RETURN OF CONTRIBUTION CLAUSE IN A DC PLAN UNDER MODIFIED CEV MODEL

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Abstract
In this research paper, a defined contribution (DC) pension plan member's optimal portfolio strategy with return of contribution clause under modified constant elasticity of variance (M-CEV) is studied. Considering investment in a risk- free asset and a risky asset modeled by a M-CEV process, a continuous time mean-variance stochastic optimal control problem consisting of members' monthly contributions, returned contributions and invested funds is formulated. Using the game theoretic method and mean variance utility, a non-linear partial differential equation (PDE) called the extended Hamilton Jacobi Bellman (HJB) is established and solved for the optimal portfolio strategy and efficient frontier using change of variable and variable separation technique. Also, theoretical analyses of the impact of the modification parameter and some other sensitive parameters on the optimal portfolio strategy were studied. Moreover, our result generalizes some existing results.

***Keywords**: Modified constant elasticity of variance, extended HJB equation, optimal portfolio strategy, return of contribution, mean variance utility, change of variable technique.*

1. Introduction

The modified constant elasticity of variance process is an improved stochastic volatility model and an extension of the CEV process. It was first developed and used by Health and Platen (2002). Considering the unstable nature of the financial market especially with the ravaging effect of the deadly corona virus pandemic which has destabilized most countries economy and businesses, the need to choose a stochastic volatility model that best fit the volatile nature of the financial market is of necessity. Examples of such volatility models include the Heston's volatility, M-CEV model etc. Some of the great features of the M-CEV model is that it captures the volatility smile effects of the stock price, it's probability can touch zero unlike Geometric Brownian Motion (GBM) which always positive, it enhance analytically tractable strategies etc.

The optimal portfolio strategy is an important aspect of study in financial mathematics and has received diverse attention from several researchers in academics and financial institutions all over the world. In (Health and Platen, 2002), the M-CEV process was used to develop a hedging and consistent pricing Process; here they considered a modified CEV model by introducing a modification parameter and proved that there exist no corresponding risk or neutral pricing measure and therefore, the classical risk neutral pricing methodology fails. Furthermore, they used the bench mark method to establish a consistent pricing and hedging framework. They also showed that nonnegative price process and benchmark duplicate the contingent claim. (Muravey, 2017) investigated optimal portfolio strategy with M-CEV model and solve for the explicit solutions of the optimal strategy in terms of confluent hyper-geometric functions using the Laplace transforms and also application to the algorithmic tradition. (Ihedioha, 2020) studied the optimal investment problem for an investor under M-CEV and Ornstein-Uhlenbeck models; he showed that the investor's optimal portfolio strategy when the Brownian motions (BM) do not correlate is less than the optimal portfolio strategy when the Brownian motions correlate.

In Recent times, the optimal portfolio strategy with return of contributions clause have been investigated by many authors. In He and Liang (2013) who studied the optimal portfolio strategy for a DC plan with return clause under mean-variance utility was studied. In (Sheng and Rong, 2014) the optimal investment problem was studied for both the accumulation and distribution phases. In their work, the risky asset followed the Heston volatility model. (Chai *et al.*, 2017) studied portfolio selection problem with return clause where the risky asset was modeled by a jump diffusion process. They also studied the effect of the jump diffusion process on the strategy and value function. (Akpanibah *et al.*, 2019) extended the work of (He and Liang, 2013) by considering investment in a risk free and two risky assets and solved for the investment strategy of the three assets. (Wang *et al.*, 2018) investigated the same problem by considering investment in one risk free asset, stock and inflation index bond where the stock market price was modelled by Heston volatility process. The optimal portfolio strategy in a pension scheme with a risk free and one risky asset was studied by (Akpanibah and Osu, 2019) when the returned contributions are with accumulated interest from the risk free asset and the price of the risky asset modeled by GBM. Also, (Akpanibah *et al.*, 2020) studied the the optimal allocation strategy for a DC plan with return of contributions with predetermined interest under Heston's volatility model.

The optimal portfolio strategy under the CEV model have been studied in (Li *et al.*, 2017; Osu *et al.*, 2018) under different assumptions. In (Li *et al.*, 2017), the equilibrium strategy in a DC plan with default risk and return of contribution clause under CEV process was studied; the stock market price was modeled by CEV process and investment in three different assets while in (Osu *et al.*, 2018) the optimal portfolio strategy was studied for a DC plan with multiple contributors and the stock market price was modeled by CEV process.

From the available literatures and to the best of our knowledge, no work has been published that studied the optimal portfolio strategy of a DC pension plan member where the stock market price is modeled by modified constant elasticity of variance (M-CEV) model and this form the basis of this research.

2. Mathematical model of the Financial Market

We assume that the market is made up of risk-free asset (cash) and risky asset (stock). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space where Ω is a real space, $\mathcal{P}$ is a probability measure and $\mathcal{F}$ is the filtration which represents information generated by the standard Brownian motion $\mathcal{Z}_t(t)$.

Let $\mathcal{S}_t(t)$ be the price risky asset (stock) whose price process is modeled by the modified constant elasticity of variance (MCEV) (Ihedioha, 2020) and is driven by the stochastic differential equation as follows

$$d\mathcal{S}_t(t) = (\eta \mathcal{S}_t + \kappa \xi^2 \mathcal{S}_t^{2\lambda+1})dt + \xi \mathcal{S}_t^{\lambda+1} d\mathcal{Z}_t. \quad (2.1)$$

where η is the instantaneous expected rate of return for the risky asset, $\kappa > 0$, is the modification factor, $\xi$is the instantaneous volatility, and λ, the elasticity parameter and satisfies the condition$\lambda < 0$. If $\kappa = 0$, then the stochastic differential equation in (2.2) reduces to that of a CEV process (Gao, 2009; Akpanibah and Ini, 2020).

Let $\mathcal{C}_t(t)$ be the price of the risk-free asset (bank security) and is given as follows

$$d\mathcal{C}_t(t) = \mathcal{R}\mathcal{C}_t dt, \tag{2.2}$$

where $\mathcal{R} > 0$ is the risk free interest rate.

Assume the fraction of the wealth invested in stock is ψ_1 and the fraction for the bank security is ψ_0 such that $\psi_0 = 1 - \psi_1$. Let the contribution rate at any time be m , the initial age of accumulation phase n_0, T is the accumulation period such that $n_0 + T$ is the end age. The actuarial symbol $\Delta_{\frac{1}{i},n_0+t}$is the mortality rate from$t$ to $t + \frac{1}{i}$, tm is the premium accumulated at time t, $tm\Delta_{\frac{1}{i},n_0+t}$ is the returned premium at time $t + \frac{1}{i}$during the accumulation phase. If we take into consideration the time interval of the accumulation phase $[t, t + \frac{1}{i}]$, the differential form of the member's wealth can be formulated thus:

$$\mathcal{A}\left(t + \frac{1}{i}\right) = \left[\mathcal{A}(t)\left(\psi_1 \frac{\mathcal{S}_{t+\frac{1}{i}}}{\mathcal{S}_t} + \psi_0 \frac{\mathcal{C}_{t+\frac{1}{i}}}{\mathcal{C}_t}\right) + m\frac{1}{i} - tm\delta_{\frac{1}{i}n_0+t}\right]\left(\frac{1}{1-\Delta_{\frac{1}{i},n_0+t}}\right) \tag{2.3}$$

$$\mathcal{A}\left(t + \frac{1}{i}\right) = \left[\begin{array}{c}\mathcal{A}(t)\left(\psi_1\left(\frac{\mathcal{S}_{t+\frac{1}{i}}}{\mathcal{S}_t} - \frac{\mathcal{S}_t}{\mathcal{S}_t} + \frac{\mathcal{S}_t}{\mathcal{S}_t}\right) + (1-\psi_1)\left(\frac{\mathcal{C}_{t+\frac{1}{i}}}{\mathcal{C}_t} - \frac{\mathcal{C}_t}{\mathcal{C}_t} + \frac{\mathcal{C}_t}{\mathcal{C}_t}\right)\right) \\ +m\frac{1}{i} - tm\delta_{\frac{1}{i}n_0+t}\end{array}\right]\left(1 + \frac{\Delta_{\frac{1}{i},n_0+t}}{1-\Delta_{\frac{1}{i},n_0+t}}\right) \tag{2.4}$$

$$\mathcal{A}\left(t + \frac{1}{i}\right) = \left[\begin{array}{c}\mathcal{A}(t)\left(\psi_1 + 1 - \psi_1 + \psi_1\left(\frac{\mathcal{S}_{t+\frac{1}{i}}}{\mathcal{S}_t} - \frac{\mathcal{S}_t}{\mathcal{S}_t}\right) + (1-\psi_1)\left(\frac{\mathcal{C}_{t+\frac{1}{i}}}{\mathcal{C}_t} - \frac{\mathcal{C}_t}{\mathcal{C}_t}\right)\right) \\ +m\frac{1}{i} - tm\delta_{\frac{1}{i}n_0+t}\end{array}\right]\left(1 + \frac{\Delta_{\frac{1}{i},n_0+t}}{1-\Delta_{\frac{1}{i},n_0+t}}\right) \tag{2.5}$$

$$\mathcal{A}\left(t + \frac{1}{i}\right) - \mathcal{A}(t) = \left[\begin{array}{c}\mathcal{A}(t)\left(\psi_1\left(\frac{\mathcal{S}_{t+\frac{1}{i}} - \mathcal{S}_t}{\mathcal{S}_t}\right) + (1-\psi_1)\left(\frac{\mathcal{C}_{t+\frac{1}{i}} - \mathcal{C}_t}{\mathcal{C}_t}\right)\right) \\ +m\frac{1}{i} - tm\delta_{\frac{1}{i}n_0+t}\end{array}\right]\left(1 + \frac{\Delta_{\frac{1}{i},n_0+t}}{1-\Delta_{\frac{1}{i},n_0+t}}\right) \tag{2.6}$$

$$\begin{cases} \Delta_{\frac{1}{i},n_0+t} = 1 - \exp\{-\int_0^{\frac{1}{i}} \zeta(n_0 + t + \varsigma)d\varsigma\} = \zeta(n_0+t)\frac{1}{i} + O(\frac{1}{i}), \frac{\Delta_{\frac{1}{i},n_0+t}}{1-\Delta_{\frac{1}{i},n_0+t}} = \zeta(n_0+t)\frac{1}{i} + O(\frac{1}{i}) \\ \frac{1}{i} \to \infty, \Delta_{\frac{1}{i},n_0+t} = \zeta(n_0+t)dt, \frac{\Delta_{\frac{1}{i},n_0+t}}{1-\Delta_{\frac{1}{i},n_0+t}} = \zeta(n_0+t)dt, m\frac{1}{i} \to mdt, \frac{\mathcal{S}_{t+\frac{1}{i}} - \mathcal{S}_t}{\mathcal{S}_t} \to \frac{d\mathcal{S}_t}{\mathcal{S}_t}, \frac{\mathcal{C}_{t+\frac{1}{i}} - \mathcal{C}_t}{\mathcal{C}_t} \to \frac{d\mathcal{C}_t}{\mathcal{C}_t} \end{cases} \tag{2.7}$$

Substituting (2.7) into (2.6) we have

$$d\mathcal{A}(t) = \left[\mathcal{A}(t)\left(\psi_1\frac{d\mathcal{S}_t}{\mathcal{S}_t} + (1-\psi_1)\frac{d\mathcal{C}_t}{\mathcal{C}_t}\right) + mdt - tm\zeta(n_0+t)dt\right](1 + \zeta(n_0+t)dt) \tag{2.8}$$

Substituting (2.1) and (2.2) into (2.8), we have

$$d\mathcal{A}(t)=\begin{bmatrix}\mathcal{A}(t)\left(\psi_1\left((\eta+\kappa\xi^2\mathcal{S}_t^{2\lambda})dt+\xi\mathcal{S}_t^{\lambda}d\mathcal{Z}_t\right)+(1-\psi_1)\mathcal{R}dt\right)\\ +m\left(1-t\zeta(n_0+t)\right)dt\end{bmatrix} \tag{2.9}$$

Since $\zeta(t)$is the force function and ω is the maximal age of the life table. From [12] The force function is given as

$$\zeta(t)=\frac{1}{n-t}\ 0\leq t<n\ . \tag{2.10}$$

This implies that

$$\zeta(n_0+t)=\frac{1}{n-n_0-t} \tag{2.11}$$

Substituting (2.11) into (2.9) and simplifying it, we have

$$d\mathcal{A}(t)=\left\{\begin{matrix}\begin{bmatrix}\mathcal{A}(t)\left(\left(\eta+\kappa\xi^2\mathcal{S}_t^{2\lambda}-\mathcal{R}\right)\psi_1+\mathcal{R}+\frac{1}{n-n_0-t}\right)\\ +m\left(\frac{n-n_0-2t}{n-n_0-t}\right)\end{bmatrix}dt+\psi_1\mathcal{A}(t)\xi\mathcal{S}_t^{\lambda}d\mathcal{Z}_t\\ \mathcal{A}(0)=a_0\end{matrix}\right\} \tag{2.12}$$

3. Extended Hamilton Jacobi Bellman Equation

In this section, we consider a member with interest to maximize his fund size and minimize the volatility of the wealth accumulated. Hence, we develop an optimal portfolio strategy with the help of mean-variance utility function as follows:

$$\mathcal{K}(t,a,s)=\sup_{\psi_1}\left\{E_{t,a,s}\mathcal{A}^{\psi_1}(T)-Var_{t,a,s}\mathcal{A}^{\psi_1}(T)\right\} \tag{3.1}$$

following the procedures in (He and Liang, 2013; Ini *et al*, 2020) and applying the game theoretic approach in (Björk and Murgoci, 2009). The control problem in (3.1) is similar to the Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{K}(t,a,s)$ such that

$$\mathcal{K}(t,a,s)\ =\sup_{\psi_1}\mathcal{L}(t,a,s,\psi_1)$$

where

$$\begin{cases}\mathcal{L}(t,a,s,\psi_1)=E_{t,a,s}\mathcal{A}^{\psi_1}(T)-\frac{\nu}{2}Var_{t,a,s}\mathcal{A}^{\psi_1}(T)\\ =\left(E_{t,a,s}\mathcal{A}^{\psi_1}(T)-\frac{\lambda}{2}\left(E_{t,a,s}[\mathcal{A}^{\psi_1}(T)^2]-(E_{t,a,s}\mathcal{A}^{\psi_1}(T))^2\right)\right.\end{cases} \tag{3.2}$$

Following (He and Liang, 2013) the optimal portfolio strategy ψ_1 satisfies:

$$\mathcal{K}(t,a,s)=\sup_{\psi_1}\mathcal{L}(t,a,s,,\psi_1^*) \tag{3.3}$$

where λ is the risk-aversion coefficient of the members

Let $d^{\psi_1}(t,a,s)=E_{t,a,s}[\mathcal{A}^{\psi_1}(T)]$, $e^{\psi_1}(t,a,s)=E_{t,a,s}[\mathcal{A}^{\psi_1}(T)^2]$ then

$$\mathcal{K}(t,a,s)\ =\sup_{\psi_1}w\left(t,a,\ d^{\psi_1}(t,a,s),e^{\psi_1}(t,a,s)\right)$$

where

$$w(t,a,s,d,e)=d-\frac{\nu}{2}(e-d^2) \tag{3.4}$$

Theorem 3.1 (verification theorem). If $\mathcal{W},\mathcal{X},\mathcal{Y}$ are three real functions:$[0,T]\times R\to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\begin{cases} \sup_{\psi} \begin{Bmatrix} \mathcal{W}_t - w_t + \begin{bmatrix} a\left(\left(\eta+\kappa\xi^2 s^{2\lambda}-\mathcal{R}\right)\psi_1+\mathcal{R}+\frac{1}{n-n_0-t}\right) \\ +m\left(\frac{n-n_0-2t}{n-n_0-t}\right) \end{bmatrix}(\mathcal{W}_a - w_a) \\ +\left(\eta s+\kappa\xi^2 s^{2\lambda+1}\right)(\mathcal{W}_s - w_s)+\frac{1}{2}\psi_1^2 a^2\xi^2 s^{2\lambda}(\mathcal{W}_{aa}-\mathcal{M}_{aa}) \\ +\frac{1}{2}\xi^2 s^{2\lambda+2}(\mathcal{W}_{ss}-\mathcal{M}_{ss})+a\psi_1\xi^2 s^{2\lambda+1}(\mathcal{W}_{as}-\mathcal{M}_{as}) \end{Bmatrix} = 0 \\ \mathcal{W}(T,a,s) = w(T,a,s,a,a^2) \end{cases} \quad (3.5)$$

where:

$\mathcal{M}_{aa} = \nu\mathcal{X}_a^2, \mathcal{M}_{xs} = \nu\mathcal{X}_a\mathcal{X}_s, \mathcal{M}_{ss} = \nu\mathcal{X}_s^2$

(3.6)

$$\begin{cases} \begin{Bmatrix} \mathcal{X}_t + \begin{bmatrix} a\left(\left(\eta+\kappa\xi^2 s^{2\lambda}-\mathcal{R}\right)\psi_1+\mathcal{R}+\frac{1}{n-n_0-t}\right) \\ +m\left(\frac{n-n_0-2t}{n-n_0-t}\right) \end{bmatrix}\mathcal{X}_a \\ +\left(\eta s+\kappa\xi^2 s^{2\lambda+1}\right)\mathcal{X}_s+\frac{1}{2}\psi_1^2 a^2\xi^2 s^{2\lambda}\mathcal{X}_{aa} \\ +\frac{1}{2}\xi^2 s^{2\lambda+2}\mathcal{X}_{ss}+a\psi_1\xi^2 s^{2\lambda+1}\mathcal{X}_{as} \end{Bmatrix} = 0 \\ \mathcal{X}(T,a,s) = a \end{cases} \quad (3.7)$$

$$\begin{cases} \begin{Bmatrix} \mathcal{Y}_t + \begin{bmatrix} a\left(\left(\eta+\kappa\xi^2 s^{2\lambda}-\mathcal{R}\right)\psi_1+\mathcal{R}+\frac{1}{n-n_0-t}\right) \\ +m\left(\frac{n-n_0-2t}{n-n_0-t}\right) \end{bmatrix}\mathcal{Y}_a \\ +\left(\eta s+\kappa\xi^2 s^{2\lambda+1}\right)\mathcal{Y}_s+\frac{1}{2}\psi_1^2 a^2\xi^2 s^{2\lambda}\mathcal{Y}_{aa} \\ +\frac{1}{2}\xi^2 s^{2\lambda+2}\mathcal{Y}_{ss}+a\psi_1\xi^2 s^{2\lambda+1}\mathcal{Y}_{as} \end{Bmatrix} = 0 \\ \mathcal{Y}(T,a,s) = a^2 \end{cases} \quad (3.8)$$

Then $\mathcal{K}(t,a,s) = \mathcal{W}(t,a,s)$, $d^{\psi_1^*}(t,a,s) = \mathcal{X}(t,a,s)$, $e^{\psi_1^*}(t,a,s) = \mathcal{Y}(t,a,s)$ for the optimal portfolio strategy ψ_1^*.

*Proof***:** see the details of the proof in (He and Liang, 2009; Liang and Huang, 2011; Zeng and Li, 2011).

4. The Optimal Portfolio Strategy and Efficient Frontier

Proposition 4.1. The optimal portfolio strategy for the risky asset and the efficient frontier of the fund members are given as

(i) $$\psi_1^* = \frac{e^{\mathcal{R}(t-T)}}{\nu a\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right)\begin{bmatrix} \eta-\mathcal{R}+\kappa\xi^2 s^{2\lambda} \\ +\frac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4 s^{4\lambda}+(1-2\lambda)\kappa^2\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{8\lambda}\kappa^2\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2} \end{bmatrix}$$

(4.1)

(ii)

$E_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1{}^*}(T)] =$

$$\begin{pmatrix} a\left(\frac{n-n_0-t}{n-n_0-T}\right)e^{\mathcal{R}(T-t))} \\ +\begin{pmatrix} \mathscr{s}^{-2\lambda}\left(\frac{(\eta-\mathcal{R})^2}{2\mathcal{R}\lambda\xi^2}+\frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{2\mathcal{R}\lambda}\right)\left[1-e^{2\mathcal{R}\lambda(t-T)}\right] \\ +\begin{pmatrix} \frac{(2\lambda+1)}{4\mathcal{R}^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\left[e^{2\mathcal{R}\lambda(t-T)}-1\right] \\ +\left(\frac{(2\lambda+1)}{2\mathcal{R}}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)+\kappa(\eta-\mathcal{R})\right)(T-t) \\ +\left(\frac{m\nu e^{\mathcal{R}(T-t)}}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2t}{\mathcal{R}}\right)-\left(\frac{m\nu}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2T}{\mathcal{R}}\right) \end{pmatrix} \end{pmatrix} \\ \times\sqrt{\frac{Var_{t,a,\mathscr{s}}\left[\mathcal{A}^{\psi_1{}^*}(T)\right]}{\begin{pmatrix} \mathscr{s}^{-2\lambda}\begin{pmatrix} (\frac{(\eta-\mathcal{R})^2}{\mathcal{R}\lambda\xi^2}+\frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}\lambda} \\ -\frac{1}{2\lambda\eta}\left(\frac{(\eta-\mathcal{R})^2}{\xi^2}+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\left[1-e^{2\eta\lambda(t-T)}\right] \\ -\begin{pmatrix} \frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} \\ +\frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}} \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda\eta}\left[1-e^{2\eta\lambda(t-T)}\right] \\ -\frac{1}{\lambda(\eta-\mathcal{R})}\left[e^{2\eta\lambda(t-T)}-e^{2\mathcal{R}\lambda(t-T)}\right] \end{pmatrix} \end{pmatrix} \\ +\begin{pmatrix} \begin{pmatrix} \frac{(2\kappa-2\lambda-1)}{4\eta^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^4\xi^2\mathscr{s}^{4\lambda}\right) \\ -\begin{pmatrix} \frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} \\ +\frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}} \end{pmatrix}\left(\frac{1}{2\eta^2\lambda}+\frac{1}{2\eta\lambda(\eta-\mathcal{R})}\right) \end{pmatrix}\left[e^{2\eta\mathcal{R}(t-T)}-1\right] \\ +\begin{pmatrix} \frac{(2\lambda+1)}{2\mathcal{R}^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right) \\ -\frac{1}{\lambda\mathcal{R}^2}\left(\kappa(\eta-\mathcal{R})^2+\kappa^4\xi^2\mathscr{s}^{4\lambda}\right) \\ +\left(\frac{(\eta-\mathcal{R})^2}{2\lambda\mathcal{R}^2\xi^2}+\frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{2\lambda\mathcal{R}^2}\right) \end{pmatrix}\left[e^{2\mathcal{R}\lambda(t-T)}-1\right] \\ +\begin{pmatrix} \left(\frac{(2\lambda+1)}{\mathcal{R}}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\right) \\ -2\left(\frac{\kappa(\eta-\mathcal{R})^2}{\mathcal{R}}+\frac{\kappa^3\xi^4\mathscr{s}^{4\lambda}}{\mathcal{R}}\right) \\ +\left(\frac{(\eta-\mathcal{R})^3}{\eta\mathcal{R}\xi^2}+\frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\eta\mathcal{R}}\right) \\ +\frac{(2\kappa-2\lambda-1)}{2\eta}\left((\eta-\mathcal{R})^2+\kappa^2\xi^4\mathscr{s}^{4\lambda}\right) \end{pmatrix}(T-t) \end{pmatrix} \end{pmatrix}}} \end{pmatrix} \quad (4.2)$$

***Proof*.** Recall that from (3.4),

$w_d = 1+\nu d, w_e = -\frac{\nu}{2}, w_{dd} = \nu, \qquad w_t = w_a = w_{aa} = w_{ad} = w_{ae} = w_{de} = w_{ee} = 0,$

Substituting (3.6) and the above equations into (3.5), we differentiate the resultant equation with respect to ψ_1 and solve for ψ_1

$$\psi_1^* = -\left[\frac{\mathcal{W}_a(\eta+\kappa\xi^2\mathscr{s}^{2\lambda}-\mathcal{R})+\xi^2\mathscr{s}^{2\lambda+1}(\mathcal{W}_{a\mathscr{s}}-\nu\mathcal{X}_a\mathcal{X}_{\mathscr{s}})}{a\xi^2\mathscr{s}^{2\lambda}(\mathcal{W}_{aa}-\nu\mathcal{X}_a^2)}\right], \quad (4.3)$$

where ψ_1^* is the optimal portfolio strategy.

substituting (4.3) into (3.5) and (3.7) we have,

$$\left\{\begin{array}{c} \mathcal{W}_t + \left[a\left(\mathcal{R} + \frac{1}{n-n_0-t}\right) + m\left(\frac{n-n_0-2t}{n-n_0-t}\right)\right]\mathcal{W}_a + \left(\eta s + \kappa\xi^2 s^{2\lambda+1}\right)\mathcal{W}_s \\ -\frac{\left(\eta+\kappa\xi^2 s^{2\lambda}-\mathcal{R}\right)^2}{2\xi^2 s^{2\lambda}}\frac{\mathcal{W}_a^2}{(\mathcal{W}_{aa}-\nu\mathcal{X}_a^2)} + \frac{1}{2}(\mathcal{W}_{ss} - \nu\mathcal{X}_s^2)\xi^2 s^{2\lambda+2} \\ -\frac{1}{2}\frac{(\mathcal{W}_{as}-\nu\mathcal{X}_a\mathcal{X}_s)^2}{\mathcal{W}_{aa}-\nu\mathcal{X}_a^2}\xi^2 s^{2\lambda+2} - s\left(\eta + \kappa\xi^2 s^{2\lambda} - \mathcal{R}\right)\frac{\mathcal{W}_{as}-\nu\mathcal{X}_a\mathcal{X}_s}{\mathcal{W}_{aa}-\nu\mathcal{X}_a^2}\mathcal{W}_a \end{array}\right\} = 0 \tag{4.4}$$

$$\left\{\begin{array}{c} \mathcal{X}_t + \left[a\left(\mathcal{R} + \frac{1}{n-n_0-t}\right) + m\left(\frac{n-n_0-2t}{n-n_0-t}\right)\right]\mathcal{X}_a + \left(\eta s + \kappa\xi^2 s^{2\lambda+1}\right)\mathcal{X}_s \\ -\frac{\left(\eta+\kappa\xi^2 s^{2\lambda}-\mathcal{R}\right)^2}{\xi^2 s^{2\lambda}}\frac{\mathcal{W}_a\mathcal{X}_a}{(\mathcal{W}_{aa}-\nu\mathcal{X}_a^2)} + \frac{1}{2}\xi^2 s^{2\lambda+2}\mathcal{X}_{ss} - s\left(\eta + \kappa\xi^2 s^{2\lambda} - \mathcal{R}\right)\frac{\mathcal{X}_{as}-\nu\mathcal{X}_a\mathcal{X}_s}{\mathcal{W}_{aa}-\nu\mathcal{X}_a^2}\mathcal{X}_a \end{array}\right\} = 0 \tag{4.5}$$

Next, we assume a solution for $\mathcal{W}(t,a,s)$and $\mathcal{X}(t,a,s)$as follows:

$$\left\{\begin{array}{c} \mathcal{W}(t,a,s) = a\mathcal{G}_1(t) + \frac{s^{-2\lambda}}{\nu}\mathcal{G}_2(t) + \frac{1}{\nu}\mathcal{G}_3(t), \mathcal{G}_1(T) = 1, \mathcal{G}_2(T) = 0, \mathcal{G}_3(T) = 0 \\ \mathcal{W}_t = \mathcal{G}_{1t}a + \frac{\mathcal{G}_{2t}s^{-2\lambda}}{\nu} + \frac{\mathcal{G}_{3t}}{\nu}, \mathcal{W}_a = \mathcal{G}_1, \mathcal{W}_{aa} = \mathcal{W}_{as} = 0, \\ \mathcal{W}_s = \frac{-2\lambda\mathcal{G}_2 s^{-2\lambda-1}}{\nu}, \mathcal{W}_{ss} = \frac{2\lambda(2\lambda+1)\mathcal{G}_2 s^{-2\lambda-2}}{\nu} \\ \mathcal{X}(t,a,s) = a\mathcal{H}_1(t) + \frac{s^{-2\lambda}}{\nu}\mathcal{H}_2(t) + \frac{1}{\nu}\mathcal{H}_3(t), \mathcal{H}_1(T) = 1, \mathcal{H}_2(T) = 0, \mathcal{H}_3(T) = 0 \\ \mathcal{X}_t = \mathcal{H}_{1t}a + \frac{\mathcal{H}_{2t}s^{-2\lambda}}{\nu} + \frac{\mathcal{H}_{3t}}{\nu}, \mathcal{X}_a = \mathcal{H}_1, \mathcal{X}_{aa} = \mathcal{X}_{as} = 0, \\ \mathcal{X}_s = \frac{-2\lambda\mathcal{H}_2 s^{-2\lambda-1}}{\nu}, \mathcal{X}_{ss} = \frac{2\lambda(2\lambda+1)\mathcal{H}_2 s^{-2\lambda-2}}{\nu} \end{array}\right. \tag{4.6}$$

Substituting (4.6) into (4.4) and (4.5), we have:

$$\left\{\begin{array}{c} \left[\mathcal{G}_{1t} + \left(\mathcal{R} + \frac{1}{n-n_0-t}\right)\mathcal{G}_1\right]a + \frac{s^{-2\lambda}}{\nu}\left[\begin{array}{c}\mathcal{G}_{2t} - 2\eta\lambda\mathcal{G}_2 + 2(\eta-\mathcal{R})\lambda\mathcal{H}_2\frac{\mathcal{G}_1}{\mathcal{H}_1} \\ +\left(\frac{(\eta-\mathcal{R})^2}{2\xi^2} + \frac{\kappa^2\xi^2 s^{4\lambda}}{2}\right)\frac{\mathcal{G}_1^2}{\mathcal{H}_1^2}\end{array}\right] \\ +\frac{1}{\nu}\left[\mathcal{G}_{3t} + m\nu\left(\frac{n-n_0-2t}{n-n_0-t}\right)\mathcal{G}_1 + \lambda(2\lambda+1)\xi^2\mathcal{G}_2 + \kappa(\eta-\mathcal{R}) + 2\kappa\xi^2\mathcal{H}_2\frac{\mathcal{G}_1}{\mathcal{H}_1}\right] \end{array}\right\} = 0 \tag{4.7}$$

$$\left\{\begin{array}{c} \left[\mathcal{H}_{1t} + \left(\mathcal{R} + \frac{1}{n-n_0-t}\right)\mathcal{H}_1\right]a + \frac{s^{-2\lambda}}{\nu}\left[\mathcal{H}_{2t} - 2\mathcal{R}\lambda\mathcal{H}_2 + \left(\frac{(\eta-\mathcal{R})^2}{\xi^2} + \frac{\kappa^2\xi^2 s^{4\lambda}}{2}\right)\frac{\mathcal{G}_1^2}{\mathcal{H}_1^2}\right] \\ +\frac{1}{\nu}\left[\mathcal{H}_{3t} + m\nu\left(\frac{n-n_0-2t}{n-n_0-t}\right)\mathcal{H}_1 + \lambda(2\lambda+1)\xi^2\mathcal{H}_2 + \kappa(\eta-\mathcal{R})\right] \end{array}\right\} = 0 \tag{4.8}$$

Since$a \neq 0, \frac{s^{-2\lambda}}{\nu} \neq 0, \frac{1}{\nu} \neq 0$, then simplifying (4.7) and (4.8), we have

$$\left\{\begin{array}{c} \mathcal{G}_{1t} + \left(\mathcal{R} + \frac{1}{n-n_0-t}\right)\mathcal{G}_1 = 0 \\ \mathcal{G}_1(T) = 1 \end{array}\right. \tag{4.9}$$

$$\begin{cases} \mathcal{G}_{2t} - 2\eta\lambda\mathcal{G}_2 + 2(\eta - \mathcal{R})\lambda\mathcal{H}_2 \frac{\mathcal{G}_1}{\mathcal{H}_1} + \left(\frac{(\eta-\mathcal{R})^2}{2\xi^2} + \frac{\kappa^2\xi^2 s^{4\lambda}}{2}\right)\frac{\mathcal{G}_1^2}{\mathcal{H}_1^2} = 0 \\ \mathcal{G}_2(T) = 0 \end{cases} \tag{4.10}$$

$$\begin{cases} \mathcal{G}_{3t} + m\nu\left(\frac{n-n_0-2t}{n-n_0-t}\right)\mathcal{G}_1 + \lambda(2\lambda+1)\xi^2\mathcal{G}_2 + \kappa(\eta - \mathcal{R}) + 2\kappa\xi^2\mathcal{H}_2\frac{\mathcal{G}_1}{\mathcal{H}_1} 0 \\ \mathcal{G}_3(T) = 0 \end{cases} \tag{4.11}$$

$$\begin{cases} \mathcal{H}_{1t} + \left(\mathcal{R} + \frac{1}{n-n_0-t}\right)\mathcal{H}_1 = 0 \\ \mathcal{H}_1(T) = 1 \end{cases} \tag{4.12}$$

$$\begin{cases} \mathcal{H}_{2t} - 2\mathcal{R}\lambda\mathcal{H}_2 + \left(\frac{(\eta-\mathcal{R})^2}{\xi^2} + \frac{\kappa^2\xi^2 s^{4\lambda}}{2}\right)\frac{\mathcal{G}_1^2}{\mathcal{H}_1^2} = 0 \\ \mathcal{H}_2(T) = 0 \end{cases} \tag{4.13}$$

$$\begin{cases} \mathcal{H}_{3t} + m\nu\left(\frac{n-n_0-2t}{n-n_0-t}\right)\mathcal{H}_1 + \lambda(2\lambda+1)\xi^2\mathcal{H}_2 + \kappa(\eta - \mathcal{R}) = 0 \\ \mathcal{H}_3(T) = 0 \end{cases} \tag{4.14}$$

Solving (4.9) – (4.14), we obtain:

$$\mathcal{G}_1(t) = \left(\frac{n-n_0-t}{n-n_0-T}\right)e^{\mathcal{R}(T-t)} \tag{4.15}$$

$$\mathcal{G}_2(t) = \begin{pmatrix} \frac{1}{4\lambda\eta}\left(\frac{(\eta-\mathcal{R})^2}{\xi^2} + \kappa^2\xi^2 s^{4\lambda}\right)\left[1 - e^{2\eta\lambda(t-T)}\right] \\ +\left(\frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} + \frac{(\eta-\mathcal{R})\kappa^2\xi^2 s^{4\lambda}}{\mathcal{R}}\right)\begin{pmatrix} \frac{1}{2\lambda\eta}\left[1 - e^{2\eta\lambda(t-T)}\right] \\ -\frac{1}{2\lambda(\eta-\mathcal{R})}\left[e^{2\eta\lambda(t-T)} - e^{2\mathcal{R}\lambda(t-T)}\right] \end{pmatrix} \end{pmatrix} \tag{4.16}$$

$$\mathcal{G}_3(t) = \begin{pmatrix} \begin{pmatrix} \frac{(2\kappa-2\lambda-1)}{8\eta^2\lambda}\left((\eta-\mathcal{R})^2 + \kappa^4\xi^2 s^{4\lambda}\right) \\ -\left(\frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} + \frac{(\eta-\mathcal{R})\kappa^2\xi^2 s^{4\lambda}}{\mathcal{R}}\right)\left(\frac{1}{4\eta^2\lambda} + \frac{1}{4\eta\lambda(\eta-\mathcal{R})}\right) \end{pmatrix}\left[e^{2\eta\mathcal{R}(t-T)} - 1\right] \\ +\begin{pmatrix} \kappa(\eta-\mathcal{R}) + \left(\frac{\kappa(\eta-\mathcal{R})^2}{\mathcal{R}} + \frac{\kappa^3\xi^4 s^{4\lambda}}{\mathcal{R}}\right) - \left(\frac{(\eta-\mathcal{R})^3}{2\eta\mathcal{R}\xi^2} + \frac{(\eta-\mathcal{R})\kappa^2\xi^2 s^{4\lambda}}{2\eta\mathcal{R}}\right) \\ -\frac{(2\kappa-2\lambda-1)}{4\eta}\left((\eta-\mathcal{R})^2 + \kappa^2\xi^4 s^{4\lambda}\right) \end{pmatrix}(T-t) \\ \left(\frac{1}{2\lambda\mathcal{R}^2}\left(\kappa(\eta-\mathcal{R})^2 + \kappa^4\xi^2 s^{4\lambda}\right) - \left(\frac{(\eta-\mathcal{R})^2}{4\lambda\mathcal{R}^2\xi^2} + \frac{\kappa^2\xi^2 s^{4\lambda}}{4\lambda\mathcal{R}^2}\right)\right)\left[e^{2\mathcal{R}\lambda(t-T)} - 1\right] \\ +\left(\frac{m\nu e^{\mathcal{R}(T-t)}}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2} - \frac{n-n_0-2t}{\mathcal{R}}\right) - \left(\frac{m\nu}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2} - \frac{n-n_0-2T}{\mathcal{R}}\right) \end{pmatrix} \tag{4.17}$$

$$\mathcal{H}_1(t) = \left(\frac{n-n_0-t}{n-n_0-T}\right)e^{\mathcal{R}(T-t))} \tag{4.18}$$

$$\mathcal{H}_2(t) = \left(\frac{(\eta-\mathcal{R})^2}{2\mathcal{R}\lambda\xi^2} + \frac{\kappa^2\xi^2 s^{4\lambda}}{2\mathcal{R}\lambda}\right)\left[1 - e^{2\mathcal{R}\lambda(t-T)}\right] \tag{4.19}$$

$$\mathcal{H}_3(t) = \begin{pmatrix} \frac{(2\lambda+1)}{4\mathcal{R}^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2 s^{4\lambda}\right)\left[e^{2\mathcal{R}\lambda(t-T)}-1\right] \\ +\left(\frac{(2\lambda+1)}{2\mathcal{R}}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2 s^{4\lambda}\right)+\kappa(\eta-\mathcal{R})\right)(T-t) \\ +\left(\frac{m\nu e^{\mathcal{R}(T-t)}}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2t}{\mathcal{R}}\right)-\left(\frac{m\nu}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2T}{\mathcal{R}}\right) \end{pmatrix} \tag{4.20}$$

Substituting (4.15) (4.16), (4.17) into (4.6) and (4.18), (4.19), (4.20) into (4.13) we have:

$$\mathcal{X}(t,a,s) = \begin{pmatrix} a\left(\frac{n-n_0-t}{n-n_0-T}\right)e^{\mathcal{R}(T-t))} \\ +\frac{1}{\nu}\begin{pmatrix} s^{-2\lambda}\left(\frac{(\eta-\mathcal{R})^2}{2\mathcal{R}\lambda\xi^2}+\frac{\kappa^2\xi^2 s^{4\lambda}}{2\mathcal{R}\lambda}\right)\left[1-e^{2\mathcal{R}\lambda(t-T)}\right] \\ +\begin{pmatrix} \frac{(2\lambda+1)}{4\mathcal{R}^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2 s^{4\lambda}\right)\left[e^{2\mathcal{R}\lambda(t-T)}-1\right] \\ +\left(\frac{(2\lambda+1)}{2\mathcal{R}}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2 s^{4\lambda}\right)+\kappa(\eta-\mathcal{R})\right)(T-t) \\ +\left(\frac{m\nu e^{\mathcal{R}(T-t)}}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2t}{\mathcal{R}}\right)-\left(\frac{m\nu}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2T}{\mathcal{R}}\right) \end{pmatrix} \end{pmatrix} \end{pmatrix} \tag{4.21}$$

$$\mathcal{W}(t,a,s) = \begin{pmatrix} a\left(\frac{n-n_0-t}{n-n_0-T}\right)e^{\mathcal{R}(T-t)} \\ +\frac{1}{\nu}\begin{pmatrix} s^{-2\lambda}\begin{pmatrix} \frac{1}{4\lambda\eta}\left(\frac{(\eta-\mathcal{R})^2}{\xi^2}+\kappa^2\xi^2 s^{4\lambda}\right)\left[1-e^{2\eta\lambda(t-T)}\right] \\ +\begin{pmatrix} \frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} \\ +\frac{(\eta-\mathcal{R})\kappa^2\xi^2 s^{4\lambda}}{\mathcal{R}} \end{pmatrix}\begin{pmatrix} \frac{1}{2\lambda\eta}\left[1-e^{2\eta\lambda(t-T)}\right] \\ -\frac{1}{2\lambda(\eta-\mathcal{R})}\left[e^{2\eta\lambda(t-T)}-e^{2\mathcal{R}\lambda(t-T)}\right] \end{pmatrix} \end{pmatrix} \\ +\begin{pmatrix} \begin{pmatrix} \frac{(2\kappa-2\lambda-1)}{8\eta^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^4\xi^2 s^{4\lambda}\right) \\ -\begin{pmatrix} \frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} \\ +\frac{(\eta-\mathcal{R})\kappa^2\xi^2 s^{4\lambda}}{\mathcal{R}} \end{pmatrix}\left(\frac{1}{4\eta^2\lambda}+\frac{1}{4\eta\lambda(\eta-\mathcal{R})}\right) \end{pmatrix}\left[e^{2\eta\mathcal{R}(t-T)}-1\right] \\ +\begin{pmatrix} \kappa(\eta-\mathcal{R})+\left(\frac{\kappa(\eta-\mathcal{R})^2}{\mathcal{R}}+\frac{\kappa^3\xi^4 s^{4\lambda}}{\mathcal{R}}\right) \\ -\left(\frac{(\eta-\mathcal{R})^3}{2\eta\mathcal{R}\xi^2}+\frac{(\eta-\mathcal{R})\kappa^2\xi^2 s^{4\lambda}}{2\eta\mathcal{R}}\right) \\ -\frac{(2\kappa-2\lambda-1)}{4\eta}\left((\eta-\mathcal{R})^2+\kappa^2\xi^4 s^{4\lambda}\right) \end{pmatrix}(T-t) \\ +\begin{pmatrix} \frac{1}{2\lambda\mathcal{R}^2}\left(\kappa(\eta-\mathcal{R})^2+\kappa^4\xi^2 s^{4\lambda}\right) \\ -\left(\frac{(\eta-\mathcal{R})^2}{4\lambda\mathcal{R}^2\xi^2}+\frac{\kappa^2\xi^2 s^{4\lambda}}{4\lambda\mathcal{R}^2}\right) \end{pmatrix}\left[e^{2\mathcal{R}\lambda(t-T)}-1\right] \\ +\left(\frac{m\nu e^{\mathcal{R}(T-t)}}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2t}{\mathcal{R}}\right) \\ -\left(\frac{m\nu}{n-n_0-T}\right)\left(\frac{2}{\mathcal{R}^2}-\frac{n-n_0-2T}{\mathcal{R}}\right) \end{pmatrix} \end{pmatrix} \end{pmatrix} \tag{4.22}$$

Differentiating (4.21) and (4.22) for $\mathcal{W}_a$, $\mathcal{W}_{as}$,$\mathcal{W}_{aa}$, $\mathcal{X}_a$, $\mathcal{X}_s$ we have

$$\begin{cases} \mathcal{W}_a = \mathcal{X}_a = a\left(\frac{n-n_0-t}{n-n_0-T}\right)e^{\mathcal{R}(T-t))} \\ \mathcal{W}_{as} = \mathcal{W}_{aa} = 0, \\ \mathcal{X}_s = \frac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4 s^{4\lambda}+(2\lambda+1)\kappa^2\xi^4 s^{6\lambda}\right)\left(1-e^{2r\lambda(t-T)}\right)+\frac{2\lambda+1}{8\lambda}\kappa^2\xi^4 s^{6\lambda}(T-t)}{\nu\mathcal{R}^2} \end{cases} \tag{4.23}$$

Substituting (4.23) into (4.3), we obtain (4.1) which complete the proof of (i)

(ii**)** Recall that

$$Var_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1{}^*}(T)] = E_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1{}^*}(T)] - (E_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1{}^*}(T)])^2$$

$$Var_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1{}^*}(T)] = \frac{2}{\nu}(\mathcal{X}(t,a,\mathscr{s}) - \mathcal{W}(t,a,\mathscr{s}))$$

Substituting (4.22) and (4.21) for $\mathcal{W}(t,a,\mathscr{s})$ and$\mathcal{X}(t,a,\mathscr{s})$ respectively in the above equation, we have

$$Var_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1{}^*}(T)] = \frac{1}{\nu^2}\begin{pmatrix} \mathscr{s}^{-2\lambda}\begin{pmatrix} (\frac{(\eta-\mathcal{R})^2}{\mathcal{R}\lambda\xi^2} + \frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}\lambda} \\ -\frac{1}{2\lambda\eta}\left(\frac{(\eta-\mathcal{R})^2}{\xi^2} + \kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\left[1 - e^{2\eta\lambda(t-T)}\right] \\ -\begin{pmatrix} \frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} \\ + \frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}} \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda\eta}\left[1 - e^{2\eta\lambda(t-T)}\right] \\ -\frac{1}{\lambda(\eta-\mathcal{R})}\left[e^{2\eta\lambda(t-T)} - e^{2\mathcal{R}\lambda(t-T)}\right] \end{pmatrix} \end{pmatrix} \\ + \begin{pmatrix} \begin{pmatrix} \frac{(2\kappa-2\lambda-1)}{4\eta^2\lambda}\left((\eta-\mathcal{R})^2 + \kappa^4\xi^2\mathscr{s}^{4\lambda}\right) \\ -\begin{pmatrix} \frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2} \\ + \frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}} \end{pmatrix}\left(\frac{1}{2\eta^2\lambda} + \frac{1}{2\eta\lambda(\eta-\mathcal{R})}\right) \end{pmatrix}\left[e^{2\eta\mathcal{R}(t-T)} - 1\right] \\ + \begin{pmatrix} \frac{(2\lambda+1)}{2\mathcal{R}^2\lambda}\left((\eta-\mathcal{R})^2 + \kappa^2\xi^2\mathscr{s}^{4\lambda}\right) \\ -\frac{1}{\lambda\mathcal{R}^2}\left(\kappa(\eta-\mathcal{R})^2 + \kappa^4\xi^2\mathscr{s}^{4\lambda}\right) \\ + \left(\frac{(\eta-\mathcal{R})^2}{2\lambda\mathcal{R}^2\xi^2} + \frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{2\lambda\mathcal{R}^2}\right) \end{pmatrix}\left[e^{2\mathcal{R}\lambda(t-T)} - 1\right] \\ + \begin{pmatrix} \left(\frac{(2\lambda+1)}{\mathcal{R}}\left((\eta-\mathcal{R})^2 + \kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\right) \\ -2\left(\frac{\kappa(\eta-\mathcal{R})^2}{\mathcal{R}} + \frac{\kappa^3\xi^4\mathscr{s}^{4\lambda}}{\mathcal{R}}\right) \\ + \left(\frac{(\eta-\mathcal{R})^3}{\eta\mathcal{R}\xi^2} + \frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\eta\mathcal{R}}\right) \\ + \frac{(2\kappa-2\lambda-1)}{2\eta}\left((\eta-\mathcal{R})^2 + \kappa^2\xi^4\mathscr{s}^{4\lambda}\right) \end{pmatrix}(T-t) \end{pmatrix} \end{pmatrix} \quad .24)$$

$$\frac{1}{\nu} =$$

$$Var_{t,x,s}[X^{d_1^*}(T)]$$

$$\sqrt{\left(\mathscr{s}^{-2\lambda}\left(\begin{array}{c}(\frac{(\eta-\mathcal{R})^2}{\mathcal{R}\lambda\xi^2}+\frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}\lambda}\\ -\frac{1}{2\lambda\eta}\left(\frac{(\eta-\mathcal{R})^2}{\xi^2}+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\left[1-e^{2\eta\lambda(t-T)}\right]\\ -\left(\begin{array}{c}\frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2}\\ +\frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}}\end{array}\right)\left(\begin{array}{c}\frac{1}{\lambda\eta}\left[1-e^{2\eta\lambda(t-T)}\right]\\ -\frac{1}{\lambda(\eta-\mathcal{R})}\left[\begin{array}{c}e^{2\eta\lambda(t-T)}\\ -e^{2\mathcal{R}\lambda(t-T)}\end{array}\right]\end{array}\right)\end{array}\right)+\left(\begin{array}{c}\left(\begin{array}{c}\frac{(2\kappa-2\lambda-1)}{4\eta^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^4\xi^2\mathscr{s}^{4\lambda}\right)\\ -\left(\begin{array}{c}\frac{(\eta-\mathcal{R})^3}{\mathcal{R}\xi^2}\\ +\frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\mathcal{R}}\end{array}\right)\left(\frac{1}{2\eta^2\lambda}+\frac{1}{2\eta\lambda(\eta-\mathcal{R})}\right)\end{array}\right)\left[e^{2\eta\mathcal{R}(t-T)}-1\right]\\ +\left(\begin{array}{c}\frac{(2\lambda+1)}{2\mathcal{R}^2\lambda}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\\ -\frac{1}{\lambda\mathcal{R}^2}\left(\kappa(\eta-\mathcal{R})^2+\kappa^4\xi^2\mathscr{s}^{4\lambda}\right)\\ +\left(\frac{(\eta-\mathcal{R})^2}{2\lambda\mathcal{R}^2\xi^2}+\frac{\kappa^2\xi^2\mathscr{s}^{4\lambda}}{2\lambda\mathcal{R}^2}\right)\end{array}\right)\left[e^{2\mathcal{R}\lambda(t-T)}-1\right]\\ +\left(\begin{array}{c}\left(\frac{(2\lambda+1)}{\mathcal{R}}\left((\eta-\mathcal{R})^2+\kappa^2\xi^2\mathscr{s}^{4\lambda}\right)\right)\\ -2\left(\frac{\kappa(\eta-\mathcal{R})^2}{\mathcal{R}}+\frac{\kappa^3\xi^4\mathscr{s}^{4\lambda}}{\mathcal{R}}\right)\\ +\left(\frac{(\eta-\mathcal{R})^3}{\eta\mathcal{R}\xi^2}+\frac{(\eta-\mathcal{R})\kappa^2\xi^2\mathscr{s}^{4\lambda}}{\eta\mathcal{R}}\right)\\ +\frac{(2\kappa-2\lambda-1)}{2\eta}\left((\eta-\mathcal{R})^2+\kappa^2\xi^4\mathscr{s}^{4\lambda}\right)\end{array}\right)(T-t)\end{array}\right)\right)} \tag{4.25}$$

Recall from theorem 3.1, the expectation is given as

$$E_{t,a,\mathscr{s}}[\mathcal{A}^{\psi_1^*}(T)] = \mathcal{X}(t,a,\mathscr{s}) \tag{4.26}$$

Substituting equation (4.22) and (4.25) into (4.26), we obtain (4.2) which complete the proof (ii).

Remark 4.1: In a case where the price of risky asset is modelled by CEV process i.e when the modification factor $\kappa = 0$, then optimal portfolio strategy and the efficient frontier reduces the result in (Li *et al*, 2017) as follows

$$\psi_2^* = \frac{e^{\mathcal{R}(t-T)}}{\nu a\xi^2\mathscr{s}^{2\lambda}}(\eta-\mathcal{R})\left[1+\frac{(\eta-\mathcal{R})\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)}{\mathcal{R}}\right]\left(\frac{\mathscr{n}-\mathscr{n}_0-T}{\mathscr{n}-\mathscr{n}_0-t}\right) \tag{4.27}$$

Remark 4.2: In a case where the price of risky asset is modelled by GBM i.e. when the modification factor $\kappa = 0$ and the elasticity parameter $\lambda = 0$, then optimal portfolio strategy and the efficient frontier reduces the result in He and Liang (2013) as follows

$$\psi_3^* = \frac{(\eta-\mathcal{R})}{\nu a\xi^2}\left(\frac{\mathscr{n}-\mathscr{n}_0-T}{\mathscr{n}-\mathscr{n}_0-t}\right)e^{\mathcal{R}(t-T)} \tag{4.28}$$

5. Theoretical Analysis

Proposition 5.1 Suppose $\eta-\mathcal{R} > 0, a > 0, \mathcal{R} > 0, \xi > 0, \kappa > 0, \nu > 0, \lambda < 0, \mathscr{s}(t) > 0, \mathscr{n}-\mathscr{n}_0-T > 0, t > 0$ then

a. $\frac{d\psi_1^*}{d\kappa} > 0$ (b) $\frac{d\psi_1^*}{d\nu} < 0$ (c) $\frac{d\psi_1^*}{da} < 0$ (d) $\frac{d\psi_1^*}{d\xi} < 0$

Proof.

Recall from equation (4.1),

$$\psi_1^* = \frac{e^{\mathcal{R}(t-T)}}{\nu a\xi^2\mathscr{s}^{2\lambda}}\left(\frac{\mathscr{n}-\mathscr{n}_0-T}{\mathscr{n}-\mathscr{n}_0-t}\right)\left[\begin{array}{c}\eta-\mathcal{R}+\kappa\xi^2\mathscr{s}^{2\lambda}\\ +\frac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4\mathscr{s}^{4\lambda}+(1-2\lambda)\kappa^2\xi^4\mathscr{s}^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{8\lambda}\kappa^2\xi^4\mathscr{s}^{6\lambda}(T-t)}{\mathcal{R}^2}\end{array}\right]$$

(a) Differentiating ψ_1^* with respect to κ , we have

$$\frac{d\psi_1^*}{d\kappa} = \frac{e^{\mathcal{R}(t-T)}}{\nu a\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right)\begin{bmatrix} \eta-\mathcal{R}+\xi^2 s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+2\mathcal{R}\kappa\xi^4 s^{4\lambda}+2(1-2\lambda)\kappa\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{4\lambda}\kappa\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix}$$

Since $\begin{bmatrix} \eta-\mathcal{R}+\xi^2 s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+2\mathcal{R}\kappa\xi^4 s^{4\lambda}+2(1-2\lambda)\kappa\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{4\lambda}\kappa\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix} > 0$ and $\frac{e^{\mathcal{R}(t-T)}}{\nu a\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right) > 0$, then $\frac{d\psi_1^*}{d\kappa} > 0$

(b) Differentiating ψ_1^* with respect to ν , we have

$$\frac{d\psi_1^*}{d\nu} = -\frac{e^{\mathcal{R}(t-T)}}{\nu^2 a\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right)\begin{bmatrix} \eta-\mathcal{R}+\kappa\xi^2 s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4 s^{4\lambda}+(1-2\lambda)\kappa^2\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{8\lambda}\kappa^2\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix}$$

Since $\begin{bmatrix} \eta-\mathcal{R}+\kappa\xi^2 s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4 s^{4\lambda}+(1-2\lambda)\kappa^2\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{8\lambda}\kappa^2\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix} > 0$
and $\frac{e^{\mathcal{R}(t-T)}}{\nu^2 a\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right) > 0$,
then $\frac{d\psi_1^*}{d\nu} < 0$

(c) Differentiating ψ_1^* with respect to a , we have

$$\frac{d\psi_1^*}{da} = -\frac{e^{\mathcal{R}(t-T)}}{a^2\nu\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right)\begin{bmatrix} \eta-\mathcal{R}+\kappa\xi^2 s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4 s^{4\lambda}+(1-2\lambda)\kappa^2\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{8\lambda}\kappa^2\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix}$$

Since $\begin{bmatrix} \eta-\mathcal{R}+\kappa\xi^2 s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+\mathcal{R}\kappa^2\xi^4 s^{4\lambda}+(1-2\lambda)\kappa^2\xi^4 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{8\lambda}\kappa^2\xi^4 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix} > 0$and
$\frac{e^{\mathcal{R}(t-T)}}{a^2\nu\xi^2 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right) > 0$,

then $\frac{d\psi_1^*}{da} < 0$

(d) Differentiating ψ_1^* with respect to ξ , we have

$$\frac{d\psi_1^*}{d\xi} = -\frac{2e^{\mathcal{R}(t-T)}}{a\nu\xi^3 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right)\begin{bmatrix} \eta-\mathcal{R}+2\kappa\xi s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+4\mathcal{R}\kappa^2\xi^3 s^{4\lambda}+4(1-2\lambda)\kappa^2\xi^3 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{2\lambda}\kappa^2\xi^3 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix}$$

Since $\begin{bmatrix} \eta-\mathcal{R}+2\kappa\xi s^{2\lambda} \\ +\dfrac{\left(\mathcal{R}(\eta-\mathcal{R})^2+4\mathcal{R}\kappa^2\xi^3 s^{4\lambda}+4(1-2\lambda)\kappa^2\xi^3 s^{6\lambda}\right)\left(1-e^{2\mathcal{R}\lambda(t-T)}\right)+\frac{1-2\lambda}{2\lambda}\kappa^2\xi^3 s^{6\lambda}(T-t)}{\mathcal{R}^2}\end{bmatrix} > 0$ and

$\frac{2e^{\mathcal{R}(t-T)}}{av\xi^3 s^{2\lambda}}\left(\frac{n-n_0-T}{n-n_0-t}\right) > 0,$

then $\quad \frac{d\psi_1^*}{d\xi} < 0$

6. Discussion

From proposition 5.1, the optimal portfolio strategy of the risky asset is an increasing function of the modification parameter and a decreasing function of the risk averse coefficient, instantaneous volatility, initial fund size. The implication of the proposition is that members with high risk averse coefficient will invest less in risky asset while members with low risk averse coefficient will invest more in the risky asset. Also, if the fund size at the early stage of investment is high, members will invest more in risk free asset and reduce that of risky asset and may increase it as retirement age draw closer. Since the instantaneous volatility represent the risk coefficient of the risky asset, if the instantaneous volatility is high, fund manager may not want to invest more in risky asset hence a confirmation of proposition 5.1 which shows that the optimal portfolio strategy is a decreasing function of the instantaneous volatility. and vice versa. Furthermore, we observed that the degree of volatility of any investment in a risky asset depends on the elasticity parameter.

From equation (2.1), we observed that an increase in the modification parameter κ will definitely increase the value of the risky asset (stock), thereby making it more attractive for the fund manager to invest more in it assuming all other parameters remain constant. If κ depreciate, the fund manager will reduce his investment in the risky asset. This is confirmed as the optimal portfolio strategy ψ_1^* increases with κ. Finally, from remark 4.1, we observed that when modification parameter $\kappa = 0$, our optimal portfolio strategy reduces to the result obtain in (Li *et al*, 2017) and from remark 4.2, we also observed that when the modification parameter (κ) and elasticity parameters (λ) are equal to zero, our optimal portfolio strategy reduces to the result in (He and Liang, 2013) which are cases of CEV model and that of GBM respectively.

7. Conclusion

This paper solves the optimal portfolio problem of DC plan member with return of contribution clause under M-CEV. A continuous time mean-variance stochastic optimal control problem which consist of the members' monthly contributions, the returned contributions and the invested funds was formulated. Also, an extended HJB equation was established and solved for the explicit solutions of optimal portfolio strategy and efficient frontier using change of variable and variable separation technique. We gave some theoretical analyses of the impact of the modification parameter, initial fund size, risk averse coefficient and the instantaneous volatility on the optimal portfolio strategy. In conclusion, our result generalizes results (He and Liang, 2013) and (Li *et al.*, 2017).

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