ONE STEP HYBRID BLOCK METHOD FOR THE SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Yunusa, S.,[1] Mohammed, G.A.,[1] Etuk, E.D.,[1] Aliyu, Y.A.[1] and Garba, U.[2]
[1]*Mathematics Department Federal College of Education (Tech), Gombe, Gombe State*
[2]*Mathematics Departmen College of Education, Billiri, Gombe State*
Correspondence author email: samyunusa@gmail.com

Abstract
In this paper, a new k-step hybrid block method was derived for the solution of first order initial value problems (IVP) of ordinary differential equations (ODEs). A continuous linear multistep method (CLMM) with variable coefficients was developed using interpolation and collocation of a polynomial approximate solution. This CLMM was evaluated at some selected off-grids points which give a class of discrete linear multistep methods (DLMMs) and was implemented as a block method. Investigations on the properties of the method such as, order, zero-stability, consistency were carried out and the results indicated that the method were of order three, A-stable and convergent. MATLAB codes were written to test the numerical performance of the block method on some linear and non-stiff IVPs of ODEs and the results showed that the one step hybrid block method (HBM) compared favorably with the existing method.

Keywords: *One step, interpolation, collocation, hybrid and block method.*

1. Introduction

We consider a numerical method for solving general first order initial value problems (IVPs) of ordinary differential equations (ODEs) of the form

$$y' = f(x, y), y(\chi_0) = y_0 \qquad (1.1)$$

whose solution is sought in the range $a \leq x \leq b,$ where a and b are finite, and f is a continuous function and satisfies Lipschitz condition for the existence and uniqueness of solution. Problems in the form (1.1) have wide application in engineering, physical sciences, medicine, molecular dynamics, quantum chemistry astrophysics, electronics and semi-discretization of wave equation etc. Linear multistep methods (LMMs) are very prevalent for solving IVPs of ODEs. They are also applicable to solving higher order ODEs. Generally, LMMs are not self-starting especially when the step number $k > 1$ hence, need starting values from single-step methods like Euler's method and Runge-Kutta family of methods. Therefore, numerical scheme was developed to solve problem (1.1) in tune with those developed by Ayinde and Ibijola (2015), Fatunla (1976), Fatunla (1988), Ibijola (1997) and Ogunrinde *et al* (2012).

2.0 Methodology

2.1 Derivation of One-Step Hybrid Block Method (HBM)

We consider the approximate solution in the form

$$y\left(x\right)=\sum_{j=0}^{m+t-1}a_jx^j \qquad (2.1)$$

with first derivative given as

$$y^{'}\left(x\right)=\sum_{j=1}^{m+t-1}ja_jx^{j-1} \qquad (2.2)$$

Substituting (2.2) into (1.1) gives

$$f\left(x,y\right)=\sum_{j=1}^{m+t-1}ja_jx^{j-1} \qquad (2.3)$$

where a_j s' are the parameters to be determined. In this method we interpolate (2.1) and collocate (2.2) at the same points $x_{n+j}, j=0,1$ and the continuous hybrid linear multistep method (CHLMM) reduces to,

$$y(x)=\left[\alpha_0\left(x\right)y_n+\alpha_1\left(x\right)y_{n+1}+\beta_0\left(x\right)f_n+\beta_1\left(x\right)f_{n+1}\right] \qquad (2.4)$$

And the D matrix becomes

$$D=\begin{bmatrix}1 & x_n & x_n^2 & x_n^3\\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3\\ 0 & 1 & 2x_n & 3x_n^2\\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2\end{bmatrix}$$

Using Maple 18 software to perform some algebraic manipulations to determine the values of a's we obtained

$$y(\xi)=\left[(2\xi+1)(\xi-1)^2\right]y_n+\left[-\xi^2(2\xi-3)\right]y_{n+1}+\left[\xi(\xi-1)^2\right]f_n+\left[\xi^2(\xi-1)\right]f_{n+1}$$

where $\xi=\frac{x-x_n}{h}$, evaluating the continuous hybrid scheme at $x=x_{n+j}, j=\frac{1}{2}$ and the first derivative at $x=x_{n+j}, j=\frac{1}{2}$ gives

$$y_{n+\frac{1}{2}}=y_n+h\left[\frac{5}{24}f_n+\frac{1}{3}f_{n+\frac{1}{2}}-\frac{1}{24}f_{n+1}\right] \qquad (2.5)$$

$$y_{n+1}=y_n+h\left[\frac{1}{6}f_n+\frac{2}{3}f_{n+\frac{1}{2}}+\frac{1}{6}f_{n+1}\right] \qquad (2.6)$$

Writing equations (2.5) - (2.6) in block form gives,

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_{n+\frac{1}{2}}\\ y_{n+1}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_{n+\frac{1}{2}}\\ y_n\end{pmatrix}$$

$$+h\begin{pmatrix} 0 & \frac{5}{24} \\ 0 & \frac{1}{6} \end{pmatrix}\begin{pmatrix} f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}+h\begin{pmatrix} \frac{1}{3} & -\frac{1}{24} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix} \qquad (2.7)$$

Equation (2.7) above is the One Step Hybrid Block Method (HBM). (2.7) was obtained when the two schemes were derived and put them in a block form, then $\left[A^{(1)}\right]^{-1}\left[A^{(1)}\right]^{-1}=A^{(1)}$, $\left[A^{(1)}\right]^{-1}\left[A^{(0)}\right]=A^{(0)}$, $\left[A^{(1)}\right]^{-1}\left[\beta^{(1)}\right]=\beta^{(1)}$, $\left[A^{(1)}\right]^{-1}\left[\beta^{(0)}\right]=\beta^{(0)}$, then we have the normalized block method (2.7) and (2.5), (2.6) was obtained from (2.7).

3.0 Analysis of the New Method

In this section, the order and error constant, consistency, zero-stability, linearstabilityand region of absolute stability of the block are obtained. The result obtained in (2.7) gives the hybrid block Method for $k=1$.

3.1 *Order and error constant of HBM*

Expanding the first row of (2.5) and (2.6) in Taylor series, the method HBM is of order $p=3>1$ and $p=4>1$ with the following error constants $\frac{1}{384}, and -\frac{1}{2880}$ respectively.

3.2 *Consistency of the HBM*

The block method (LMM) is consistent since it has order $p=3>1$.

3.3 *Zero stability of the HBM*

The method (2.7) is said to be zero stable if the roots $z_s, s=1,2,3,...n$ of the first characteristics polynomial $\bar{\rho}(\lambda)$ defined by $\bar{\rho}(\lambda)=\det\left[\lambda A^{(1)}-A^{(0)}\right]=0$

where

$$A^{(1)}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta^{(1)}=\begin{pmatrix} \frac{1}{3} & -\frac{1}{24} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix},$$

$$A^{(0)}=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \beta^{(0)}=\begin{pmatrix} 0 & \frac{5}{24} \\ 0 & \frac{1}{6} \end{pmatrix}$$

ZERO STABILITY

$$\bar{\rho}(\lambda)=\det\left[\lambda A^{(1)}-A^{(0)}\right]=0$$

$$\left(\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right) = \begin{pmatrix}\lambda & -1\\ 0 & \lambda-1\end{pmatrix},$$

Determinant: $\lambda^2 - \lambda = \lambda(\lambda - 1)$

Therefore, $\lambda = 0$, $(\lambda - 1) = 0$,

$\lambda_1 = 0$, $\lambda_2 = 1$

Thus, the HBM is zero stable and since it is consistent with order $p = 3 > 1$, hence by Henrici(1962) the method is convergent.

3.4 Linear Stability of the HBM

$$M(z) = -\left[(A^{(1)} - zB^{(1)})^{-1}(A^{(0)} + zB^{(0)})\right]$$

$$\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - z\begin{pmatrix}\frac{1}{3} & -\frac{1}{24}\\ \frac{2}{3} & \frac{1}{6}\end{pmatrix}\right)^{-1}\left(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} + z\begin{pmatrix}0 & \frac{5}{24}\\ 0 & \frac{1}{6}\end{pmatrix}\right) =$$

$$\begin{pmatrix}0 & -z\dfrac{\frac{1}{6}z+1}{2z^2-12z+24} - (z-12)\dfrac{\frac{5}{24}z+1}{z^2-6z+12}\\ 0 & 8z\dfrac{\frac{5}{24}z+1}{z^2-6z+12} - 4\left(\frac{1}{6}z+1\right)\dfrac{z-3}{z^2-6z+12}\end{pmatrix}$$

eigenvalues: $\dfrac{1}{z^2-6z+12}(\lambda^2+6z+12), 0$

3.5 Region of Absolute Stability of the HBM.

The stability polynomial is given by $-z\left(\frac{2}{3}w^2 + \frac{1}{3}w\right) + w^2 - w$
which is plotted using MATLAB $R2015a$ software and the absolute stability region of the HBM is shown in fig.1.

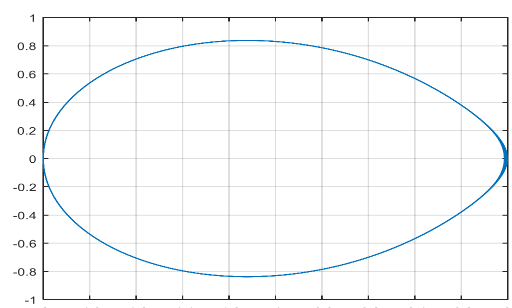

Figure 1: Region of Absolute Stability of HBM

4.0 Numerical Experiments

The following numerical experiments are performed with the aid of MATLAB $R2015a$ software package in order to further affirm the earlier established convergence of the Absolute Error of One Step Hybrid Block method (HBM)

Example 1. Solve $y' = y, y(0) = 1, h = 0.1, 0 \le x \le 1$

Exact solution $y(x) = e^x$

Table 1: Numerical Result of Example 1 of HBM

X_n	Exact Solution	Computed Result
0.1	1.105170918075648	1.105170902716915
0.2	1.221402758160170	1.221402724212121
0.3	1.349858807576003	1.349858751298409
0.4	1.491824697641270	1.491824614712790
0.5	1.648721270700128	1.648721156137448
0.6	1.822118800390509	1.822118648456899
0.7	2.013752707470477	2.013752511572436
0.8	2.225540928492468	2.225540681062964
0.9	2.459603111156950	2.459602803523574
1.0	2.718281828459046	2.718281450695203

Table 2: Comparison of Absolute Error of Example 1

X_n	Absolute error	Ayinde et al 2015
0.1	1.535873273006416e-08	1.226221039551945e-05
0.2	3.394804903855686e-08	1.355183832019158e-05
0.3	5.627759436244162e-08	1.497709759790133e-05
0.4	8.292848030500011e-08	1.655225270247307e-05
0.5	1.145626802312449e-07	1.829306831546695e-05
0.6	1.519336099153890e-07	2.021696710463594e-05
0.7	1.958980404559441e-07	2.234320409577606e-05
0.8	2.474295039966989e-07	2.469305938346267e-05
0.9	3.076333761065087e-07	2.729005110868599e-05
1.0	3.777638424296015e-07	3.016017083767864e-05

Example 2. Solve $y' = x^2 + y, y(0) = 1, 0 \le x \le 1$

Theoretical solution $y(x) = -2 - 2x - x^2 + 3e^x, h = 0.1$

Table 3: Numerical Result of Example 2 of HBM

X_n	Exact Solution	Computed Result

0.1	1.105512754226943	1.105512708150745
0.2	1.224208274480510	1.224208172636362
0.3	1.359576422728009	1.359576253895226
0.4	1.515474092923811	1.515473844138370
0.5	1.696163812100385	1.696163468412344
0.6	1.906356401171527	1.906355945370697
0.7	2.151258122411429	2.151257534717308
0.8	2.436622785477404	2.436622043188892
0.9	2.768809333470850	2.768808410570721
1.0	3.154845485377138	3.154844352085609

Table 4: Comparison of Absolute Error of Example 2

X_n	Absolute error	Ayinde et al 2015
0.1	4.607619819019249e-08	2.452442079081685e-05
0.2	1.018441475597598e-07	2.710367664016111e-05
0.3	1.688327833093695e-07	2.995419519646880e-05
0.4	2.487854413590895e-07	3.310450540472409e-05
0.5	3.436880409157794e-07	3.658613663071186e-05
0.6	4.558008297461669e-07	4.043393420927188e-05
0.7	5.876941204796538e-07	4.468640819110803e-05
0.8	7.422885115460076e-07	4.938611876692534e-05
0.9	9.229001287636152e-07	5.458010221648380e-05
1.0	1.133291528621072e-06	6.032034167668954e-05

Example 3. Solve $y' = 2xy,\ y(0) = 1,\ 0 \le x \le 1$

Theoretical solution $y(x) = e^{x^2},\ h = 0.1$

Table 5: Numerical Result of Example 3 of HBM

X_n	Exact Solution	Computed Result
0.1	1.010050167084168	1.010050166943508
0.2	1.040810774192388	1.040810770102466
0.3	1.094174283705210	1.094174259898869
0.4	1.173510870991810	1.173510786698273
0.5	1.284025416687741	1.284025183607879
0.6	1.433329414560340	1.433328856618014
0.7	1.632316219955379	1.632315000678282
0.8	1.896480879304952	1.896478370294535
0.9	2.247907986676472	2.247903028542661
1.0	2.718281828459046	2.718272294360717

Table 6: Comparison of Absolute Error of Example 3

X_n	Absolute error	Ayinde et al 2015

0.1	1.406603722386990e-10	0.189949832915832
0.2	4.089922400751789e-09	0.171452718345096
0.3	2.380634156473604e-08	0.155641862311982
0.4	8.429353703931497e-08	0.141505280153339
0.5	2.330798622995900e-07	0.128038189693506
0.6	5.579423261181660e-07	0.114124930711879
0.7	1.219277097064264e-06	0.098392007074560
0.8	2.509010416540392e-06	0.079005915228979
0.9	4.958133810628596e-06	0.053376460234161
1.0	9.534098329044838e-06	0.017703811500527

Example 4. Solve $y' = 2xy + 4x,\ y(0) = 1,\ 0 \le x \le 1$

Theoretical solution $y(x) = 3e^{x^2} - 2,\ h = 0.1$

Table 7: Numerical Result of Example 4 of HBM

X_n	Exact Solution	Computed Result
0.1	1.030150501252504	1.030150500830523
0.2	1.122432322577165	1.122432310307397
0.3	1.282522851115632	1.282522779696607
0.4	1.520532612975431	1.520532360094820
0.5	1.852076250063224	1.852075550823638
0.6	2.299988243681021	2.299986569854043
0.7	2.896948659866137	2.896945002034846
0.8	3.689442637914855	3.689435110883605
0.9	4.743723960029415	4.743709085627982
1.0	6.154845485377138	6.154816883082148

Table 8: Comparison of Absolute Error of Example 4

X_n	Absolute error	Ayinde et al 2015
0.1	4.219808946714920e-10	0.189949832915832
0.2	1.226976742429997e-08	0.171452718345096
0.3	7.141902491625274e-08	0.155641862311982
0.4	2.528806108959003e-07	0.141505280153338
0.5	6.992395866767254e-07	0.128038189693506
0.6	1.673826978354498e-06	0.114124930711878
0.7	3.657831291192792e-06	0.098392007074561
0.8	7.527031250287308e-06	0.079005915228979
0.9	1.487440143321805e-05	0.053376460234161
1.0	2.860229498935496e-05	0.017703811500527

4.1 Discussion of Result

The results of the numerical examples for HBM in this research were compared with that of Ayinde and Ibijola (2015).

We consider one step hybrid block method (HBM) incorporating one off-grid point at step size $h = 0.1$. The convergence, consistency, order and error constant, and region of absolute stability of the block method including its hybrid forms have been determined. The region of absolute stability displayed in figure 1 shows that the method is A-stable, HBM is of order $p \geq 3\ and > 4$. We considered three numerical examples to test the efficiency and accuracy of our method. Table 2 shows the Error constant of HBM compared with that of Ayinde and Ibijola (2015). From the absolute error it shows that our HBM gives a better approximation than that of the existing methods for the solution of ordinary differential equation. For example, 1.535873273006416e-08 for HBM when compared with 1.226221039551945e-05 for Ayinde et al (2015) we discovered that the absolute error of our method is minimal compared with their own.

Furthermore, from the result obtained in example 2 and 3, it is evident that our method HBM performed better than that of Ayinde et al (2015). Thus, the method is consistent, convergent and zero stable, it also has large stability region, A-stable, making it able to cope effectively with non-stiff problems. Hence the method derived is efficient and computationally reliable.

4.2 Summary and Conclusion

In this research, we derived one step one off grid point hybrid block method for the solution of initial value problems of ordinary differential equations. This was achieved because of the good stability properties of the new block method which performed accurately well and is efficient compared to some existing methods.

An order three hybrid block method with one off grid point was derived and implemented as a self-starting method for ordinary differential equation.

The order and error constant, consistency, zero stability and region of absolute stability of the block method including its hybrid form was determined for HBM. Since the stability regions encroaches into the left half of the complex Z-plane shows that the block method is A-stable.

We considered three numerical examples to test the efficiency and accuracy of our methods. Table 2 shows the Error constant of HBM compared with that of Ayinde $et\ al$ (2015), from the absolute error it shows that our HBM gives a better approximation than that of the existing method for the solution of ordinary differential equation. Thus, the method is consistent, convergent and zero stable, they also have large stability region. The method is A-stable, making them able to cope effectively with stiff and non-stiff problems. Hence the methods derived are efficient and computationally reliable.

References

Abdullahi, A., Chollom, J. P & Yahaya, Y. A. (2014). A family of Two-Step block generalized Adams Methods for the solution of Non-Stiff IVP in ODEs. *Journal of Mathematical Association of Nigeria (ABACUS), 41(2A):* 67-77.

Aboiyar, T., Luga, T. &Iyorter, B. V. (2015). Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions. *American Journal of Applied Mathematics and Statistics, Vol. 3, No. 6,* 220-225.

Adesanya, A. O., Odekunle, M. R., Alkali, M. A., & Abubakar, A. B. (2013). Starting the five steps Stomer-Cowell method by Adams-Bashforth method for the solution of the first order ordinary differential equations. *African Journal of Mathematics and Computer Science Research, Vol. 6(**5**),* pp. 89-93.

Ayinde & Ibijola (2015). A new numerical method for solving first order differential equations. *American Journal of Applied Mathematics and Statistics, Vol. 3, No. 4,* 156-160.

Fatunla, S. O. (1976). A New Algorithm for the Numerical Solution of ODEs. Computers and Mathematics with Applications, 2, 247-253.

Fatunla, S. O. (1988). Numerical Methods for initial Value Problems in Ordinary Differential Equations, Academic Press, San Diego, U. Ibijola, E. A., (1997). New Schemes for Numerical Integration of Special Initial Value Problems in Ordinary Differential Equations. Ph.d Thesis, University of Benin, Nigeria.

Ogunrinde, R.B., Fadugba, S.E., and Okunlola J. T. (2012). On some Numerical methods for solving initial value problem in ODEs. IOSR Journal of Mathematics (IOSRJM). 1(3), 25-31.

Lambert, J. D. (1991). Numerical methods for Ordinary Differential Systems: the initial value problem. John Wiley & Sons, Inc., New York.