ON THE VARIETY OF UNARY SEMIGROUPS WITH ASSOCIATE INVERSE SUBSEMIGROUP

C. Udeogu[1] and **U. E Obasi** [2]
[1,2]*Department of Mathematics, Micheal Okpara University of Agriculture, Umudike Umuahia, Abia State, Nigeria*
Email: 1999nkedozie@gmail.com, sirurchobasi@gmail.com

Abstract
*In this paper, two examples which naturally rise the question of whether the whole class of unary semigroups with an associate inverse subsemigroup is a variety of unary semigroups were presented. Due to the importance of this result to achieving $s** = s*$, a veritable tool leading to the conclusion was achieved.*

Keywords: *Unary Semigroups, Idempotent, Associate Inverse Semigroup*
MSC Subject Classification: *08A60, 6F05*

1. Introduction

In a semigroup S, an element $t \in S$ is an associate of $s \in S$ if $s = sts$ [1]. The concept of an associate inverse subsemigroup of a regular semigroup was introduced in [2] and extends the concept of an associate subgroup of a semigroup first presented in [3]. An axiomatic characterisation of a regular semigroup containing an associate inverse semigroup was established in [2]. An associate inverse subsemigroup of a regular semigroup S is a subsemigroup $S*$ of S containing a least associate $x*$ of each $x \in S$, in relation to the natural partial order $\leq$. Due to a simple characterisation of inverse semigroups in terms of the natural partial order on an arbitrary semigroup, such a semigroup $S*$ is necessarily inverse. The concept of semigroups have been discussed in [4], [5] and [6]. In [1], the authors showed that the class of regular semigroups containing an inverse subsemigroup can be considered as a variety of unary semigroups. A closer look in Example 1.1 found in [1] show that the result $s* = s***$ was wrongly derived. However, this paper will present the correct derivation and results obtained in [1]

2. Preliminaries

Theorem 1.1. [[2], Theorem 2.1] Let S be a regular semigroup. The following are equivalent:

(i) S is inverse;
(ii) For all $a \in S$, the set $\{x \in S : a = axa\}$ contains a least element with respect to the natural partial order.

Theorem 1.2. [[2] Theorem 2.5] A regular semigroup S contains an associate inverse subsemigroup if and only if it has a unary operation $x \to x*$, satisfying, for all $s, t \in S,$

(i) $s = ss*s$;
(ii) $s*t* = (s*t*)**$;
(i) $s = st*s \Rightarrow s* \leq t*$.

This characterisation extends the axiomatic characterisation of semigroups with an associate semigroup, established in [[6], Theorem 3.1].

3. Main Result

As noted in [1], some subclasses of the class of all semigroups with associate inverse subsemigroup can be defined in terms of identities and therefore, form varieties of unary semigroups. The examples given to buttress the above point were the following:

Example 3.1. Let S be a semigroup with unary operation $x \to x*$ satisfying, for all $s, t \in S$,
(1) $s = ss*s$;
(2) $s*t* = (s*t*)**$;
(3) $s*s**t*t** = t*t**s*s**$;
(4) $(st)* = t*s*$.
In what follows, it was noted in [1] that by (1) and (4), $S* = \{s* \mid s \in S\}$ is a regular semigroup of S. Also, for each $s \in S$,

$$s* = s*(s**s*) \quad (1)$$
$$= s*(ss*)* \quad (2)$$
$$= (s*(ss*)*)** \quad (3)$$
$$= (s*s**s*)* \quad (4)$$
$$= s*** \quad (5)$$

A closer look at the above derivation shows that the result it was expected to yield is $s* = s**$ and not $s* = s***$, hence, prompting our intervention.

We hereby present the right derivation at this point, then represent every other subsequent result in [1],

$$s* = s*(s**\ s*) \quad (6)$$
$$= s*\ (ss*)* \quad (7)$$
$$= (s*\ (ss*)*)** \quad (8)$$
$$= ((ss*)**s**)* \quad (9)$$
$$= ((s**\ s*)*\ s**)* \quad (10)$$
$$= ((s**s***)s**)* \quad (11)$$
$$= s*** \quad (12)$$

Moreover, for each idempotent $s*$ of $S*$, we have

$$s** = s**s***s** \quad (13)$$
$$= s**s*s** \quad [s*** = s*]$$
$$= s**s*s*s** \quad [s* \text{ is idempotent}]$$
$$= s*s**s**s* \quad [(14) \text{ and } s*** = s*]$$
$$= s*(s*s*)*s* \quad (15)$$
$$= s*s**s* \quad [s* \text{ idempotent}]$$
$$= s* \quad (16)$$

and so $s* = s*s* = s*s**$. It follows from (3) that the subsemigroup $S*$ is inverse. We show next that s∗ is the least associate of s ∈ S∗, since for any associate $t*$ of s, we have

$$s = st*s \Rightarrow s* = s*t**s* \quad (17)$$
$$\Rightarrow s** \leq t** \quad [S* \text{ inverse; theorem 1.1}]$$

$\Rightarrow\ s*\leq\ t*$ [$S*$ inverse and $s*=\ (s**)^{-1}$]
Hence S is a regular semigroup with associate inverse subsemigroup $S*=\ \{s*:s\in\ S\}$.
Observe that this class of semigroups contains the class of inverse semigroups. In fact, S is an inverse semigroup if and only if S satisfies (1)-(4) and $s**=\ s$ for all $s,t\ \in\ S$.

Example 3.2. Let S be a semigroup with a unary operation $x\ \to\ x*$ satisfying,
for all $s,t,u\ \in\ S$,
(1) $s\ =\ ss*s$;
(2) $s*t*=\ (s*t*)**$;
(3) $s*s**t*t**=\ t*t**s*s**$;
(4) $(su*t)*=\ t*u**s*$.
In example 3.1, $S*$ is an inverse an inverse subsemigroup of S. Also, for any associate $t*$ of s, we have by (4),
$s*=\ (st*s)*=\ s*t**s*$
and in example 1.1, $s*\leq\ t*$. Thus S is a regular semigroup with associate inverse subsemigroup $S*=\ \{s*:s\ \in\ S\}$.

4. Conclusion

This class of semigroups contains the class of completely simple semigroups. In fact, if S is the Rees matrix semigroup, $M(G;\ I;\ \Lambda;\ P)$ with P normalized and $(i,g,\lambda)*=\ (1,g^{-1},1)$, routine calculations shows that S satisfies (1)-(4)

References

Billhardt, B., Giraldes, E., Marques-Smith, P. and Mendes Martins, P. (2013). *The Variety of Unary Semigroups With Associate Inverse Subsemigroup,* Semigroup Forum. 89. 10.1007/s00233-013-9511-x.

Billhardt, B., Giraldes, E., Marques-Smith, P. and Mendes Martins, P. (2009). *Associate inverse subsemigroups of regular semigroups.* Semigroup Forum 79:101-118

Blyth, T.S., Giraldes, E., Paula M. O., and Marques-Smith, P. (1994). Associate subgroups of orthodox semigroups. Glasgow Math J. 36:163-171

Blyth, T. S., and Mendes Martins, P. (1997). *On associate subgroups of regular semigroups.* Communications in Algebra 25 (7):2147-2156

Howie, J.W. (1995). *Fundamentals of semigroup theory*. Claredon Press, Oxford.

Mendes Martins, P. and Petrich, M. (2008). *Unary Semigroups with an Associate Subgroup,* Communications in Algebra, 36 (6):1999-2013

SENSITIVITY ANALYSIS OF BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION ARISING IN FINANCIAL MODELLING

Hamza Abubakar,[1] **Mohammed Lawal Danrimi**[2] and **Surajo Yusuf**
[1,3]*Department of Mathematics, Isa Kaita College of Education, P.M.B. 5007 Dutsin-Ma, Katsina Nigeria*
[2]*Department of Accounting, Umaru Musa Yar'Adua University, Katsina*
Email: zeeham4u2c@yahoo.com mldanrimi.kt@gmail.com

Abstract
Option pricing theory estimates the value of an options contract by assigning a price, known as a premium, based on the calculated probability that the contract will finish in the money (ITM) at expiration. Essentially, option pricing theory provides an evaluation of an option's fair value, which traders incorporate into their strategies. Models used to price options account for variables such as underlying price (S), *strike price* (X), *risk-free interest rate* (r), *stock volatility* (σ) *and time to expiration* (T) *to theoretically value an option. These variables involve complex activity in determining the pair price of derivative that required a proper understanding of the sensitivities letters which are essential in measuring risk in option values due to the uncertainty of underlying asset price movements and profit/loss guideposts in options pricing strategies is synonymous with flying a plane without the ability to study the instruments. This paper attempts to provide simple derivations of Sensitivities (Greek) letters for European options within the Black-Scholes (BS) model. The relationship that exist between Delta, Theta, and Gamma in BS model have been derived with relatively simple and easy proofs of the BS model Greek letters.*

Keywords: *Sensitivities, Black-Scholes, option pricing, Call option, Put option, Greek letters.*

Introduction

The Black and Scholes model is the cornerstone of modern option pricing theory. This model, as well as various other option pricing models based on the Black and Scholes model, rely on the assumptions that investors know the value of the underlying asset's volatility, and that they agree on this value. These assumptions are unrealistic and very problematic, as they imply that investors agree on the value of the option, and furthermore they imply that the option is redundant. The development of options pricing theory is intimately related to notions associated with *stochastic processes.* Options as financial means can be used in many possible ways for creating various opportunities for an attractive investment. Pricing of Option is one of the many challenges in the theory of financial mathematics as it is called now. It all started in the seventies with the celebrated model of Merton, Black, and Scholes. Options are derivative contract where the holder may choose to forfeit the contract. She has the right to exercise, but not the obligation. A call option of an underlying asset gives the holder the right to buy the asset at a predetermined price (the *strike price)* at a specified time in the future. For a European option, this time point is fixed at the maturity time. The American option allows the holder to buy the asset for the strike price at any time up to the maturity date. The payoff function of the American option is presented in equation (1) as follows.

$$V_t = (S_t - X)^+ \quad (1)$$

where V_t is the payoff function, X is the strike price and S_t is the spot price of the underlying assets at the exercise date. The put option works in the opposite way of the call, allowing the holder to sell the underlying asset at the strike price. The payoff is given in equation (2) as follows.

$$V_t = (X - S_t)^+ \quad (2)$$

The celebrated Black-Scholes(Black & Scholes, 1973) model offers an elegant and effective way for option pricing and option hedging by providing an analytic solution to the option price model and its associated Greek letters, even though this model could make certain pricing bias in the realistic market (Backus *et al*., 2011). The Black-Scholes formula thus has been regarded as a benchmark for option valuation and option hedging and accepted by many financial professionals including practitioners who seek to manage their risk exposure(Kim & Kim, 2004). Typically, an options trader would use the Greek letters under the Black-Scholes framework (Black-Scholes Greeks) as a benchmark for properly adjusting option position so that all risks are acceptable(Feunou & Okou, 2019). Greek letter measures the sensitivity of an option price concerning the change in the value of a given underlying parameter such as the underlying asset's price, value, and time (Kumar, 2018). The option hedge ratio is defined as the rate of change of option price to the underlying price (Song *et al*., 2019).

In mathematical finance, the Greeks are the quantities representing the sensitivities of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The name is used because the most common of these sensitivities are often denoted by Greek letters. Collectively these have also been called the risk sensitivities, risk measures, or hedge parameters (Jeong *et al*., 2016).

Generally, the derivations of Black-Scholes Greek letters are quite mathematically involved because the computation of partial derivatives even complicated integrals are required (Chen et al., 2010). For example, the hedge ratio of Black-Scholes option's delta is commonly derived either by taking the partial derivative of the option price formula concerning an underlying price via the Chain rule (Cox et al., 1979) or instead by differentiating the original formula which expresses the option's value as a discounted risk-neutral expectation(Turner, 2010). The former needs to calculate all involved complicated partial derivative including the derivative of the standard normal distribution function, and the latter involves the derivative of an integral due to the discounted risk-neutral expectation, both are not easy to follow. This article provides simple derivations for five Greek letters of call and puts options under the Black-Scholes model framework. The proofs are succinct and easily-understood. Trading options and option strategies are based on risk factors and it can be predicted by calculating Black-Scholes and its Greeks letters.

Black-Scholes Option Pricing Model

For simplicity, and yet without any loss of generality, this article just considers that case in which the underlying asset, say stock, pays no dividends. Assume the price of the underlying stock in equation (1) and (2) follows a geometric Brownian motion as follows.

$$dS_t = \mu S_t dt + \sigma S_t W_t \tag{3}$$

where the parameters have their standard meaning as S_t denote the stock's price at time t and W_t defined as a standard Wiener process; μ and σ are the expected growth rate and the standard deviation of returns of underlying stock respectively. Given this stochastic process, the Black-Scholes option pricing formulas can be written as:

$$d_1 = \frac{In\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)}{\sqrt{\tau}} \tag{4}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{In\left(\frac{S_t}{X}\right) + 2\tau\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)}{\sqrt{2\tau\left(\frac{\sigma^2}{2}\right)}} \tag{5}$$

$$= \frac{In\left(\frac{S_t}{X}\right) + 2\tau\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)}{\sqrt{2\tau\left(\frac{\sigma^2}{2}\right)}} \tag{6}$$

From equation (4) to (6) we can deduce the following equations:

$$= Xe^{-\alpha x - \lambda^2\tau} e^{-\lambda x - \lambda^2\tau} N(d_1) - Xe^{\alpha x - \lambda^2\tau} e^{x\alpha - \alpha^2\tau} N(d_2) \tag{7}$$

$$= Xe^{x(\alpha - \lambda^2)} N(d_1) - Xe^{\lambda^2\tau} e^{\alpha^2\tau} N(d_2) \tag{8}$$

$$= Xe^{x} N(d_1) - Xe^{\tau\left(\frac{2r}{\sigma^2}\right)} N(d_2) \tag{9}$$

$$= Xe^{x} N(d_1) - Xe^{r(T-t)} N(d_2) \tag{10}$$

$$= S_t N(d_1) - Xe^{-r(T-t)} N(d_2) \tag{11}$$

Therefore, the Black-Scholes Call and Put option formula can be generated using equations (4) to (12) as follows.

$$V_c(S_t, t) = S_t N(d_1) - Xe^{-r(T-t)} N(d_2) \tag{12}$$

$$V_P(S_t, t) = S_t N(-d_1) - Xe^{-r(T-t)} N(d_2) \tag{13}$$

where V_c and V_p are the call and put option prices, respectively; S_0 , X, r and τ respectively stand for the current price of underlying stock, option's strike price, annual continuously compounded risk-free interest rate and the time to maturity. $N(\cdot)$ denotes the cumulative distribution function of standard normal.

Price Hedging Parameters

Considers the sensitivity of option price to the underlying parameters, such as asset prices, volatility, interest rates, and so on. Changes in the values of these parameters will certainly change the values of the options considerably. A portfolio consisting of options is liable to changes of these parameters and, thus, should be hedged, and the risk it is exposed to should be minimized. It is important to compute the sensitivity of options' prices to parameters such as the spot price or the volatility. The partial derivatives concerning the relevant parameters are called the Greeks: Equations can be derived for these by directly differentiating the partial differential equation concerning appropriate boundary conditions. The Greek letters *Delta* (Δ) ,*Gamma* (Γ) ,*Theta* (Θ), *rho* (ρ) and *Vega* (A) are use when measuring black-Scholes option pricing sensitivities described as follows.

i. The *Delta* (Δ) of a financial derivative is the rate of change of the option value with respect to the value of the underlying security, in symbols,

$$\Delta = \frac{\partial V}{\partial S_t}$$

ii. The *Gamma* (Γ) of a derivative is the sensitivity of option value concerning S_t , in symbols

$$\Gamma = \frac{\partial^2 \pi}{\partial S_t^2}$$

iii. The *Theta* (Θ) of a European claim with a value function $V(S_t, t)$ is

$$\Theta = \frac{\partial \pi}{\partial t}$$

iv. The *rho* (ρ) of derivative security is the rate of change of the value of the derivative security concerning the interest rate, in symbols

$$\rho = \frac{\partial \pi}{\partial \tau}$$

v. The *Vega* (A) of derivative security is the rate of change of the value of the derivative concerning the volatility of the underlying asset, in symbols.

$$\mathrm{A} = \frac{\partial \pi}{\partial \sigma}$$

Hedging is the attempt to make a portfolio value immune to small changes in the underlying asset value (or its parameters). **Hedging** against investment risk means strategically using **financial** instruments or market strategies to offset the risk of any adverse price movements. Put another way, investors **hedge** one investment by making a trade-in another. A risk reduction, therefore, will always mean a reduction in potential profits (Stowell, 2010).

Sensitivities of Black-Scholes Formula

In this section, all the proofs of Greek letters for both call and put options are provided in order. Finally, the relationship between Delta, Theta, and Gamma was provided for easy understanding of the model. To derive these Greek letters, the following lemmas are necessary and sufficient.

Lemma 1. *From the relationship between d_1 and d_2 shown in equations (13) and (14) respectively, it holds that:*

$$\frac{\partial d_2}{\partial S_0} = \frac{\partial d_1}{\partial S_0} \tag{14}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial S_0} - \sqrt{\tau} \tag{15}$$

$$\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r} \tag{16}$$

Proof: From the given relationship $d_2 = d_1 - \sigma\sqrt{\tau}$ in Equation (4). These Equations (5)-(13) are immediate.

Lemma 2. *The relationship between the values of density function d_2 and d_1 can be expressed as* $S_0 N(d_1) = Xe^{-r(T-t)} N(d_2)$

Proof: First, consider the computation of $d_1 - d_2$ as follows.

$$d_2^2 - d_1^2 = (d_2 - d_1)(d_2 + d_1) \tag{17}$$

$$= \left(-\sigma\sqrt{\tau}\right)\left(2d_1 - \sigma\sqrt{\tau}\right) \tag{18}$$

$$= \left(-\sigma\sqrt{\tau}\right)\left[\frac{2In\left(\frac{S_0}{X}\right) + 2\tau\left(r + \frac{\sigma^2}{2}\right)}{\sqrt{\tau}} - \left(\sigma\sqrt{\tau}\right)\right] \tag{19}$$

$$= 2\left[In\left(\frac{S_0}{X}\right) + r\tau\right] \tag{20}$$

Equations (14)-(20) are employed to derive equation (21). By the definition of density function in Equation (21) as follows.

$$N(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2}} \tag{21}$$

$$In\left(\frac{N(d_1)}{N(d_2)}\right) = \frac{d_1^2}{2} + \frac{d_2^2}{2} \tag{22}$$

$$= \frac{1}{2}(d_2^2 - d_1^2) \tag{23}$$

Taking the exponential form from both sides and rearranging the terms, Equation (24) can easily be achieved.

$$= -\left[In\left(\frac{S_0}{X}\right) + r\tau \right] \quad (24)$$

Derivation of Black-Scholes Model Sensitivities

Sensitivity Analysis of the option theory

The sensitivity analysis is the most important factor in the market analysis and the parameters involved in market analysis are known collectively as the '*Greeks' delta, gamma, theta*, *Vega,* and *Rho*. Technically speaking these are partial derivatives of the option pricing model (Chen et al., 2010). This means that they measure the change in the calculated option value for a given change in one of the inputs, all other inputs remaining constant. The expressions of the Greek letters can be derived in order as follows:

Proposition 1: The expressions of Greek letters for Black-Scholes Call and Put Options are as follows:

For Call Option:

$$\Delta = \frac{\partial V}{\partial S_0} = N(d_1) \quad (25)$$

$$\Theta = \frac{\partial \pi}{\partial \tau} = rXe^{-r\tau}N(d_2) - \frac{\sigma}{2\sqrt{\tau}}S_0 N(d_1) \quad (26)$$

$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{1}{S_0 \sigma \sqrt{\tau}} N(d_1) \quad (27)$$

$$\nu = \frac{\partial V_c}{\partial \sigma} = \frac{1}{\sqrt{\tau}} S_0 N(d_1) \quad (28)$$

$$\rho = \frac{\partial V_c}{\partial r} = \tau X e^{-r\tau} N(d_1) \quad (29)$$

Put-call parity relation presented in Equation (4) can be used to derive a Put option utilizing Equations (25) to (29) as follows.

$$V_P = Xe^{-r\tau} + V_C - S_t \quad (30)$$

$$\Delta = \frac{\partial V_P}{\partial S_0} = N(d_1) - 1 \quad (31)$$

$$\Theta = -\frac{\partial V_P}{\partial \tau} = -\frac{\sigma}{2\sqrt{\tau}} S_0 N(d_1) + rXe^{-r\tau} N(-d_2) \quad (32)$$

$$\Gamma = -\frac{\partial \Delta}{\partial S_0} = -\frac{1}{S_0 \sigma \sqrt{\tau}} S_0 N(d_1) \quad (33)$$

$$\nu = \frac{\partial V_P}{\partial \sigma} = \sqrt{\tau} S_0 N(d_1) \qquad (34)$$

$$\rho = \frac{\partial V_P}{\partial r} = -\tau X e^{-r\tau} N(-d_2) \qquad (35)$$

Proof: for call option valuation formula, the formula is given in Equation (5) as:

$$V_c(S_t,t) = S_t N(d_1) - rXe^{-r(T-t)} N(-d_2) \qquad (36)$$

Derivation of The Delta (Δ), Delta-hedging ratio,

$$\Delta = \frac{\partial V_C}{\partial S_t} = (d_1) \qquad (37)$$

$$= N(d_1) - S_t N(d_1) \frac{\partial d_1}{\partial S_t} - Xe^{-r(T-t)} S_t N(d_1)(d_2) \frac{\partial d_2}{\partial S_t} \quad (38)$$

$$= N(d_1) + S_t \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{1}{S_0 \sigma \sqrt{\tau}} - Xe^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{1}{S_0 \sigma \sqrt{\tau}} \qquad (39)$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi\sigma\sqrt{\tau}}} e^{-\frac{d_1^2}{2}} \left[1 - \frac{Xe^{-r\tau}}{S_t} e^{-\left(\frac{d_1 \sigma \sqrt{\tau} - \sigma^2 \tau}{2}\right)} \right] (40)$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi\sigma\sqrt{\tau}}} e^{-\frac{d_1^2}{2}} \left[1 - \frac{Xe^{-r\tau}}{S_t} e^{-\left(in\left(\frac{S_t}{X}\right) + (r + \frac{\sigma^2}{2}) \right)\tau - \frac{\sigma^2 \tau}{2}} \right] \qquad (41)$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi\sigma\sqrt{\tau}}} e^{-\frac{d_1^2}{2}} \left[1 - \frac{Xe^{-r\tau}}{S_t} e^{\left(in\left(\frac{S_t}{X}\right) + r\tau \right)} \right] \qquad (42) = N(d_1) \qquad (43)$$

Derivation of the Theta (Θ), the time decay factor, we have,

$$\Theta = -\frac{\partial V_C}{\partial \tau} = -\frac{\partial [S_t N(d_1) - rXe^{r\tau} N(-d_2)]}{\partial \tau} \qquad (44)$$

$$\Theta = -S_t N(d_1) \frac{\partial d_1}{\partial \tau} + rXe^{r\tau} \left[-rN(d_2) - N(d_2) \frac{\partial d_2}{\partial \tau} \right] \quad (45)$$

Based on Equation (15), we have,

$$\Theta = -S_t N(d_1) + rXe^{r\tau} \left[-rN(d_2) - N(d_2) \frac{\partial d_1}{\partial \tau} - \frac{1}{2} \frac{\sigma}{\sqrt{\tau}} N(d_1) \right] \qquad (46)$$

$$\Theta = \left[-S_t N(d_1) + Xe^{r\tau} N(d_2) \right] \left[\frac{\partial d_1}{\partial \tau} - Xe^{r\tau} - rN - \frac{1}{2} \frac{\sigma}{\sqrt{\tau}} N(d_2) \right] \qquad (47)$$

Based on Equations (17) to (21), we have,

$$\Theta = -rXe^{-r\tau}N(d_2) - \frac{1}{2}\frac{\sigma}{\sqrt{\tau}}N(d_1) \tag{48}$$

Derivation of Gamma (Γ), *the convexity factor*, we have,

$$\Gamma = -\frac{\partial \Delta}{\partial S_t} = -\frac{\partial N(d_1)}{S_t} \tag{49}$$

$$\Gamma = N(d_1)\frac{\partial (d_1)}{\partial S_t} \tag{50}$$

$$\Gamma = N(d_1)\left[\frac{\dfrac{\partial(In\left(\dfrac{S_t}{X}\right) + \tau(r + \dfrac{\sigma^2}{2})}{\sigma\sqrt{\tau}}}{\partial S_t}\right] \tag{51}$$

$$= \frac{1}{S_t\sigma\sqrt{\tau}}N(d_1) \tag{52}$$

Derivation of the Vega (ν), the volatility factor, we have,

$$\nu = -\frac{\partial V_C}{\partial \sigma} = \frac{\partial[S_tN(d_1) + Xe^{-r\tau}N(-d_2)]}{\partial \sigma} \tag{53}$$

$$= -S_tN\frac{\partial d_1}{\partial \sigma} - Xe^{-r\tau}N(d_2)\frac{\partial d_1}{\partial \sigma} \tag{54}$$

Based on Equation (15), we have,

$$= S_tN(d_1)\frac{\partial d_1}{\partial \sigma} - Xe^{-r\tau}N(-d_2)\left[\frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}\right] \tag{55}$$

$$= S_tN(d_1) - Xe^{-r\tau}N(-d_2)\left[\frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}Xe^{-r\tau}N(-d_2)\right] \tag{56}$$

Based on Equation (17), we have Vega as follows

$$\nu = \sqrt{\tau}S_tN(d_1) \tag{57}$$

Derivation of The Rho (ρ), the interest rate factor,

$$\rho = \frac{\partial V_C}{\partial r} = \frac{\partial[S_tN(d_1) + Xe^{-r\tau}N(-d_2)]}{\partial r} \tag{58}$$

$$= S_tN\frac{\partial d_1}{\partial r} - Xe^{-r\tau}\left[N(d_2)\frac{\partial d_2}{\partial r} - \tau N(d_2)\right] \tag{59}$$

Based on Equation (16), we have,

$$= \left[S_t N(d_1) - Xe^{-r\tau} N(d_2) \right] \left[\frac{\partial d_2}{\partial r} - \tau Xe^{-r\tau} N(d_2) \right] \quad (60)$$

Based on Equation (15), we have,

$$= \tau Xe^{-r\tau} N(d_2) \quad (61)$$

Relationship Between Delta, Theta and Gamma

From the Black-Scholes differential equation, we can see a useful relationship between *Delta*, *Theta*, *and* (Hruška, 2015). Delta, Theta, and Gamma are defined as derivatives considered in the Black-Scholes differential equation.

The parity relation for the European option is defined in Equation (30), then employ the above results for the call option, we can immediately obtain the Greek letters for the put option as demonstrated in Equations (30)-(35). Finally, the relationship between Delta, Theta, and Gamma satisfies the well-known Black-Scholes partial differential equation has been presented. It is shown in the following Corollary.

Corollary 1 *With proven Proposition, Delta, Theta, and Gamma satisfies the Black-Scholes partial differential*

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + rS_t \frac{\partial V}{\partial \tau} = rV \quad (62)$$

That is,

$$\Theta = rS_t\Delta + \frac{1}{2}\sigma^2 S_t^2 \Gamma = rV \quad (63)$$

$$\frac{\partial V_C}{\partial S_t} - \frac{\partial V_P}{\partial S_t} = 1 \quad (64)$$

$$\frac{\partial^2 V_C}{\partial S_t^2} - \frac{\partial^2 V_P}{\partial S_t^2} = \frac{1}{S_t \sigma \sqrt{\tau}} N(d_1) \quad (65)$$

Therefore,

$$Gamma_{Call\ option} = Gamma_{Put\ option} \quad (66)$$

where the parameter V denotes the value of the option, either C for a call option or P for Put options.

Conclusion

The main factors in option sensitivity and risk are reflected in the Black-Scholes option pricing model. The Sensitivities in Black-Scholes option pricing modelling help to provide important measurements of an option position's risks and potential rewards. Once you have a clear understanding of the basics, you can begin to apply this to your current strategies. It is not enough to just know the total capital at risk in an options position. To understand the probability of a trade making money, it is essential to be able to determine a variety of risk-exposure measurements. Given any impression that option valuers continuously balance their investment to maintain neutrality was wrong, and so on as would be suggested by the continuous mathematical derivation and proofs in this paper. In reality, transaction costs make frequent balancing expensive. Rather than trying to reduce risks that are involved, option pricing usually concentrates on assessing risks and deciding whether they are acceptable. Traders tend to use Delta, Gamma, and Vega measures to quantify the different aspects of risk in their investment.

The Greek letters are used to understand to identify the market price fluctuation or simply it is used to calculate risk sensitivities towards price changes. It educates the investors on how to behave in the options market. If a firm grasp on your Greeks will help you judge what option is the best to trade, based on your outlook for the underlying. If you don't contend with the Greeks, though, you could be flying into your next options trade blind.

References

Backus, D. K., Foresi, S., & Wu, L. (2011). Accounting for Biases in Black-Scholes. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.585623

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*. https://doi.org/10.1086/260062

Chen, H.-Y., Lee, C.-F., & Shih, W. (2010). Derivations and Applications of Greek Letters: Review and Integration. In *Handbook of Quantitative Finance and Risk Management*. https://doi.org/10.1007/978-0-387-77117-5_33

Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*. https://doi.org/10.1016/0304-405X(79)90015-1

Feunou, B., & Okou, C. (2019). Good Volatility, Bad Volatility, and Option Pricing. *Journal of Financial and Quantitative Analysis*. https://doi.org/10.1017/S0022109018000777

Hruška, J. (2015). Delta-gamma-theta Hedging of Crude Oil Asian Options. *Acta Universitatis Agriculturae et Silviculturae Mendelianae Brunensis*. https://doi.org/10.11118/201563061897

Jeong, D., Yoo, M., & Kim, J. (2016). Accurate and Efficient Computations of the Greeks for Options Near Expiry Using the Black-Scholes Equations. *Discrete Dynamics in Nature and Society*. https://doi.org/10.1155/2016/1586786

Kim, I. J., & Kim, S. (2004). Empirical comparison of alternative stochastic volatility option pricing models: Evidence from Korean KOSPI 200 index options market. *Pacific Basin Finance Journal*. https://doi.org/10.1016/S0927-538X(03)00042-8

Kumar, A. (2018). A STUDY ON RISK HEDGING STRATEGY: EFFICACY OF OPTION GREEKS. In *Abhinav National Monthly Refereed Journal of Research in Commerce & Management*.

Song, H., Han, S. K., Jeong, S. H., Lee, H. S., & Oh, K. J. (2019). Using genetic algorithms to develop a dynamic guaranteed option hedge system. *Sustainability (Switzerland)*. https://doi.org/10.3390/su11154100

Stowell, D. P. (2010). Investment Banks, Hedge Funds, and Private Equity. In *Investment Banks, Hedge Funds, and Private Equity*. https://doi.org/10.1016/C2009-0-19176-6

Turner, E. (2010). The black-scholes model and extensions. *Working Paper*.