above LPP in system (2.1) can be written as:

$$\left.\begin{array}{l} \text{Minimize } z^* = b_1y_1 + b_2y_2 + \cdots + b_my_m \\ \text{Subject to} \\ a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\ -------------- \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \\ \qquad y_1 \ge 0, y_2 \ge 0, \cdots, y_m \ge 0 \end{array}\right\} \quad (2.2)$$

The next step is to combine the LPP in system (2.1) and its dual LPP in system (2.2) together to form a super LPP.

In order to combine the above two LPPs, both of them have to be of the same type. That is both of them have to be of maximization objective function type or the minimization objective function type.

Hence, the dual LPP (2.2) in maximization objective function type is:

$$\left.\begin{array}{l} \text{Maximize } -z^* = -b_1y_1 - b_2y_2 - \cdots - b_my_m \\ \text{Subject to} \\ -a_{11}y_1 - a_{21}y_2 - \cdots - a_{m1}y_m \le -c_1 \\ -a_{12}y_1 - a_{22}y_2 - \cdots - a_{m2}y_m \le -c_2 \\ ----------------- \\ -a_{1n}y_1 - a_{2n}y_2 - \cdots - a_{mn}y_m \le -c_n \\ \qquad y_1 \ge 0, y_2 \ge 0, \cdots, y_m \ge 0 \end{array}\right\} \tag{2.3}$$

The combination of the LPP in system (2.1) and its dual LPP in system (2.3) produces a super LPP.

2.3 Super Linear Programming Problem

The super LPP formed by combining the LPP (2.1) and its dual (2.3) becomes:

$$\left.\begin{array}{l} \text{Maximize } z - z^* = -b_1y_1 - \cdots - b_my_m + c_1x_1 + \cdots + c_nx_n \\ \text{Subject to} \\ \qquad 0y_1 + \cdots + 0y_m + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ \qquad 0y_1 + \cdots + 0y_m + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ ------------------------------ \\ \qquad 0y_1 + \cdots + 0y_m + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ \quad -a_{11}y_1 - \cdots - a_{m1}y_m + 0x_1 + \cdots + 0x_n \le -c_1 \\ \quad -a_{12}y_1 - \cdots - a_{m2}y_m + 0x_1 + \cdots + 0x_n \le -c_2 \\ ------------------------------ \\ \quad -a_{1n}y_1 - \cdots - a_{mn}y_m + 0x_1 + \cdots + 0x_n \le -c_n \\ \qquad\qquad y_1 \ge 0, \cdots y_m \ge 0, x_1 \ge 0, \cdots x_n \ge 0 \end{array}\right\} \tag{2.4}$$

Though this super LP problem can be solved to optimality, it is not yet in the form of a game. Since we know the z - z* = 0, we replace the objective function by the equivalent constraint;

$$-b_1y_1 - \cdots - b_my_m + c_1x_1 + \cdots + c_nx_n \ge 0 \tag{2.5}$$

Where the > sign has been redundantly inserted in the full knowledge that no feasible solution will ever require its presence. The complete set of linear constraints in (2.4) including (2.5) define the optimum solution (x,y,z-z^*) of our original problem, its dual, and of the new all-inclusive super problem.

As we look at this maximum problem, we instantly note its skew symmetric (anti symmetric) form. The a_{ij}'s appear twice, once with a plus sign and once in transposed form a_{ji} with a minus sign. The b's and c's both appear twice, once in the vertical and once in the horizontal, and with opposite algebraic signs. Moreover, to keep the constraints in conventional form, with the sign $\leq$ being used rather than $\geq$, it was necessary to introduce negative signs.

Our super LP problem is a maximum problem that is self-dual. This is shown in system (2.6) which is the same as the super LPP in system (2.4).

$$\left.\begin{array}{l}
\text{Min } b_1y_1+\cdots+b_my_m-c_1x_1-\cdots-c_nx_n \\
\text{Subject to} \\
0y_1+\cdots+0y_m-a_{11}x_1-\cdots-a_{1n}x_n \geq -b_1 \\
------------------------ \\
0y_1+\cdots+0y_m-a_{m1}x_1-\cdots-a_{mn}x_n \geq -b_m \\
a_{11}y_1+\cdots+a_{m1}y_m+0x_1+\cdots+0x_n \geq c_1 \\
------------------------ \\
a_{1n}y_1+\cdots+a_{mn}y_m+0x_1+\cdots+0x_n \geq c_n \\
\qquad y_1,y_2,\cdots,y_m,x_1x_2,\cdots,x_n \geq 0
\end{array}\right\} \tag{2.6}$$

This shows that the primal super problem in (2.4) is self-dual because of its special structure of anti-symmetry and zeros. If the right-hand coefficients of the constraints of (2.4) are all multiplied by v, then the best z-z* will be multiplied by v. However, any previous optimal solution $(y_1,\ldots,y_m,x_1,\ldots x_n)$ will be equal to $(vy_1,\ldots,vy_m,vx_1,\ldots vx_n)$. we can now regard v as one of our variables just like the *x's* and *y's*. Thirdly, we can transfer the right-hand coefficients of the constraints over to the left-hand side. The only advantage in doing this lies in the fact that a game is always written without any right-hand coefficients.

We now have n + m + 1 new variables, namely, $(vy_1,\ldots,vy_m,vx_1,\ldots vx_n,v)$, and the full statement of our super problem, as given by(2.4) and (2.5) can be written as:

$$\left.\begin{array}{l}
Max\ z-z^* = -b_1vy_1-\cdots-b_mvy_m+c_1vx_1+\cdots+c_nvx_n+0 \\
s.t. \\
0vy_1+\cdots+0vy_m+a_{11}vx_1+\cdots+a_{1n}vx_n-b_1v \leq 0 \\
---------------------------- \\
0vy_1+\cdots+0vy_m+a_{m1}vx_1+\cdots+a_{mn}vx_n-b_mv \leq 0 \\
-a_{11}vy_1-\cdots-a_{m1}vy_m+0vx_1+\cdots+0vx_n+c_1v \leq 0 \\
---------------------------- \\
-a_{1n}vy_1-\cdots-a_{mn}vy_m+0vx_1+\cdots+0vx_n+c_nv \leq 0 \\
vy_1 \geq 0,\cdots,vy_m \geq 0, vx_1 \geq 0,\cdots,vx_n \geq 0, v>0
\end{array}\right\} \tag{2.7}$$

where only the ratios $\frac{vy_i}{v}$ and $\frac{vx_i}{v}$ are significant.

Thus, the skew-symmetric game with pay-off matrix from player 2 to player 1 is as follows:

$$\left.\begin{array}{l} 0 \ \cdots \ 0 \quad a_{11}\cdots a_{1n} \quad -b_1 \\ ------------------ \\ 0 \ \cdots \ 0 \quad a_{m1}\cdots a_{mn} \quad -b_m \\ -a_{11}\cdots -a_{m1} \quad 0 \ \cdots \ 0 \quad +c_1 \\ ------------------ \\ -a_{1n}\cdots -a_{mn} \quad 0 \ \cdots \ 0 \quad +c_n \\ b_1 \ \cdots \ b_m -c_1 \cdots -c_n \quad 0 \end{array}\right\} \tag{2.8}$$

Suppose that player 2 must find its optimal mixed strategy for the probabilities of playing the different columns, $(Y_1, ..., Y_n, Y_{n+1}, ..., Y_{n+m}, Y_{n+m+1})$ since neither player has any advantage denied the other, consequently, player 2's optimal mixed strategy Y_i must be such as to result in player 1 receiving not more than the value of the game or zero for any pure strategy that player 1 plays against player 2's optimal Y_j. We write down this condition for each of player 1's pure strategies, with V = 0 and *Y's* replacing the *x's*. The problem becomes:

$$\left.\begin{array}{l} Max \ \ z - z^* = -b_1Y_1 - \cdots - b_mY_m + c_1Y_{m+1} + \cdots + c_nY_{m+n} + Y_{m+n+1} \\ s.t. \\ 0Y_1 + \cdots + 0Y_m + a_{11}Y_{m+1} + \cdots + a_{1n}Y_{m+n} - b_1Y_{m+n+1} \le 0 \\ ----------------------------- \\ 0Y_1 + \cdots + 0Y_m + a_{m1}Y_{m+1} + \cdots + a_{mn}Y_{m+n} - b_mY_{m+n+1} \le 0 \\ -a_{11}Y_1 - \cdots - a_{m1}Y_m + 0Y_{m+1} + \cdots + 0Y_{m+n} + c_1Y_{m+n+1} \le 0 \\ ----------------------------- \\ -a_{1n}Y_1 - \cdots - a_{mn}Y_m + 0Y_{m+1} + \cdots + 0Y_{m+n} + c_nY_{m+n+1} \le 0 \\ Y_1 + \cdots + Y_m + Y_{m+1} + \cdots + Y_{m+n} + Y_{m+n+1} = 1 \\ Y_1 \ge 0, \cdots, Y_m \ge 0, Y_{m+1} \ge 0, \cdots, Y_{m+n+1} \ge 0 \end{array}\right\} \tag{2.9}$$

Except for the inessential normalization condition

$$Y_1 + \cdots + Y_{m+n+1} = 1$$

and the condition $Y_{m+n+1} \ge 0$ rather than > 0, it is obvious that the game inequalities of (2.9) are identical with the super-linear programming problems inequalities of (2.7) provided we identify the $Y_{j's}$ with our previous *x's, y's,* and *v*. Thus,

$$Y_1 = vy_1, \cdots, Y_m = vy_m, Y_{m+1} = vx_1, \cdots, Y_{m+n} = vx_n, Y_{m+n+1} = v$$

This basic identity between the game and programming inequalities can be verified.

A procedure for solving our original linear-programming problem can now be given;

Using the a's, b's, and c's of the programming problem, set up the associated skew-symmetric game of the type (2.8).

By any known device, solve this game for an optimal mixed strategy $(Y_1, \ldots, Y_{m+n+1})$ or $(vy_1, \ldots, vx_1, \ldots, v)$, where $Y_{m+n+1} \neq 0$. If the original linear programming and its dual have optimal x's and y's that are all finite, then the final component $Y_{m+n+1} = v$ will never equal zero for any optimal strategy.

3.0 Numerical illustration of Conversion of LPP to Skew-Symmetric Game and its Computer Solution

Consider the LPP below;

$$\left.\begin{array}{l} \text{Maximize } z = 4x_1 + 3x_2 + 6x_3 \\ s.t \\ 3x_1 + x_2 + 3x_3 \leq 30 \\ 2x_1 + 2x_2 + 3x_3 \leq 40 \\ x_1, x_2, x_3 \geq 0 \end{array}\right\} \quad (3.1)$$

The dual of the LPP (3.1) can be written as;

$$\left.\begin{array}{l} \text{Minimize } z^* = 30y_1 + 40y_2 \\ s.t \\ 3y_1 + 2y_2 \geq 4 \\ y_1 + 2y_2 \geq 3 \\ 3y_1 + 3y_2 \geq 6 \\ \quad y_1, y_2, y_3 \geq 0 \end{array}\right\} \quad (3.2)$$

The optimal solutions to both the primal and its dual are as follows:

For the Primal:

$x_1 = 0, x_2 = 10, x_3 = 6.67, z = 70$

For the Dual:

$y_1 = 1, y_2 = 1, z^* = 70$

The next step in the conversion process is to convert the above LPP in system (3.2) to the maximization objective type so that we can easily combine the LPP and its dual to form the super LPP.

The dual LPP (3.2) in maximization type becomes.

$$\left.\begin{array}{l} \text{Maximize } -z^* = -30y_1 - 40y_2 \\ s.t \\ -3y_1 - 2y_2 \le -4 \\ -y_1 - 2y_2 \le -3 \\ -3y_1 - 3y_2 - \le -6 \\ \quad y_1, y_2 \ge 0 \end{array}\right\} \quad (3.3)$$

The super LPP formed by combining LPP (3.1) and its dual LPP(3.3) becomes

$$\left.\begin{array}{l} \text{Max } z - z^* = -30y_1 - 40y_2 + 4x_1 + 3x_2 + 6x_3 \\ \text{subject to} \\ 0y_1 + 0y_2 + 3x_1 + x_2 + 3x_3 \le 30 \\ 0y_1 + 0y_2 + 2x_1 + 2x_2 + 3x_3 \le 40 \\ -3y_1 - 2y_2 + 0x_1 + 0x_2 + 0x_3 \le -4 \\ \quad -y_1 - 2y_2 + 0x_1 + 0x_2 + 0x_3 \le -3 \\ -3y_1 - 3y_2 + 0x_1 + 0x_2 + 0x_3 \le -6 \\ y_1, y_2 \ge 0, x_1, x_2, x_3 \ge 0 \end{array}\right\} \quad (3.4)$$

For ease of computer usage, the above super LPP in system (3.4) can be recoded and rewritten as

$$\left.\begin{array}{l} \text{Max } z - z^* = -30\bar{x}_1 - 40\bar{x}_2 + 4\bar{x}_3 + 3\bar{x}_4 + 6\bar{x}_5 \\ \text{Subject to} \\ 0\bar{x}_1 + 0\bar{x}_2 + 3\bar{x}_3 + \bar{x}_4 + 3\bar{x}_5 \le 30 \\ 0\bar{x}_1 + 0\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4 + 3\bar{x}_5 \le 40 \\ -3\bar{x}_1 - 2\bar{x}_2 + 0\bar{x}_3 + 0\bar{x}_4 + 0\bar{x}_5 \le -4 \\ \quad -\bar{x}_1 - 2\bar{x}_2 + 0\bar{x}_3 + 0\bar{x}_4 + 0\bar{x}_5 \le -3 \\ -3\bar{x}_1 - 3\bar{x}_2 + 0\bar{x}_3 + 0\bar{x}_4 + 0\bar{x}_5 \le -6 \\ \qquad \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5 \ge 0 \end{array}\right\} \quad (3.5)$$

Note: $\bar{x}_1 = y_1, \bar{x}_2 = y_2, \bar{x}_3 = x_1, \bar{x}_4 = x_2$ *and* $\bar{x}_5 = x_3$ are the decision variables

The optimal solution to the super LPP (3.5) is obtained and stated below:

The optimal solutions to the super LPP as obtained using the Program Simplex can be stated in terms of the decision variables as follows;

$$y_1(=\bar{x}_1) = 0, y_2(=\bar{x}_2) = 0, x_1(=\bar{x}_3) = 0, x_2(=\bar{x}_4) = 10, x_3(=\bar{x}_5) = 6.67 \text{ and } z = 70$$

The super LPP (3.4) can be stated in the form below by multiplying the constraints including the right-hand, by a positive constant say v, and transferring the right-hand coefficients of the constraints over to the left hand side. The reason for this transformation is to get the correct structure of a LP problem resulting from a game payoff matrix in which the R.H.S. values of the inequality constraints are equal to the value of the game, in this case z-z*=0. We let $Y_1 = vy_1, Y_2 = vy_2, Y_3 = vx_1 \; and \; Y_4 = vx_2, Y_5 = vx_3, Y_6 = v$

The super LPP in system (3.4) can be further rewritten as:

$$\left.\begin{array}{l} Maximize\; R = -30Y_1 - 40Y_2 + 4Y_3 + 3Y_4 + 6Y_5 \\ s.t \\ 0Y_1 + 0Y_2 + 3Y_3 + Y_4 + 3Y_5 - 30Y_6 \le 0 \\ 0Y_1 + 0Y_2 + 2Y_3 + 2Y_4 + 3Y_5 - 40Y_6 \le 0 \\ -3Y_1 - 2Y_2 + 0Y_3 + 0Y_4 + 0Y_5 + 4Y_6 \le 0 \\ -Y_1 - 2Y_2 + 0Y_3 + 0Y_4 + 0Y_5 + 3Y_6 \le 0 \\ -3Y_1 - 3Y_2 + 0Y_3 + 0Y_4 + 0Y_5 + 6Y_6 \le 0 \\ \qquad Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 = 1 \\ \qquad Y_1, Y_2, Y_3, Y_4, Y_5 \ge 0 \text{ and } Y_6 > 0 \end{array}\right\} \qquad (3.6)$$

It should b[2]e noted that an additional constraint, $Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 = 1$ is added to form the super LPP (3.6)

Using Program FullSimplex, the optimal solution to the system (3.6) is thus obtained:

$$Y_1 = 0.05, Y_2 = 0.05, Y_3 = 0, Y_4 = 0.51$$

$$Y_5 = 0.34, Y_6 = 0.05 = v$$

$$\therefore y_1 = \frac{Y_1}{v} = \frac{0.05}{0.05} = 1, \; y_2 = \frac{Y_2}{0.05} = 1, \; x_1 = \frac{Y_3}{0.05} = 0$$

$, x_2 = \frac{Y_4}{0.05} = 10.2, \; x_3 = \frac{Y_5}{0.05} = 6.8$ and objective function value is 0.07.

The coefficients of the LP problem can be written using matrices of different dimensions as follows:

0	A	-b	0
$-a^T$	0	C	0
B	-c	0	0
1	1	1	1

Figure 1

The combined payoff matrix enclosed in the bolded line constitute the skew symmetric game problem.

4.0 Discussion of Results

In this paper, the researchers carried out the conversion of LPP to skew-symmetric game through the formation of a super LPP. In our attempt to transform LPP into a game we first examined the structure of an LPP that results from the transformation of a game, that is, when a game is transformed into an LPP. The LP model is supposed to have a certain structure. One of which is that the LHS must have columns of variables and the RHS must have only one constant value. In our attempt to convert LPP to game problem, we took this structure into consideration. The first appearance of our super LPP in system (3.4) did not fit into that structure. We then went further to multiply the super LPP (3.4) by a positive constant say v, and this enables us to move the RHS to the left. The values of the RHS which were different then became zero. This process gave us the confidence of identifying the skew- symmetric matrix game which was given a general representation in Figure 1.

5.0 Conclusion

So far, researchers have considered the reverse process of converting from linear programming to game problem. The researchers converted LPP to skew-symmetric game. The super LPP of the skew-symmetric game is sparse and thus makes solving by computer easier.

6.0 References

Dorfman, R., Samuelson A. Paul, Solow M. Robert (2012). *Linear Programming and Economic Analysis*: Courier Corporation.

Ekoko, P.O. (2004). *Expanding the Scope of Usage of Computer in Solving Practical LP Problems:* Journal of Institute of Maths and Computer Sciences (Comp. Sc. Ser.) Vol. 15, No. 1, pp. 65-72.

Felix Brandt, Felix Fischer and Markus Holzer. (2009). *Symmetries and the complexity of pure Nash equilibrium. Journal of Computer and System Sciences Vol. 75, Issue 3*, pp 163-177.

Gale, D. and Stewart, F.M. (1951), *Infinite Games with Perfect Information*: Oxford University Press.

Kim, D.S, Lee, J.W (1999).*Generalized Symmetric Duality for Multiobjective Variational Problems With Invexity*. *Journal of Mathematical Analysis and Applications*. Vol. 234 pp. 147-164.

Kim, D.S, Lee, J.W (1998). *Symmetric Duality for Multiobjective Variational Problems With Invexity*. *Journal of Mathematical Analysis and Applications.* Vol. 218, pp. 34- 48

Kim, D.S, Lee, G.M and Lee, J.W.(1997). *Symmetric Duality for Multiobjective Variational Problems with Pseudo-Invexity, in: Nonlinear Analysis and Convex Analysis*. Vol.1, pp. 106-117, Kyoto University.

Leigh Metcalf and William Casey. (2016). *Cyber security and Applied Mathematics*: pp 95-112 retrieved from https://www.sciencedirect.com/science/article/pii/B9780128044520000063

Okafor, Folakemi. M, Adiri, Emmanuel I. and Adetunji K.I (2018). ABACUS, Journal of the Mathematical Association of Nigeria. (Mathematics Science Series). Vol. 45, No. 1 pp. 130-140.