COEFFICIENT PROPERTIES OF A K-UNIFORMLY CONVEX FUNCTIONS ASSOCIATED WITH A Q-DERIVATIVE OPERATOR

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Abstract
The work investigated some properties of a uniformly convex functions associated with a q-derivative operator. Properties studied include growth and distortion theorem of the subclass under consideration. The results showed that the operator generalized the coefficients of the subclass.
KEYWORDS: q-number, q-symmetric derivative, uniformly convex functions,

1.1 INTRODUCTION

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1.1)$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$ in the complex plane with the usual normalization $f(0) = f'(0) - 1$. Here S denotes the subclass of A consisting of analytic and univalent functions.

The q-derivative operator is a linear operator with a wide range of applications in Fractal, dynamical systems, quantum groups and q-deformed super-algebras, see [1,2] .Study of various classes of analytic functions are made possible using the concept of q-calculus. Other areas of applications are hypergeometric functions, complex variables [3,4].

Definition 1.1 [1]

Let $q \in (0,1)$ and let $\lambda \in C$. The $q-number$, denoted $[\lambda]_q$,we define as

$$[\lambda]_q = \frac{1-q^{\lambda}}{1-q} \qquad 1.2$$

In the case when $\lambda = n \in \mathrm{N}$,we obtain $[\lambda]_q = 1 + q + q^2 + \cdots + q^{n-1}$, and when $q \to 1^-$ then $[n]_q = n$. The symmetric $q-number$,denoted

$$[\tilde{n}]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad 1.3$$

Definition 1.2 [1]

The $q-derivative$ of a function f, defined on a subset of C, is given by

$(D_q f)(z)=\dfrac{f(z)-f(qz)}{(1-q)z}$, for $z\neq 0$ and equal to $f'(0)$ if $z=0$.

Given the power series $f(z)=z+a_2z^2+\cdots$

then

$$(D_q f)(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1} \qquad 1.4$$

Definition 1.3 [1].

The symmetric $q-derivative$ $\tilde{D}_q f$ of a function is defined as follows:

$(\tilde{D}_q f)(z)=\dfrac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z}$ for $z=0$, and equal to $f'(0)$ for $z=0$.

when $f(z)=z+a_2z^2+\cdots$ then

$$(\tilde{D}_q f)(z)=1+\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1} \qquad 1.5$$

Definition 1.5 [1].

Let $0\leq k<\infty$ and $0\leq\alpha<1$. By $k-ST_q(\alpha)$ we denote the class of functions $f\in A$ satisfying the condition

$$\mathrm{Re}\left(\frac{z(\tilde{D}_q f(z))}{f(z)}\right)>k\left|\frac{z(\tilde{D}_q f(z))}{f(z)}-1\right|+\alpha \qquad (z\in D) \qquad 1.6$$

Let P be the Caratheodory class of functions with positive real part consisting of all functions p analytic in D satisfying $p(0)=1$, and $\Re(p(z))>0$. Letting $p(z)=\dfrac{z(\tilde{D}_q f)(z)}{f(z)}$

condition (1.7) may be written in the form $\Re p(z)>k\left|p(z)-1\right|+\alpha$ $(z\in D)$ or $p\prec p_{k,\alpha}$, is a function with a positive real part, that maps the unit disk onto a domain $\Omega_{k,\alpha}$ described by the inequality $\Re p(z)>k\left|p(z)-1\right|+\alpha$. Note that $\Omega_{k,\alpha}$ is a domain bounded by a conic section ,symmetric about real axis and contained in a right half plane.[5]

The representation of $p_{k,\alpha}=1+P_1z+P_2z^2+\cdots$

Definition 1.6 (k-uniformly convex function with respect to q-symmetric derivative operator).

Let $0 \le k < \infty$ and $0 \le \beta < 1$. By $k - \widetilde{U}CV_q(\beta)$ we denote the class of functions $f \in A$ satisfying the condition

$$\mathrm{Re}\left(\frac{z(\widetilde{D}_q f)'(z)}{(\widetilde{D}_q f)(z)} + 1 \right) > k \left| \frac{z(\widetilde{D}_q f)'(z)}{(\widetilde{D}_q f)(z)} \right| + \beta \qquad , z \in E \tag{1.8}$$

The method adopted to obtain the coefficient inequality for the class of $k - \widetilde{U}CV_q$ is due to [1]

2.1 COEFFICIENTS ESTIMATES

Theorem 2.1

Let $0 < q < 1$ and $f \in S$ be given by (1.6) . If the inequality

$$\sum_{n=2}^{\infty} [\tilde{n}]_q \left[n(k+1) - k + \beta \right] |a_n| < 1 - \beta$$

(2.1)

holds true for some k $(0 \le k < \infty)$ and β $(0 \le \beta < 1)$, then $k - \widetilde{U}CV_q(\beta)$.

PROOF.

Since $k - \widetilde{U}CV_q(\beta)$ then definition (1.6) is satisfied. Thus, using the fact that $\mathrm{Re}\, f(z) > \beta$ implies that $|f(z) - 1| < 1 - \beta$ on definition (1.6) results in the inequality

$$(k+1) \left| \frac{z(D_q)'(z)}{(D_q f)(z)} \right| < 1 - \beta$$

(2.2)

Given that $(\widetilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} [\tilde{n}]_q a_n z^{n-1}$ and $(D_q)'(z) = \sum_{n=2}^{\infty} [\tilde{n}]_q a_n (n-1) z^{n-2}$

Substituting in equation (2.2) yields

$$(k+1) \left| \frac{\sum_{n=2}^{\infty} [n]_q a_n (n-1) z^{n-2}}{1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}} \right| < (k+1) \left| \frac{\sum_{n=2}^{\infty} [n]_q a_n (n-1) z^{n-2}}{1 - \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}} \right| < 1 - \beta$$

(2.3)

Upon letting $z \to 1^-$ and after some easy computation on the right side of (2.3) gives

$$\sum_{n=2}^{\infty}[\tilde{n}]_q\left[n(k+1)-k+\beta\right]a_n\Big| < 1-\beta$$

(2.4)

Theorem 2.2 will dwell on finding the necessary and sufficient condition for a function in the class $k-\tilde{U}CV_q(\beta)$.The method applied here is due to [1].

Theorem 2.2

Let $0 \le k < \infty$, $0 < q < 1$. and $0 < \alpha < 1$. A necessary and sufficient condition for f of the form

$f(z) = z - a_2 z^2 - \cdots (a_n \ge 0)$ to be in the class $k-\tilde{U}CV_q(\beta)$ is that

$$\sum_{n=2}^{\infty}[\tilde{n}]_q\left[n(k+1)-k+\beta\right]a_n \le 1-\beta \qquad (2.5)$$

The result is sharp, equality holds for the functions f given by

$$f(z) = z - \frac{1-\beta}{[\tilde{n}]_q\left[n(k+1)-k+\beta\right]} z^n$$

Proof:

If $f \in k-UCV_q(\alpha)$ then from the right-side of inequality (2.3) in theorem (2.1) one can write

$$\left|\frac{1-\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1}}{1-\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1}}\right| > (k+1)\left|\frac{\sum_{n=2}^{\infty}[\tilde{n}]_q a_n (n-1) z^{n-2}}{1-\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1}}\right| \qquad (2.6)$$

$$\frac{1-\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1} - \beta - \sum_{n=2}^{\infty}[\tilde{n}]_q a_n \beta z^{n-1}}{1-\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1}} \ge \frac{\sum_{n=2}^{\infty}[\tilde{n}]_q a_n (k+1)(n-1) z^{n-2}}{1-\sum_{n=2}^{\infty}[\tilde{n}]_q a_n z^{n-1}} \qquad (2.7)$$

Now, clearing the denominator of (2.7) and choosing values of z on the real axis so that $\tilde{D}_q f(z)$ is real and letting $z \to 1^-$ through the real values gives

$$\sum_{n=2}^{\infty}[\tilde{n}]_q\left[n(k+1)-k+\beta\right]a_n \le 1-\beta \qquad (2.8)$$

which is the required result.

The next theorem gives the growth properties of the class $k-\tilde{U}CV_q(\beta)$

Theorem 2.3

Let $0 \le k < \infty$, $0 < q < 1$. Let the function f defined by $f(z) = z - a_2 z^2 - \cdots$ $(a_n \ge 0)$ be in the class $k-\widetilde{U}CV_q(\beta)$,Then for $|z| = r < 1$ it holds

$$r - \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} r^2 \le |f(z)| \le r + \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} r^2 \quad (2.9)$$

Equality in (2.09) holds true for the functions given by

$$f(z) = z + \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} z^2$$

(2.10)

Proof. Given that $k-\widetilde{U}CV_q(\beta)$ result of theorem (2.2) can be expressed in the form

$$\frac{(q^2+1)}{q}\left[2(k+1)-k+\beta\right]\sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty}[n]_q\left[n(k+1)-k+\beta\right]|a| \le 1-\beta$$

which yields $\sum_{n=2}^{\infty} a_n \le \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]}$.

(2.11)

Therefore,

$$|f(z)| \le |z| + \sum a_n |z|^n \le r + \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} r^2$$

and

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^n \ge r - \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} r^2.$$

Letting $r \to 1^-$ gives the required results.

Theorem 2.4

Let $0 \le k < \infty, 0 < q < 1$ and $0 \le \alpha < 1$.Let the function f with the representation $f(z) = z - a_n z^2 - \cdots$ $(a_n \ge 0)$ be a member of the class $k-\widetilde{U}CV_q(\beta)$. Then for $|z| = r < 1$.

$$1 - \frac{2q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} \le |f'(z)| \le 1 + \frac{2q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} \quad (2.12)$$

Proof.

Since $f(z) = z + \frac{q(1-\beta)}{(q^2+1)[2(k+1)-k+\beta]} z^2$,

Differentiating f and applying triangle inequality for the modulus will yield

$$|f'(z)| \le 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \le 1 + r\sum_{n=2}^{\infty} n a_n \quad (2.13)$$

and $$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq 1 - r\sum_{n=2}^{\infty} na_n \qquad (2.14).$$

Following (2.13) ,(2.14) and consequences of (2.11) the equation in (2.12) is obtained.

3.0 CONCLUSION

The class of k-uniformly convex and starlike functions have been studied by several authors such as [1] , [2] to mention but a few. These researchers used different differential and integral operators to obtain various bounds. Specifically, q-symmetric derivative operator was used by [1] to obtain the coefficient bounds,growth theorem**,** and proved the necessary and sufficient condition for a class of k-uniformly starlike function associated with q-symmetric derivative operator . Following a method due to [1] the coefficient estimates of k-uniformly convex functions and some results were obtained using simple partial differential calculus, binomial theorem, q-symmetric derivative operator. The operator in a similar pattern generalizes the coefficient properties of this subclass, see [6].

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