A STUDY OF LYAPUNOV STABILITY ANALYSIS OF SOME THIRD ORDER NON-LINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract
Lyapunov functions enable analyzing the stability of dynamic systems described by ordinary differential equations without finding the solution of such equations. Stability is one of the properties of solutions of any differential equation. A dynamical system in a state of equilibrium is said to be stable. In other words, a system has to be in a stable state before it can be asymptotically stable which means that stability does not necessarily imply asymptotic stability but asymptotic stability implies stability. For nonlinear systems, devising a Lyapunov function is not an easy task to solve in general. In this paper, we construct a suitable Lyapunov functions for some third order non-linear ordinary differential equations. We present three possible Lyapunov functions by finding quadratic form function for the appropriate third order linear differential equation, and construct Lyapunov functions for some non-linear ordinary differential equations by analogy with the linear system.

Keywords*: Lyapunov function, Asymptotic Stability, Linear ODE, Nonlinear ODE, Third Order ODE.*

1. Introduction

Lyapunov functions are scalar functions used to prove the stability of equilibrium of ordinary differential equation. It is very important to stability theory in dynamical systems and control theory. Lyapunov function is a necessary and sufficient condition for stability. There are many ways to construct Lyapunov functions and much remains to be done in this regard since there isn't universal constructive method for finding simple and explicit Lyapunov functions which helps to determine the stability or instability of a system as well as acting as control.

Lyapunov himself (Lyapunov, 1992) indicated the method of constructing the Lyapunov functions for an independent linear system. In (Oziraner and Rumiantsev, 1972), they constructed a Lyapunov function for a second order system with one non-linear function. His method consists of finding a quadratic form function for linear system, and analogy in the subsequent selection, a Lyapunov function for the non-linear system. The method gives us also the possibility of obtaining the Lyapunov function in a rather small proximity of the equilibrium position of non-linear system (Barbashin,1960). Further results in the construction of Lyapunov functions were obtained in the connection with the so called Aizerman problem (Pai,1995). The formulation of this problem prompts the idea of constructing Lyapunov function for non-linear systems by analogy with linear systems. (Joshi and Srirangarajan HR, 1976) have studied the singular points of third order non-linear ordinary differential equations. According to them, only one singular point of each of different equation is stable and the rest of singular points are unstable. Also, they have not investigated the stability character of the system over some regions around a singular point.

For an autonomous polynomial system of differential equations, how to compute the Lyapunov function at equilibria is a basic problem. In (Nguyen and Mori, 2006), the author transformed the problem of computing the Lyapunov function into a quantifier elimination problem. The disadvantage of the method is that the computation complexity of quantifier elimination is doubly exponential in the number of total variables. In order to avoid this problem, (She et al, 2009) propose a symbolic method; they first construct a special semi algebraic system using the local properties of a Lyapunov function as well as its derivative and solving these inequations using cylindrical algebraic decomposition (CAD). Lyapunov functions are indispensable in verifying some quantitative properties solution in ordinary differential equation. The problems encounter in the applications are in the construction of appropriate Lyapunov function for some non-linear ordinary differential equation (Ezeilo, and Ogbu, 2010). Lyapunov function (Okereke, 2016) is a good tool in the stability analysis of dynamical system.

In this work, we construct the possible Lyapunov functions for some third order non-linear ordinary differential equations using Lyapunov stability analysis.

1. Statement of the Problems. Preliminaries, Definitions

Consider the third-order differential equation

$$\dddot{x}+a\ddot{x}+b\dot{x}+cx=0 \qquad (2.1)$$

where $a,b,c>0$ are constants. The equation (2.1) is equivalent to the following three differential equations

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z \qquad (2.2)\\ \dot{z} &= -az-by-cx\end{aligned}$$

The differential equations (2.2) have negative real parts if and only if a,b,c are positive. There is need to have a positive definite continuous quadratic function V and another positive quadratic form U such that

$$\dot{V}=-U$$

along the solution paths of (2.1) or (2.2). It is our interest therefore to construct a Lyapunov function that would ultimately satisfy (2.2).

Let

$$\dot{x}=f(x,t) \quad x(t_0)=x_0 \quad x\in\mathbb{R} \qquad (2.3)$$

were $f(x,t)$ satisfies the standard conditions for the existence and uniqueness of solutions.

Definition 1

The equilibrium point $x^*=0$ of (2.3) is stable at $t=t_0$ if for any $\epsilon>0$, there exist $\delta(t_0,\epsilon)>0$ such that

$$\|x(t_0)\|<\delta \Rightarrow \|x(t)\|<\epsilon$$

for all $t \geq t_0$. Uniform stability is a concept which guarantees that the equilibrium point is not losing stability. Therefore, uniformly stable equilibrium point x^*, δ not be a function of t_0, so that $\| x(t_0) \| < \delta \Rightarrow \| x(t) \| < \epsilon$ may hold for all t_0

Definition 2: Asymptotic Stability.

An equilibrium points $x^* = 0$ of (2.3) is asymptotic stable at $t = t_0$ if

(i) $x^* = 0$ is stable.

(ii) $x^* = 0$ is locally attractive ie there exist $\delta(t_0)$ such that

$\| x(t_0) \| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$

Theorem 1

Let $V: \mathbb{R} \to \mathbb{R}$ be defined together with $\dfrac{\partial V}{\partial x_i}$ on an open set $\Omega \in \mathbb{R}, 0 \in \Omega$. If there exist a continuous positive definite function $V(x)$ defined on Ω with $\dot{V} \leq 0$, then the trivial solution $x \equiv 0$ is stable. If in addition $\dot{V}$ is negative definite on Ω, then $x \equiv 0$ is asymptotic stable.

2. Methodology and Discussions

The ordinary differential equation under investigation is

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$$

resulting to the third scalar first order shown as follows:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \qquad . \\ \dot{z} &= -az - by - cx \end{aligned}$$

Consider the system of third- dimensional linear equation

$$\dot{X} = AX$$

where A is positive definite metrics.

$$2V(x, y, z) = X^T AX = A_1x^2 + A_2y^2 + A_3z^2 + 2A_4xy + 2A_5xz + 2A_6yz \tag{3.1}$$

such that

$$\begin{aligned} \dot{V} &= -cA_5x^2 - (bA_6 - A_4)y^2 - (aA_3 - A_6)z^2 \\ &-(cA_6 + bA_5 - A_1)xy - (cA_3 + aA_5 - A_4)xz - (aA_6 + bA_3 - A_2 - A_5)yz \end{aligned} \tag{3.2}$$

We have the following possibilities:

$$\begin{aligned}
\dot{V} &\propto -x^2 \\
\dot{V} &\propto -y^2 \\
\dot{V} &\propto -z^2 \\
\dot{V} &\propto -(x^2+y^2+z^2)
\end{aligned} \qquad (3.3)$$

CASE I

If $\dot{V} = -cA_5x^2$,

then

$$\begin{aligned}
cA_5 &> 0 \\
bA_6 - A_4 &= 0 \\
aA_3 - A_6 &= 0 \\
cA_6 + bA_5 - A_1 &= 0 \\
cA_3 + aA_5 - A_4 &= 0 \\
aA_6 + bA_3 - A_2 - A_5 &= 0
\end{aligned} \qquad (3.4)$$

Therefore,

$$\begin{aligned}
A_1 &= \frac{(a^2c + ab^2 - cb)A_5}{k} \\
A_2 &= \frac{(a^3 + c)A_5}{k} \\
A_3 &= \frac{aA_5}{k} \\
A_4 &= \frac{a^2bA_5}{k} \\
A_6 &= \frac{a^2A_5}{k}
\end{aligned} \qquad (3.5)$$

where $ab - c = k > 0$. Substituting (3.5) in (3.1) and set $A_5 = 1$ since $A_5 > 0$.

$$\begin{aligned}
2V(x,y,z) &= \frac{(a^2c+ab^2-cb)}{k}x^2 + \frac{(a^3+c)}{k}y^2 + \frac{a}{k}z^2 + \frac{2a^2b}{k}xy + 2xz + \frac{2a^2}{k}yz \\
&= \frac{a^2c}{k}x^2 + bx^2 + \frac{a^3+c}{k}y^2 + \frac{a}{k}z^2 + \frac{2a^2b}{k}xy + 2xz + \frac{2a^2}{k}yz \\
&= bx^2 + \frac{2a^2b}{k}xy + \frac{a^3+c}{k}y^2 + \frac{a^2c}{k}x^2 + 2xz + \frac{a}{k}z^2 + \frac{2a^2}{k}yz
\end{aligned}$$

$$= b\left(x + \frac{a^2y}{k}\right)^2 + \left(\frac{k(a^3+c) - ba^4}{k}\right)y^2 + \frac{a^2c}{k}\left(x + \frac{kz}{a^2c}\right)^2 + \left(\frac{a^3c - k^2}{ka^2c}\right)z^2 + \frac{2a^2}{k}zy$$

$$= b\left(x + \frac{a^2y}{k}\right)^2 + \left(\frac{k(a^3+c) - ba^4}{k}\right)y^2 + \frac{a^2c}{k}\left(x + \frac{kz}{a^2c}\right)^2 + \left(\frac{a^3c - k^2}{ka^2c}\right)\left(z + \frac{a^4cy}{a^3c - k^2}\right)^2 - \frac{a^6c}{k(a^3c - k^2)}y^2 \qquad (3.6)$$

This is positive definite and the corresponding time derivative

$$\dot{V}(x,y,z) = -cx^2.$$

Therefore

$$2V_1(x,y,z) = \frac{a^2c}{k}x^2 + bx^2 + \frac{a^3+c}{k}y^2 + \frac{a}{k}z^2 + \frac{2a^2b}{k}xy + 2xz + \frac{2a^2}{k}zy \qquad (3.7)$$

is a Lyapunov function for the third order differential equation (2.1) since $c > 0$ and $ab - c = k > 0$

CASE II

If $\dot{V} = -(bA_6 - A_4)y^2$, then

$$\begin{aligned}
cA_5 &= 0\\
bA_6 - A_4 &> 0\\
aA_3 - A_6 &= 0\\
cA_6 + bA_5 - A_1 &= 0\\
cA_3 + aA_5 - A_4 &= 0\\
aA_6 + bA_3 - A_2 - A_5 &= 0
\end{aligned}$$

such that

$$\begin{aligned}
A_5 &= 0\\
A_1 &= caA_3\\
a_2 &= (a^2 + b)A_3\\
A_4 &= cA_3\\
A_6 &= aA_3\\
(ab - c)A_3 &> 0
\end{aligned} \qquad (3.8)$$

Substituting (3.8) in (3.1) and set $A_3 = 1$

$$\begin{aligned}
2V(x,y,z) &= cax^2 + (a^2+b)y^2 + z^2 + 2cxy + 2ayz\\
&= cax^2 + 2cxy + a^2y^2 + by^2 + 2ayz + z^2
\end{aligned}$$

$$= ac\left(x + \frac{y}{a}\right)^2 + \left(a^2 - \frac{c}{a}\right)y^2 + b\left(y + \frac{az}{b}\right)^2 + \left(\frac{b-a^2}{b}\right)z^2$$

which is positive definite and the time derivative of V ;

$$\dot{V}(x,y,z) = -(ab-c)y^2$$

Since $ab - c > 0$,

$$2V_2(x,y,z) = cax^2 + (a^2+b)y^2 + z^2 + 2cxy + 2ayz \qquad (3.9)$$

is a Lyapunov function for (2.1).

CASE III

If $\dot{V} = -(aA_3 - A_6)z^2$, then

$$\begin{aligned}
cA_5 &= 0 \\
bA_6 - A_4 &= 0 \\
aA_3 - A_6 &> 0 \\
cA_6 + bA_5 - A_1 &= 0 \\
cA_3 + aA_5 - A_4 &= 0 \\
aA_6 + bA_3 - A_2 - A_5 &= 0
\end{aligned}$$

which implies

$$\begin{aligned}
A_5 &= 0 \\
A_1 &= cA_6 \\
a_2 &= (a + \frac{b^2}{c})A_6 \\
A_4 &= bA_6 \qquad\qquad (3.10) \\
A_3 &= \frac{b}{c}A_6 \\
(\frac{ab}{c} - 1)A_6 &> 0
\end{aligned}$$

Setting $A_6 = 1$ and substitute (3.10) in (3.1),

$$\begin{aligned}
2V(x,y,z) &= cx^2 + \left(a + \frac{b^2}{c}\right)y^2 + \frac{b}{c}z^2 + 2bxy + 2yz \\
&= cx^2 + 2bxy + ay^2 + \frac{b^2}{c}y^2 + 2yz + \frac{b}{c}z^2 \\
&= c\left(x + \frac{by}{c}\right)^2 + \left(\frac{ac-b^2}{c}\right)y^2 + \frac{b^2}{c}\left(y + \frac{cz}{b^2}\right)^2 + \left(\frac{b^3-c^2}{cb^2}\right)z^2
\end{aligned}$$

which is positive definite and the corresponding time derivative

$$\dot{V}(x,y,z) = -(\frac{ab}{c} - 1)z^2$$

and

$(\frac{ab}{c}-1)>0.$ Therefore,

$$2V_3(x,y,z)=cx^2+(a+\frac{b^2}{c})y^2+\frac{b}{c}z^2+2bxy+2yz \qquad (3.11)$$

is suitable Lyapunov function for (2.1).
REMARK

The sum of two or more Lyapunov functions result to another Lyapunov function that satisfies the Lyapunov criteria.

3. LyApunov Function Candidates for Third Order Non-Linear Equation

Consider the third order non-linear differential equation

$$\dddot{x}+a\ddot{x}+g(\dot{x})+cx=0 \qquad (4.1)$$

where $a,c>0$ are constants. Equivalent with the form

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= -az-g(y)-cx\end{aligned}$$

We have the following theorems

Theorem 2

Suppose $ab-c>0$, and the function $g(y)$ is continuously differentiable satisfying the following; $\frac{g(y)}{y}=b>0, g(0)=0, yg(y)>0,$ then the trivial solution of the system (4.1) is asymptotically stable.

Proof

Write $G(y)=\int_0^y g(s)ds$

and the Lyapunov function analogous to (3.9) as

$$2V(x,y,z) = cax^2 + a^2y^2 + 2G(y) + z^2 + 2cxy + 2ayz \qquad (4.2)$$

$$= cax^2 + a^2y^2 + 2\int_0^y \frac{sg(s)}{s}ds + z^2 + 2cxy + 2ayz$$

$$\geq cax^2 + a^2y^2 + by^2 + z^2 + 2cxy + 2ayz$$

$$= by^2+2ayz+z^2+cax^2+2cxy+a^2y^2$$

$$= b\left(y+\frac{az}{b}\right)^2 + ca\left(x+\frac{y}{a}\right)^2 - \frac{a^2z^2}{b} - \frac{cy^2}{a} \geq 0$$

The function V is positive definite and the corresponding derivative with respect to t along (4.1) is

$$\dot{V}(x,y,z) = (cax + cy)y + (a^2y + g(y) + cx + az)z + (z + ay)(-az - g(y) - cx)$$

$= -(ayg(y) - cy^2)$
$= -(ab - c)y^2$
Therefore, imitating theorem 1, it follows that the trivial solution of (4.1) is asymptotic stable

Theorem 3
Suppose there exist $h > 0$, $q > 0$ and $ab - c = k > 0$ such that $|x| \leq h$, $|z| \leq q$, the function $g(y)$ is continuously differentiable, and satisfies the following condition; $(b - g'(y) \geq \varepsilon > 0$, $\dfrac{g(y)}{y} = b > 0$, then the trivial solution of (4.1) is asymptotic stable.

Proof
In the domain $|x| \leq h$, and $|z| \leq q$, write $G(y) = \int_0^y g(s)ds$ and the Lyapunov function analogous to (3.11) as

$$2V(x, y, z) = \frac{2b}{c}G(y) + 2xg(y) + cx^2 + ay^2 + \frac{b}{c}z^2 + 2yz \quad (4.3)$$

$= 2\frac{b}{c}\int_0^y g(s)ds + 2xg(y) + cx^2 + ay^2 + \frac{b}{c}z^2 + 2yz$
$\geq 2\dfrac{b^2}{c}y^2 + 2xg(y) + cx^2 + ay^2 + \dfrac{b}{c}z^2 + 2yz$

$$= c\left(x + \frac{g(y)}{c}\right)^2 + \frac{b}{c}\left(z + \frac{cy}{b}\right)^2 - \frac{g^2(y)}{c} - \frac{cy^2}{b}$$

which is positive definite. The derivative with respect to time is
$\dot{V}(x, y, z) = (g(y) + cx)y + \left(\dfrac{b}{c}g(y) + xg'(y) + ay + z\right)z + \left(\dfrac{b}{c}z + y\right)(-az - g(y) - cx)$
$= xzg'(y) + z^2 - \dfrac{ab}{c}z^2 - bxz$
$\leq -\left(b - g'(y)\right)|x||z| - \left(\dfrac{ab}{c} - 1\right)z^2$
$\leq -\varepsilon hq - \left(\frac{ab}{c} - 1\right)z^2$
$\leq -\dfrac{k}{c}z^2$

Therefore, the trivial solution of (4.1) is asymptotic stable.

4. Other Forms Of Third Order Non-Linear Differential Equation
Consider the third order nonlinear differential equation

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + f(x) = 0 \quad (5.1)$$

which is equivalent to the first order ordinary differential equations

$$\dot{x} = y$$

$$\dot{y} = z$$
$$\dot{z} = -az - g(y) - f(x)$$

Theorem 4

Suppose $f(x)$ and $g(y)$ are continuously differentiable, and satisfying the following conditions: $\frac{g(y)}{y} = b > 0,\quad \frac{f(x)}{x} = c > 0$ and $ab - f'(x) \geq \grave{o} > 0$, then the trivial solution of (5.1) is asymptotically stable.

Proof

Write $F(x) = \int_0^x f(\eta)d\eta \quad G(y) = \int_0^y g(s)ds$

and the Lyapunov function analogous to (3.9) is

$$2V(x,y,z) = a^2y^2 + z^2 + 2ayz + 2yf(x) + 2aF(x) + 2G(y) \tag{5.2}$$
$$= a^2y^2 + z^2 + 2ayz + 2yf(x) + 2a\int_0^x f(\eta)d\eta + 2\int_0^y g(s)ds$$
$$\geq a^2y^2 + z^2 + 2ayz + 2yf(x) + acx^2 + by^2$$
$$= a^2y^2 + 2ayz + z^2 + by^2 + 2yf(x) + acx^2$$
$$= a^2\left(y + \frac{z}{a}\right)^2 + b\left(y + \frac{f(x)}{b}\right)^2 + acx^2 - \frac{f^2(x)}{b}$$

of course, a positive definite function.

The corresponding derivative with respect to time along (5.1) is

$$\dot{V}(x,y,z) = \big(yf'(x) + af(x)\big) + \big(a^2y + az + f(x) + g(y)\big)z + (z + ay)\big(-az - g(y) - f(x)\big)$$
$$= y^2f'(x) - aby^2$$
$$= -\big(ab - f'(x)\big)y^2$$
$$= -\varepsilon y^2$$

It follows that the trivial solution of equation (5.1) is asymptotically stable.

5. The Most General Form if Third Order Non-Linear Equation

Consider the nonlinear equation

$$\dddot{x} + h(\ddot{x}) + g(\dot{x}) + f(x) = 0 \qquad (6.1)$$

equivalent to

$$\dot{x} = y$$
$$\dot{y} = z$$
$$\dot{z} = -ah(z) - g(y) - f(x)$$

Theorem 5

Suppose there exist $h > 0,\ q > 0,\ l > 0$ and $ab - c = k > 0$ such that $|x| \leq h,\ |z| \leq q,\ |y| \leq l$, the functions $g(y),\ f(x),$ and $h(z)$ continuously differentiable, and satisfies the following conditions: $h'(z)-a\geq \epsilon>0, \quad b-g'(y)\geq\delta>0$, $\dfrac{g(y)}{y}=b>0, \quad \dfrac{f(x)}{x}=c>0, \quad \dfrac{h(z)}{z}=a>0$, then the trivial solution of (6.1) is asymptotically stable.

Proof

Let $|\ x|\leq h, \quad |\ z|\leq q, \quad |\ y|\leq l$. Write

$G(y)=\int_0^y g(s)ds \quad F(x)=\int_0^x f(\eta)d\eta \quad H(z)=\int_0^z h(\gamma)d\gamma$. The Lyapunov function analogous to (3.7) as

$$2V(x,y,z)=bx^2+\frac{a^3+c}{k}y^2+2xz+\frac{2a}{k}yh(z)+\frac{2a^2x}{k}g(y)+\frac{2}{k}H(z)+\frac{2a^2}{k}F(x)$$

$$=bx^2+\frac{a^3+c}{k}y^2+2xz+\frac{2a}{k}yh(z)+\frac{2a^2x}{k}g(y)+\frac{2}{k}\int_0^z h(\gamma)d\gamma+\frac{2a^2}{k}\int_0^x f(\eta)d\eta$$

$$\geq bx^2+\frac{a^3}{k}y^2+\frac{c}{k}y^2+2xz+\frac{2a}{k}yh(z)+\frac{2a^2x}{k}g(y)+\frac{a}{k}z^2+\frac{ca^2x^2}{k}$$

$$\geq \frac{ca^2x^2}{k}+bx^2+\frac{a^3}{k}y^2+\frac{c}{k}y^2+\frac{a}{k}z^2+\frac{2a^2bxy}{k}+2xz+\frac{2a^2zy}{k}$$

$$=b\left(x+\frac{a^2y}{2}\right)^2+\left(\frac{k(a^3+c)-ba^4}{k}\right)y^2+\frac{a^2c}{k}\left(x+\frac{kz}{a^2c}\right)^2$$

$$+\frac{a^3c-k^2}{ka^2c}\left(z+\frac{a^4cy}{a^3c-k^2}\right)^2-\frac{a^2c}{k(a^3c-k^2)}y^2$$

V is positive definite and the corresponding time derivative along (6.1) is

$$\dot{V}=\left(\frac{a^2}{f(x)}+bx+\frac{a^2}{k}g(y)+z\right)y+\left(\frac{a^3y}{k}+\frac{cy}{k}+\frac{a^2xg'(y)}{k}+\frac{ah(z)}{k}\right)z$$

$$+\left(\frac{h(z)}{k}+x+\frac{ayh'(z)}{k}\right)\big(-h(z)-g(y)-f(x)\big)$$

$$=\left(\frac{a^2c-ach'(z)}{k}\right)xy+\left(\frac{ab+a^3-a^2h'(z)}{k}\right)yz+\left(\frac{a^2g'(y)-ak}{k}\right)xz+\left(\frac{a^2b-abh'(z)}{k}\right)y^2$$

$$-\frac{g(y)h(z)}{k}-\frac{f(x)h(z)}{k}-cx^2$$

$$=-\frac{ac}{k}\big(h'(z)-a\big)xy-\frac{a^2}{k}\big(h'(z)-a\big)yz-\frac{a^2}{k}\big(b-g'(y)\big)xz-\frac{ab}{k}\big(h'(z)-a\big)y^2-cx^2$$

$$\leq -\frac{ab}{k}\epsilon |x||y| - \frac{a^2}{k}\epsilon |y||z| - \frac{a^2}{k}\delta |x||z| - \frac{ab}{k}\epsilon y^2 - cx^2$$

$$\leq -\frac{ab}{k}\epsilon pl - \frac{a^2}{k}\epsilon lq - \frac{a^2}{k}\delta pq - \frac{ab}{k}\epsilon y^2 - cx^2$$

$$\leq -\frac{ab}{k}\epsilon y^2 - cx^2$$

It follows that the trivial solution of (6.1) is asymptotically stable.

6. COnclusion

The construction of Lyapunov function by stability analysis method is the fact that any positive definite quadratic form $V(x)$, there exist another positive definite $U(x)$ such that $\dot{V} = -U$ is satisfied. Therefore, Lyapunov functions enable analyzing the stability of dynamic systems described by ordinary differential equations without finding the solution of such equations. The ordinary differential equations considered in a state of equilibrium are asymptotically stable, hence stable. Although many methods for finding Lyapunov functions are available, Lyapunov stability analysis provides possible Lyapunov functions for ordinary differential equations. Lyapunov stability analysis is a good instrument to determine the qualitative behavior of the non-linear ordinary differential equations near the equilibrium point in analogy with the linear system.

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