Abacus (Mathematics Science Series) Vol. 49, No 1, April. 2022

On The Primitivity And Solubility Of Dihedral Groups Of Degree 4p That Are Not p-Groups By Numerical Approach

J. O. Ben[1], S. Hamma[2] and M. I. Bello[3]

[1]Department of Mathematics and Statistics, Federal University, Wukari , Taraba State, Nigeria. [2,3]Department of Mathematical Sciences, Abubaka Tafawa Balewa University, Bauchi , Bauchi State, Nigeria. benjohnsonnig@yahoo,com

ABSTRACT*: Let G be a dihedral group of degree 4p, p an odd prime. We investigated the primitivity and solubility of G using the concepts of Group Theory. We applied the computational group theory (GAP) to enhance and validate our work.*

KEYWORDS*:*
Permutation Groups, Primitivity, Solubility, Dihedral Group and p-Groups.

1. Introduction

In mathematics, dihedral groups denoted by D_n are the groups of symmetries of a regular polygon, which consist of rotations and reflections (Cameron, 2013). Dihedral groups are good examples of finite permutation groups and have series of applications especially in sciences and engineering.

Conventionally, we write

$$D_n = \langle r,\ f \mid r^n = f^2 = 1,\ fr = r^{n-1}f\ =\ r^{-1}f\rangle$$

And we say that D_n is the group generated by the elements r and f subject to the conditions

$$r^n = f^2 = 1;\ f\,r = r^{n-1}f\ =\ r^{-1}f,$$

and the 2n distinct elements of D_n are

$$1, r, r^2, \ldots, r^{n-1}, f, r\,f, r^2f, \ldots, r^{n-1}f.$$

Here r is a rotation about the centre of the polygon through angle $2\pi^c/n$ and f is a reflection about an axis of symmetry of the polygon.

According to (Cameron, 2013) a group is said to solvable if it has a normal series

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\} \ldots \ldots \ldots (1)$$

Such that each of its factor group

$$\frac{G_i}{G_{i+1}}, \qquad 0 \leq i \leq n$$

is an abelian group.

The above series (1) then is referred to as a solvable series of G.

When a group G acts on a set Ω, a typical point α is moved by elements of G to various other points. The set of these images is called the orbit of α under G and we denote it by $\alpha^G \coloneqq \{ \alpha^g | \, g \in G \}$. A group G acting on a set Ω is said to be transitive on Ω if it has one orbit which is equal to the entire set, that is $\alpha^G = \Omega$ for all $\alpha \in \Omega$. Equivalently, G is transitive if for every pair of point $\alpha, \ \beta \ \in \ \Omega$ there exists $g \in \Omega$ such that $\alpha^g = \beta$. If a group is not transitive then it is called intransitive.

A permutation group G acting on a non empty set Ω is called primitive if G acts transitively on Ω and G preserves no non trivial partition of Ω where non-trivial partition means a partition that is not a partition into singleton set or partition into one set Ω. In other words, a group G is said to be primitive on a set Ω if the only sets of imprimitivity are the trivial ones otherwise G is imprimitive on Ω. Transitive and Primitive finite permutation groups can be thought of as the building blocks of finite permutation groups, and questions about finite permutation groups can often be reduced to the primitive case (Fawcett, 2009).

Transitive and primitive permutation groups of special degrees have received much attention in the academic research space. Transitive and primitive p-subgroup of dihedral groups of degree pq, where p, q are any two distinct odd prime numbers were considered by Hamma and Haruna (2009), while more recently, Audu and Hamma (2010) discussed the transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most p^2 using the concept of p-groups. They used the standard program – The Groups, Algorithms and Programming (GAP) to validate their results while Hamma and Aliyu, (2010) worked “On transitive and primitive dihedral groups of degree $at\ most\ 2^r$ $(r \geq 2)$". Also, Hamma and Mohammed (2012) discussed the transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most p^3. They proved theorems and validate them using the Groups, Algorithms and Programming (GAP). Cai and Zhang, (2015) presented “A Note on Primitive Permutation Groups of Prime Power Degree”. Fengler, (2018) in his published work explored on “Transitive Permutation Groups of Prime Degree” Studies concerning solubility include: Thanos (2006) who proved that $If\ |G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime number is solvable; Bello *et al.* (2017) used the concept of p-groups to construct locally solvable groups using two permutation groups by Wreath products. Gandi and Hamma, (2019) who investigated solvable and Nilpotent concepts on Dihedral Groups of an even degree regular polygon.

In this paper, we obtained detailed description of the unique structure of dihedral groups of degree 4p that are not p-groups and investigated their primitivity and solubility using numerical approach.

In Section 2 we give some basic definitions, concepts and results which are required here. The main result of this paper covering all the dihedral groups of degree 4p are stated in Section 3.

2. Preliminaries

The following are some fundamental definitions and results that will be required.

2.1 p-Groups

2.1.1 p-Group (Sylow, 1872)

A finite group G is said to be a p-group if its order is a power of p, where p is prime.

2.1.2 p-Subgroup (Sylow, 1872)

A subgroup H of a group G $(H \leq G)$ is called a p-subgroup G if H itself is a p-group, this is, $|H| = p^r$, for some $r \geq 0$ for all $H \in G$.

2.1.3 Sylow p-Subgroup (Sylow, 1872)

Let G be a group. If G is finite and $|G| = p^r m,\ r \geq 1$ where p and m are co-prime and $H \leq G$ such that $|H| = p^r$, we say that H is a Sylow p-subgroup of G.

Clearly, a Sylow p-subgroup is maximal among all p-subgroups of G.

According to Sylow theorem, if n divides $|G|$, then G has a subgroup of order n provided that n is a prime power.

This result is a sufficient condition for a subgroup to exist and is one of the basic tools in modern finite group theory.

2.1.4 Sylow Theorems (Sylow, 1872)

Let G be a finite group of order n.

1. If p is a prime such that p^k is a divisor of |G| for some $k \geq 0$, then G contains a subgroup of order p^k.
2. All Sylow p-subgroups of G are conjugate, and any p-subgroup of G is contained in a Sylow p-subgroup.
3. Let $n = mp^k$, with (m, p) = 1, and let n_p be the number of Sylow p-subgroups of G. Then $n_p \mid m$ and $n_p \equiv 1 \pmod p$.

2.2 Transitivity

Let G be a permutation group on Ω, where Ω is a finite set.

1. We say that G is $\frac{1}{2}$ – transitive if all the orbits have the same size.
2. Suppose that G has just one orbit Ω. then for all r ϵ Ω, $r^G = \Omega$ and as such for any $\alpha, \beta \epsilon \Omega$ there exists $g \epsilon G$ such that $\alpha^g = \beta$, and G is said to be transitive on Ω
3. The group G is said to be k-fold transitive (or, simply k-transitive) on Ω if, for any sequences

   $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_i \neq \alpha_j$ when $i \neq j$; $\beta_1, \beta_2, \ldots, \beta_k$ such that $\beta_1 \neq \beta_j$ when $i \neq j$ of k elements of Ω, there exists $g \epsilon$ G such that

$$\alpha_i^g = \beta_i \text{ for } 1 \leq i \leq k.$$

Thus for $k = 2$ we have that for $\alpha_1, \alpha_2, \beta_1, \beta_2$ in Ω with $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ there exists $g \in G$ such that;

$$\alpha_1^g = \beta_1,\ \alpha_2^g = \beta_2$$

and we say that G is doubly transitive.

If $k \geq 2$ then $k-$ transitivity implies $(k-1)-$ transitively.

4. Let G act on itself by right multiplication. Then, $\Omega = G$. If $\alpha = x,\ \beta = y$ in Ω and we take $g = x^{-1}y$; then

$$\alpha^g = x\,(x^{-1}y) = y = \beta.$$

and so G is transitive.

Let $H \leq G$ and let G act on right cosets of H in G.

Then G is transitive on $\Omega := (G : H)$. For if $\alpha, \beta \in \Omega$,

then $\alpha = H_x$, $\beta = H_y$ for some $x, y \in G$, and if we take $g := x^{-1}y$ then we have

$$\alpha^g = (Hx)x^{-1}y = Hy = \beta$$

2.2.5 Lemma

Let G be a dihedral group of any order, then G is transitive.

Proof

For given α_i, α_j as any two vertices of the regular polygon with $i < j$, we readily see that $(\alpha_1\alpha_2 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_n)^{j-i}$ is the rotation about the centre of the polygon through angle $2\pi/n$, (where n is the number of edges of the polygon) which takes α to αj . As such G is transitive.

2.3 Primitivity

A permutation group G acting on a non empty set Ω is said to be primitive on a set Ω if and only if it preserves the trivial block system otherwise G is imprimitive on Ω. For example, the group

$S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is primitive as $\{1,2\}^{(1\,2\,3)} = \{2,3\}$ implying that $\Delta^g \neq \Delta$ and $\Delta^g \cap \Delta \neq \emptyset$ for $\Delta = \{1,2\}$.

On the other hand a subset Δ of Ω is said to be a set of imprimitivity for the action of G on Ω, if for each $g \in G$, either $\Delta^g = \Delta$ or Δ^g and Δ are disjoint. In particular, Ω itself, the 1-element subsets of Ω and the empty set are obviously sets of imprimitivity which are called trivial set of imprimitivity.

The group of symmetry $D_4 = \{(1), (1234), (13)(24), (1432), (13), (24), (12)(34), (14)(23)\}$ of the square with vertices 1,2,3,4 is imprimitive. For take $G_1 = \{(1), (24)\}$. Let $H = \{(1), (13), (24), (13)(24)\}$ which is a normal subgroup of G. Then H is a group greater than G_1, but not equal to G.

2.3.1 Theorem (Passman, 1968)

Let G be a transitive permutation group of prime degree on the set **Ω**. Then G is primitive

Proof

First and foremost, G been transitive, permutes the sets of imprimitivity where all the sets Ω_i have the same order. But $\Omega = \cup |\Omega_i|$, Ω_i being the sets of imprimitivity. As $|\Omega|$ is prime we have that either each $|\Omega_i| = 1 \ or \ \Omega$ or the only sets of imprimitivity. So, G is primitive.

2.3.2 Theorem (Passman, 1968).

Let G be a non-trivial transitive permutation group on Ω. Then G is primitive if and only if $G_\alpha, \alpha \in \Omega$ is a maximal subgroup of G or equivalently G is imprimitive if and only if there is a subgroup H of G properly lying between $G_\alpha(\alpha \in \Omega)$ and G.

Proof

Let the group G be imprimitive and Ψ be a non-trivial subset of the imprimitivity of G.
Let $H = \langle g \in G | \Psi^g = \Psi\rangle$
It is obvious H is a proper subgroup of G because $\Psi \subset \Omega$ and G is transitive.
Now choose $\alpha \in \Psi$. If $g \in G$ then $\alpha \in \Psi \cap \Psi^g$ and so $\Psi = \Psi^g$.
Hence $G_\alpha \le H \le G$
Since $|\Psi| \neq 1$, choose $\beta \in \Psi$ such that $\beta \neq \alpha$. By transitivity of G, there exists some $h \in G$ with $\alpha^h = \beta$ so that $h \in G_\alpha$. Now $\beta \in \Psi \cap \Psi^h$, so $\Psi = \Psi^h$ and $h \in H - G_\alpha$. Thus $H \neq G_\alpha$
Conversely, suppose that $G_\alpha < H < G$ for some subgroup H.
Let $\Psi = \alpha^H$. Since $H > G_\alpha |\Psi| \neq 1$
Now if $\Psi = \Omega$, then H is transitive on Ω and hence $|\Omega| = |H: G_\alpha|$ showing that H = G, a contradiction.
Hence, $\Psi \neq \Omega$
Now we shall show that Ψ is a subset of imprimitivity of G.
Let $g \in G$ and $\beta \in \Psi \cap \Psi^g$ then $\beta = \alpha^h = \alpha^{h\,g}$ for some $h, h \in H.$
Hence $\alpha^{h\,gh^{-1}} = \alpha.$ so $h\,gh^{-1} \in G_\alpha < H$
This shows that $g \in H$
Thus $\Psi = \Psi^g$. Hence Ψ is a non-trivial subset of imprimitivity
So G is imprimitive.

2.4 Solubility

2.4.1 Definition

A finite group will be called Soluble if and only if it contains a normal series such that all the quotients are abelian groups.

2.4.2 Theorem (Cameron, 2013)

A group G is solvable if and only if it has a solvable series.

Proof

Suppose G is solvable. Then by the definition of "solvable," in the derived series of commutator subgroups we have $G^{(n)} = (1)$, for some $n \in N$. By Theorem 3.3.18, in the series $G > G^{(1)} > G^{(2)} > ... > G^{(n)} = (1)$, we have that $G^{(i+1)}$ is normal in $G^{(i)}$ and $G^{(i)}/G^{(i+1)}$) is abelian. So the series is subnormal (because each subgroup is normal in each previous subgroup) and is also solvable (since the quotient groups are abelian).

Now suppose $G = G_0 > G_l > ... > G_n = (l)$ is s solvable series. Then G_i/G_{i+1} is abelian (by definition of solvable series) for $0 \leq i \leq n - l$. By theorem 3.3.18, $G_{i+l} > (G_i)'$ for $0 \leq i \leq n - 1$. Since in the derived series of commutator subgroups we have $G > G^{(1)} > G^{(2)} > ... > G^{(n)}$, then

$G_1 > G_0' = G' = G^{(1)}$

$G_2 > G_1' = (G^{(1)})' = G^{(2)}$

$G_3 > G_2' = (G^{(2)})' = G^{(3)}$

$G_{i+1} > G'_i = (G^{(i)})' = G^{(i+1)}$

$G_n > G'_{n+1} = (G^{(n-1)})' = G^{(n)}$

But $G_n = (1)$ so it must be that $G^{(n)} = (1)$ and G is solvable.

2.4.3 Corollary (Thonas, 2006)

If G has only one p-Sylow subgroup H then H is normal.

2.4.4 Corollary (Thonas, 2006)

If $H \trianglelefteq G$ and $\left|\frac{G}{H}\right| = p$ or p^2 then $\frac{G}{H}$ is abelian

2.4.5 Corollary (Thonas, 2006)

Let G be a finite group and H a Sylow p-subgroup of G. Then H is the only Sylow p-subgroup of G if and only if H is normal in G.

Proof:

By Sylow theorem, the Sylow p-subgroups of G are the elements of the sets $\{g^{-1}Hg \mid g \in G\}$ and this reduces to a singleton set if and only if $g^{-1}Hg = H$ for all $g \in G$; that is precisely when H is normal in G.

2.4.6 Proposition (Thonas, 2006)

Suppose G is a solvable group and H is a subgroup of G that is, $H \leq G$. Then

1. H is solvable.
2. If $H \lhd G$, then G/H is solvable.

Proof

Start from a series with abelian slices. G: $G_0 \rhd G_l \rhd ... \rhd G_n = (l)$ Then $H = H \cap G_0 \rhd H \cap G_1 \rhd … \rhd H \cap G_n = \{1\}$. When H is normal, we use the canonical projection π: $G \to G/$

H to get G/ H = π (G_0) $\rhd \ldots \pi$(G_n) = {1} ; the quotients are abelian as well, so G / H is still solvable.

2.4.7 Theorem (Thanos, 2006)

If $|G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime is solvable.

Proof. By induction on *k*.

1st Step. For k = 1 our group is a cyclic group of prime order thus it is solvable by definition.

2nd step. Let the statement hold for all $n \leq k$.

3d Step. We will prove that it holds for k = n + 1. By corollary 3 since *G* is a p-group *Z-(G)*≠ $\{e\}$. Also *Z(G)* is a normal subgroup of G and *Z(G)* is abelian. Thus Z (G) is solvable. Now G/Z(G) is again a p-group or trivial. If it is trivial then *G* = *Z(G)* thus G is abelian hence it is solvable. If it is not trivial then $|G/Z(G)| \leq p^n$. So by the -inductive step it is solvable. Using the tool theorem G is also solvable and we are done.

2.4.8 Theorem (Cameron, 2013)

If *G* is a group and *H* is a normal subgroup of *G* such that *H* is solvable and *G*/*H* is solvable then *G* is solvable.

3. Results and Discussion

Throughout this section, $\Omega = \{\ 1,2,3, \ldots \ldots, $4p} where p is an odd prime number.

3.1 Theorem (Main Result)

Let *G* be a dihedral group of degree 4p, where p is an odd prime number. Then *G* is (i) imprimitive and (ii) soluble.

Proof:

That G is transitive follows easily from Lemma 2.2.5. Next name the vertices of *G* as 1,2,3,…,4p and let *l* be the line of symmetry joining the middle of the vertices 1 and 4p with the middle of the vertices $\frac{4p}{2}$ *and* $\frac{4p+4}{2}$ *so that* $\alpha = (1, 4p)(2, 4p-1)(3, 4p-2) \ldots \ldots \left(\frac{4p}{2}, \frac{4p+4}{2}\right)$ is the reflection in l (as in figure 2).

Then $G_{(1)} = \left\{(1), (2, 4p-1)(3, 4p-2) \ldots \left(\frac{4p}{2}, \frac{4p+4}{2}\right)\right\}$ is the stabilizer of the point 1. We readily see that $G_{(1)}$ is a non-identity proper subgroup of *G* which has $H = \left\{(1),\ (1, 4p), (2, 4p-1), (3, 4p-2), \ldots, \left(\frac{4p}{2}, \frac{4p+4}{2}\right),\ \alpha\right\}$ as a subgroup properly lying between $G_{(1)}$ and *G*. that is, $G_{(1)} \leq H \leq G$. Thus by virtue of Theorem 3.3.2, *G* is imprimitive, proving (1).

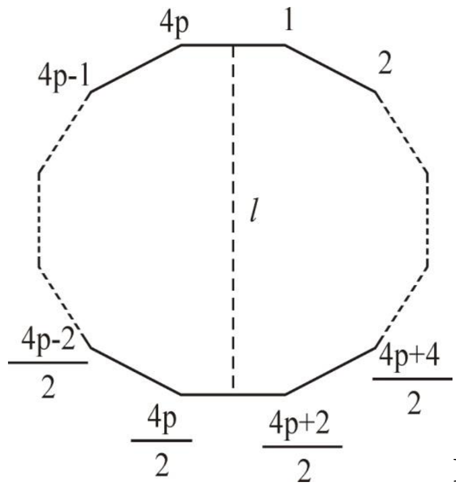

Figure 2

(ii) Now, the order $|G| = 2(4p) = 8p$.
Case 1: $p = 3$ or $p > 7$
Let $N_p(G)$ be the number of Sylow p-subgroups of the group G.
By Sylow theorem 2.1.4, we have

$$N_p(G) \equiv 1 \pmod p \text{ and } N_p(G) \mid 8.$$

It follows from this constraints that we have $N_p(G) = 1$.
Let $K = Syl_p(G)$ be the Sylow p-subgroup of G. Then $K \leq G$ with $|K| = p$. K is unique and it's normal in G by corollary 2.4.5. Since $|K| = p$, we have that K is a p-Group and by theorem 2.4.7 is Solvable. Also $|G{:}K| = 2^3$ implies that G/K is a p-Group hence Solvable by theorem 2.4.7. By theorem 2.4.8, we have that G is solvable.
Case 2: $p = 3$(where $N_p = 1$ or 4) or 7 (where $N_p = 1$ or 8)

For $p = \{3, 7\}$, $|G| = 24$ and 56 respectively and since the smallest simple non-abelian group has order 60, the group G must also be solvable in these exceptional cases.

3.2 Primitivity and Solubility of Dihedral Group of Degree 4p (p = 5)

Let $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$. Let $G = D_{20}$, the dihedral group of degree 20 acting on Ω. G is (i) imprimitive and (ii) soluble.

Illustration (1):

(i) From the description of the elements of a dihedral group, the dihedral group of degree 20,
G = {(1), (2,20)(3,19)(4,18)(5,17)(6,16)(7,15)(8,14)(9,13)(10,12), (1,2)(3,20)(4,19)(5,18)(6,17) (7,16)(8,15)(9,14)(10,13)(11,12), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20), (1,3)(4,20) (5,19)(6,18)(7,17)(8,16)(9,15)(10,14)(11,13), (1,3,5,7,9,11,13,15,17,19) (2,4,6,8,

10,12,14, 16, 18,20), (1,4)(2,3)(5,20)(6,19)(7,18)(8,17)(9,16)(10, 15)(11,14)(12,13), (1,4,7,10, 13,16,19,2,5,8, 11,14,17, 20,3,6,9,12,15,18), (1,5)(2,4)(6,20)(7,19)(8,18)(9,17)(10,16)(11,15) (12,14), (1,5,9,13, 17) (2,6,10, 14, 18)(3,7,11,15,19)(4,8,12,16,20), (1,6)(2,5)(3,4)(7,20)(8,19) (9,18) (10,17) (11,16)(12,15)(13,14), (1,6, 11,16)(2,7,12,17)(3,8,13,18)(4,9,14,19)(5,10,15,20), (1,7)(2,6)(3,5) (8,20) (9,19) (10,18)(11,17)(12,16) (13,15), (1,7,13,19,5,11,17,3,9,15)(2,8,14,20,6,12,18,4,10,16), (1,8)(2,7)(3,6)(4,5)(9,20)(10,19) (11, 18)(12,17)(13,16)(14,15), (1,8,15,2,9,16,3,10,17,4, 11,18, 5,12,19,6,13,20,7,14), (1,9)(2,8)(3,7)(4,6) (10,20)(11,19)(12,18)(13,17)(14,16), (1,9,17,5,13) (2,10,18,6,14)(3,11,19,7,15)(4,12,20,8,16), (1,10) (2,9)(3,8)(4,7)(5,6)(11,20) (12,19) (13,18) (14, 17)(15,16), (1,10,19,8,17,6,15,4,13,2,11,20,9,18,7,16,5,14,3,12), (1,11)(2,10)(3,9)(4,8)(5,7) (12, 20)(13,19)(14,18)(15,17), (1,11)(2,12)(3,13)(4,14)(5,15)(6,16)(7,17)(8,18)(9,19)(10,20), (1,12) (2,11)(3,10)(4,9)(5,8)(6,7)(13,20)(14,19)(15,18)(16,17), (1,12,3, 14,5,16,7,18,9,20,11,2, 13,4,15, 6,17,8,19,10), (1,13)(2,12)(3,11)(4,10)(5,9)(6,8)(14,20)(15,19)(16,18), (1,13,5,17,9) (2,14,6, 18,10)(3,15,7,19,11)(4,16,8,20,12), (1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7,8) (15,20)(16,19) (17, 18), (1,14,7,20,13,6,19,12,5,18,11,4,17,10,3,16,9,2,15,8), (1,15)(2,14)(3,13)(4,12)(5,11)(6,10) (7,9)(16,20)(17,19), (1,15,9,3,17,11,5,19,13,7)(2,16,10,4,18,12,6,20,14,8), (1,16)(2,15)(3,14) (4,13)(5,12)(6,11)(7,10)(8,9)(17,20)(18,19), (1,16,11,6)(2,17,12,7)(3,18,13,8)(4,19,14,9)(5,20, 15,10), (1,17)(2,16)(3,15) (4,14)(5,13) (6,12)(7,11)(8,10)(18,20), (1,17,13,9,5)(2,18,14,10,6) (3,19,15,11,7)(4,20,16,12,8), (1,18)(2,17)(3,16)(4,15)(5,14)(6,13)(7,12)(8,11)(9,10)(19,20), (1,18, 15,12,9,6,3,20,17,14,11,8,5,2,19,16,13,10,7,4), (1,19)(2,18)(3,17)(4,16) (5,15) (6,14)(7,13) (8,12) (9,11), (1,19,17,15,13,11,9,7,5,3)(2,20,8,16,14,12,10,8,6,4), (1,20,19,18,17,16,15,14,13, 12,11, 10,9,8,7,6,5,4,3,2), (1, 20)(2,19) (3,18)(4,17)(5,16)(6,15)(7,14)(8,13)(9,12)(10,11)}.

Now $|G|$ = 40 and $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$ is the set of the points of G. It follows from Lema 2.2.5 that G is transitive as the orbit $\alpha^G = \Omega \ \forall \alpha \in \Omega$.

Also the stabilizer of the point 1 in G is $G_{\{1\}}$ = {(1), (2,20)(3,19)(4,18)(5,17) (6,16)(7,15) (8, 14) (9,13)(10,12)} which is obviously a non-identity proper subgroup of G. We readily see from the subgroups of G that the group (G) has a $\mathrm{H} = \{(1), (2,20)(3,19)(4,18)(5,17)(6,16)(7,15)(8,14)(9,13)(10,12), (1,11)(2,12)(3,13)(4,14)(5,15)(6,16)(7,17)(8,18)(9,19)(10,20)\}$ properly lying between $G_{\{1\}}$ and G that is, $G_{\{1\}} < H < G$. Thus by virtue of Theorem 2.3.2, G is imprimitive.

(ii) $|G| = 40 = 2^3 \cdot 5$.

By theorem 2.1.4, the number of Sylow 2-subgroups of G denoted $N_2 \equiv 1 \bmod 2$. N_2 divides $|G|$ and 5. Therefore $N_2 = \{1,5\}$
Let $H = Syl_2(W)$ be the Sylow 2-subgroup. Routine calculation shows that H = {(1), (2,12)(3,11)(4,10)(5,9)(6,8), (1,4)(2,3)(5,12)(6,11)(7,10)(8,9), (1,4,7,10)(2,5,8,11)(3,6,9,12), (1,7) (2, 6)(3,5)(8,12)(9,11), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12), (1,10)(2,9)(3,8)(4,7)(5,6)(11,12), (1,10,7,4) (2,11,8,5)(3,12,9,6)} $\leq G$ with $|H| = 8$. It follows that H is not normal in G.
Also the number of Sylow 5-subgroups of G denoted N_5 is given by $N_5 \equiv 1 \bmod 5$ and $N_5|8$.
It follows from the constraints that $N_5 = 1$.

Let $K = Syl_5(G)$ be the Sylow 5-subgroup of G. Then K = {(1), (1,9,17,5,13)(2,10,18,6,14)(3, 11, 19,7,15)(4,12,20,8,16), (1,17,13,9,5)(2,18,14,10,6)(3,19,15,11,7)(4,20,16,12,8), (1,5,9, 13,17) (2,6,10,14,18)(3,7,11,15,19)(4,8,12,16,20), (1,13,5,17,9)(2,14,6,18,10)(3,15,7,19,11) (4,16,8,20, 12)} $\leq G$ with $|K| = 5$. K is unique and it's normal in G by corollary 2.4.5. Since $|K| = 5$, we have that K is a p-Group and by theorem 2.4.7 is Solvable. Also $|G:K| = 2^3$ implies that G/K is a p-Group hence Solvable by theorem 2.4.7. By theorem 2.4.8, we have that G is solvable.

3.3 GAP Result (Validation)

```
 GAP        GAP 4.11.1 of 2021-03-02
            https://www.gap-system.org
            Architecture: x86_64-pc-cygwin-default64-kv7
Configuration:  gmp 6.2.0, GASMAN, readline
Loading the library and packages ...
gap>
gap> #  THE DIHEDRAL GROUP OF SYMMETRY, D(12)
gap>
gap> D12 :=
GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11,12),(2,12)(3,11)(4,10)(5,9)(6,8)]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12), (2,12)(3,11)(4,10)(5,9)(6,8) ])
gap> for i in D12 do
> Print(i, " ");
> od;;
gap> Order(D12);
24
gap> IsTransitive(D12);
```

```
true
gap> IsPrimitive(D12);
false
gap> IsSolvable(D12);
true
gap> SD12 := AllSubgroups(D12);;
gap> Size(SD12);
34
gap> S2 := SylowSubgroup(D12,2);
Group([ (2,12)(3,11)(4,10)(5,9)(6,8), (1,10,7,4)(2,11,8,5)(3,12,9,6),
(1,7)(2,8)(3,9)(4,10)(5,11)(6,12) ])
gap> for i in S2 do
> Print(i," ");
> od;
() ( 2,12)( 3,11)( 4,10)( 5, 9)( 6, 8) ( 1, 7)( 2, 6)( 3, 5)( 8,12)( 9,11) ( 1, 7)( 2, 8)( 3, 9)(
4,10)( 5,11)( 6,12)( 1, 4, 7,10)( 2, 5, 8,11)( 3, 6, 9,12) ( 1, 4)( 2, 3)( 5,12)( 6,11)( 7,10)( 8,
9) ( 1,10)( 2, 9)( 3, 8)( 4, 7)( 5, 6)(11,12) ( 1,10, 7, 4)( 2,11, 8, 5)( 3,12, 9, 6)
gap> Order(S2);
4
gap> IsNormal(D12,S2);
false
gap> S3 := SylowSubgroup(D12, 3);
Group([ (1,5,9)(2,6,10)(3,7,11)(4,8,12) ])
gap> for i in S3 do
> Print(i," ");
> od;
() ( 1, 9, 5)( 2,10, 6)( 3,11, 7)( 4,12, 8) ( 1, 5, 9)( 2, 6,10)( 3, 7,11)( 4, 8,12)
gap> Order(S3);
3
gap> IsNormal(D12,S3);
true
gap>
gap> # THE DIHEDRAL GROUP OF SYMMETRY, D(20)
gap>
gap> D20 :=
GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20),(2,20)(3,19)(
4,18)(5,17)(6,16)(7,15)(8,14)(9,13)(10,12)]);Group([
```

```
(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20),
(2,20)(3,19)(4,18)(5,17)(6,16)(7,15)(8,14)(9,13)(10,12)
 ])
gap> for i in D20 do
> Print(i, " ");
> od;;
gap> Order(D20);
40
gap> IsTransitive(D20);
true
gap> IsPrimitive(D20);
false
gap> IsSolvable(D20);
true
gap> SD20 := AllSubgroups(D20);;
gap> Size(SD20);
48
gap> S2 := SylowSubgroup(D20,2);
Group([ (2,20)(3,19)(4,18)(5,17)(6,16)(7,15)(8,14)(9,13)(10,12),
(1,6,11,16)(2,7,12,17)(3,8,13,18)(4,9,14,19)
  (5,10,15,20), (1,11)(2,12)(3,13)(4,14)(5,15)(6,16)(7,17)(8,18)(9,19)(10,20) ])
gap> for i in S2 do
> Print(i," ");
> od;
() ( 2,20)( 3,19)( 4,18)( 5,17)( 6,16)( 7,15)( 8,14)( 9,13)(10,12) ( 1,11)( 2,10)( 3, 9)( 4, 8)( 5, 7)(12,20)(13,19)
(14,18)(15,17) ( 1,11)( 2,12)( 3,13)( 4,14)( 5,15)( 6,16)( 7,17)( 8,18)( 9,19)(10,20) ( 1,16,11, 6)( 2,17,12, 7)
( 3,18,13, 8)( 4,19,14, 9)( 5,20,15,10) ( 1,16)( 2,15)( 3,14)( 4,13)( 5,12)( 6,11)( 7,10)( 8, 9)(17,20)(18,19) ( 1, 6)
( 2, 5)( 3, 4)( 7,20)( 8,19)( 9,18)(10,17)(11,16)(12,15)(13,14) ( 1, 6,11,16)( 2, 7,12,17)( 3, 8,13,18)( 4, 9,14,19)
( 5,10,15,20)
gap> Order(S2);
4
gap> IsNormal(D20,S2);
false
```

```
gap> S5 := SylowSubgroup(D20, 5);
Group([ (1,13,5,17,9)(2,14,6,18,10)(3,15,7,19,11)(4,16,8,20,12) ])
gap> for i in S5 do
> Print(i," ");
> od;
() ( 1, 9,17, 5,13)( 2,10,18, 6,14)( 3,11,19, 7,15)( 4,12,20, 8,16) ( 1,17,13, 9, 5)( 2,18,14,10, 6)( 3,19,15,11, 7)
( 4,20,16,12, 8) ( 1, 5, 9,13,17)( 2, 6,10,14,18)( 3, 7,11,15,19)( 4, 8,12,16,20) ( 1,13, 5,17, 9)( 2,14, 6,18,10)
( 3,15, 7,19,11)( 4,16, 8,20,12)
gap> Order(S5);
5
gap> IsNormal(D20,S5);
true
gap> # THE DIHEDRAL GROUP OF SYMMETRY, D(28)
gap>
gap> D28 := GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28),(2,28)(3,27)(4,26)(5,25)(6,24)(7,23)(8,22)(9,21)(10,20)(11,19)(12,18)(13,17)(14,16)]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28), (2,28)(3,27)(4,26)(5,25)(6,24)
  (7,23)(8,22)(9,21)(10,20)(11,19)(12,18)(13,17)(14,16) ])
gap> for i in D28 do
> Print(i, " ");
> od;;
gap> SD28 := AllSubgroups(D28);;
gap> Size(SD28);
62
gap> S2 := SylowSubgroup(D28,2);
Group([ (2,28)(3,27)(4,26)(5,25)(6,24)(7,23)(8,22)(9,21)(10,20)(11,19)(12,18)(13,17)(14,16), (1,22,15,8)(2,23,16,9)
  (3,24,17,10)(4,25,18,11)(5,26,19,12)(6,27,20,13)(7,28,21,14), (1,15)(2,16)(3,17)(4,18)(5,19)(6,20)(7,21)(8,22)(9,23)
  (10,24)(11,25)(12,26)(13,27)(14,28) ])
gap> for i in S2 do
```

```
> Print(i," ");
> od;
() ( 2,28)( 3,27)( 4,26)( 5,25)( 6,24)( 7,23)( 8,22)(
9,21)(10,20)(11,19)(12,18)(13,17)(14,16) ( 1,15)( 2,14)( 3,13)
( 4,12)( 5,11)( 6,10)( 7, 9)(16,28)(17,27)(18,26)(19,25)(20,24)(21,23) ( 1,15)( 2,16)( 3,17)(
4,18)( 5,19)( 6,20)
( 7,21)( 8,22)( 9,23)(10,24)(11,25)(12,26)(13,27)(14,28) ( 1, 8,15,22)( 2, 9,16,23)(
3,10,17,24)( 4,11,18,25)
( 5,12,19,26)( 6,13,20,27)( 7,14,21,28) ( 1, 8)( 2, 7)( 3, 6)( 4, 5)(
9,28)(10,27)(11,26)(12,25)(13,24)(14,23)(15,22)
(16,21)(17,20)(18,19) ( 1,22)( 2,21)( 3,20)( 4,19)( 5,18)( 6,17)( 7,16)( 8,15)(
9,14)(10,13)(11,12)(23,28)(24,27)
(25,26) ( 1,22,15, 8)( 2,23,16, 9)( 3,24,17,10)( 4,25,18,11)( 5,26,19,12)( 6,27,20,13)(
7,28,21,14)
gap> Order(S2);
4
gap> IsNormal(D28,S2);
false
gap> S7 := SylowSubgroup(D28,7);
Group([ (1,21,13,5,25,17,9)(2,22,14,6,26,18,10)(3,23,15,7,27,19,11)(4,24,16,8,28,20,12)
])
gap> for i in S7 do
> Print(i," ");
> od;
() ( 1, 9,17,25, 5,13,21)( 2,10,18,26, 6,14,22)( 3,11,19,27, 7,15,23)( 4,12,20,28, 8,16,24) (
1,17, 5,21, 9,25,13)
( 2,18, 6,22,10,26,14)( 3,19, 7,23,11,27,15)( 4,20, 8,24,12,28,16) ( 1,25,21,17,13, 9, 5)(
2,26,22,18,14,10, 6)
( 3,27,23,19,15,11, 7)( 4,28,24,20,16,12, 8) ( 1, 5, 9,13,17,21,25)( 2, 6,10,14,18,22,26)( 3,
7,11,15,19,23,27)
( 4, 8,12,16,20,24,28) ( 1,13,25, 9,21, 5,17)( 2,14,26,10,22, 6,18)( 3,15,27,11,23, 7,19)(
4,16,28,12,24, 8,20)
( 1,21,13, 5,25,17, 9)( 2,22,14, 6,26,18,10)( 3,23,15, 7,27,19,11)( 4,24,16, 8,28,20,12)
gap> Order(S7);
7
gap> IsNormal(D28,S7);
true
```

```
gap> #  THE DIHEDRAL GROUP OF SYMMETRY, D(44)
gap>
gap>                                          D44                                          :=
GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,2
5,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44),(2,44)(3,43)(4,42)(5,41)(6,
40)(7,39)(8,38)(9,37)(10,36)(11,35)(12,34)(13,33)(14,32)(15,31)(16,30)(17,29)(18,28)(1
9,27)(20,26)(21,25)(22,24)]);
Group([
(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,
33,34,35,36,37,38,39,40,41,42,43,44),
(2,44)(3,43)(4,42)(5,41)(6,40)(7,39)(8,38)(9,37)(10,36)(11,35)(12,34)
  (13,33)(14,32)(15,31)(16,30)(17,29)(18,28)(19,27)(20,26)(21,25)(22,24) ])
gap> for i in D44 do
> Print(i, " ");
> od;;
gap> Order(D44);
88
gap> IsTransitive(D44);
true
gap> IsPrimitive(D44);
false
gap> IsSolvable(D44);
true
gap> SD44 := AllSubgroups(D44);;
gap> Size(SD44);
90
gap> S2 := SylowSubgroup(D44,2);
<permutation group of size 8 with 3 generators>
gap> for i in S2 do
> Print(i," ");
> od;
()  (  2,44)(  3,43)(  4,42)(  5,41)(  6,40)(  7,39)(  8,38)(
9,37)(10,36)(11,35)(12,34)(13,33)(14,32)(15,31)(16,30)(17,29)(18,28)(19,27)(20,26)(21,
25)
(22,24)  (  1,23)(  2,22)(  3,21)(  4,20)(  5,19)(  6,18)(  7,17)(  8,16)(
9,15)(10,14)(11,13)(24,44)(25,43)(26,42)(27,41)(28,40)(29,39)(30,38)(31,37)(32,36)(33,
35) ( 1,23)( 2,24)( 3,25)( 4,26)( 5,27)( 6,28)( 7,29)
```

(8,30)(9,31)(10,32)(11,33)(12,34)(13,35)(14,36)(15,37)(16,38)(17,39)(18,40)(19,41)(20,42)(21,43)(22,44) (1,12,23,34)(2,13,24,35)(3,14,25,36)
(4,15,26,37)(5,16,27,38)(6,17,28,39)(7,18,29,40)(8,19,30,41)(9,20,31,42)(10,21,32,43)(11,22,33,44) (1,12)(2,11)(3,10)(4, 9)(5, 8)(6, 7)(13,44)(14,43)(15,42)(16,41)(17,40)(18,39)(19,38)(20,37)
(21,36)(22,35)(23,34)(24,33)(25,32)(26,31)(27,30)(28,29) (1,34)(2,33)(3,32)(4,31)(5,30)(6,29)(7,28)(8,27)(9,26)(10,25)(11,24)(12,23)
(13,22)(14,21)(15,20)(16,19)(17,18)(35,44)(36,43)(37,42)(38,41)(39,40) (1,34,23,12)(2,35,24,13)(3,36,25,14)(4,37,26,15)(5,38,27,16)(6,39,28,17)(7,40,29,18)(8,41,30,19)(9,42,31,20)(10,43,32,21)
(11,44,33,22)
gap> Order(S2);
4
gap> IsNormal(D44,S2);
false
gap> S11 := SylowSubgroup(D44, 11);
Group([
(1,37,29,21,13,5,41,33,25,17,9)(2,38,30,22,14,6,42,34,26,18,10)(3,39,31,23,15,7,43,35,27,19,11)(4,40,32,24,16,8,44,36,28,20,12)])
gap> for i in S11 do
> Print(i," ");
> od;;
() (1, 9,17,25,33,41, 5,13,21,29,37)(2,10,18,26,34,42, 6,14,22,30,38)(3,11,19,27,35,43, 7,15,23,31,39)(4,12,20,28,36,44, 8,16,24,32,40) (1,17,33, 5,21,37, 9,25,41,13,29)
(2,18,34, 6,22,38,10,26,42,14,30)(3,19,35, 7,23,39,11,27,43,15,31)(4,20,36, 8,24,40,12,28,44,16,32) (1,25, 5,29, 9,33,13,37,17,41,21)(2,26, 6,30,10,34,14,38,18,42,22)(3,27, 7,31,11,35,15,39,19,43,23)
(4,28, 8,32,12,36,16,40,20,44,24) (1,33,21, 9,41,29,17, 5,37,25,13)(2,34,22,10,42,30,18, 6,38,26,14)(3,35,23,11,43,31,19, 7,39,27,15)(4,36,24,12,44,32,20, 8,40,28,16)
(1,41,37,33,29,25,21,17,13, 9, 5)(2,42,38,34,30,26,22,18,14,10, 6)(3,43,39,35,31,27,23,19,15,11, 7)(4,44,40,36,32,28,24,20,16,12, 8) (1, 5, 9,13,17,21,25,29,33,37,41)(2, 6,10,14,18,22,26,30,34,38,42)
(3, 7,11,15,19,23,27,31,35,39,43)(4, 8,12,16,20,24,28,32,36,40,44) (1,13,25,37, 5,17,29,41, 9,21,33)(2,14,26,38, 6,18,30,42,10,22,34)(3,15,27,39, 7,19,31,43,11,23,35)

(4,16,28,40, 8,20,32,44,12,24,36) (1,21,41,17,37,13,33, 9,29, 5,25)(2,22,42,18,38,14,34,10,30, 6,26)(3,23,43,19,39,15,35,11,31, 7,27)(4,24,44,20,40,16,36,12,32, 8,28) (1,29,13,41,25, 9,37,21, 5,33,17)
(2,30,14,42,26,10,38,22, 6,34,18)(3,31,15,43,27,11,39,23, 7,35,19)(4,32,16,44,28,12,40,24, 8,36,20) (1,37,29,21,13, 5,41,33,25,17, 9)(2,38,30,22,14, 6,42,34,26,18,10)
(3,39,31,23,15, 7,43,35,27,19,11)(4,40,32,24,16, 8,44,36,28,20,12)
gap> Order(S11);
11
gap> IsNormal(D44,S11);
true
gap>

4. Conclusion and Recommendation

4.1 Conclusion

This study showed that Dihedral group of degree 4p where p is an odd prime number is (i) imprimitive and (ii) soluble.

4.2 Recommendation

This study can be extended by considering for further research, one or a combination of two or more of other theoretic properties such as simplicity, nilpotency, regularity, etc of same algebraic structures.

References

Audu, M. S. and Hamma, S. (2010). On Transitive and Primitive Dihedral Groups of Degree at most p^2. *Advances in Applied Science Research.* Volume 1, Issue 2. Pp. 65-75.

Bello, M., Danjuma, M., Musa, S. and Kashim, M. R. (2017). Construction of Transitive Supersolvable permutation groups. *Journal of Natural Sciences Research*. ISSN 2224-3186, Vol.7 No.24.

Cai, Q. and Zhang, H. (2015). A Note on Primitive Permutation Groups of Prime Power Degree. *Journal of Discrete Marthematics*. Volume 2015, Article ID 194741. Pp. 4.

Cameron, P. J. (2013). Notes on finite group theory. *Bulletin of the London Mathematical Society*. Volume 13, Number 1, pp. 1-22.

Fawcett, J. M. (2009). The O'Nan-Scott Theorem for Fnite Primitive Permutation Groups, and Fnite Representability. *Masters Thesis, University of Waterloo*.

Fengler, S. (2018). Transitive Permutation Groups of Prime Degree. *Masters (Published) Thesis,* RWTH Aachen University, North Rhine-Westphalia, Germany.

Gandi, T. I. and Hamma, S. (2019). Investigating solvable and Nilpotent concepts on Dihedral Groups of an even degree regular polygon. *Frontiers of knowledge, International Journal of pure and applied sciences*. ISSBN: 2635-3393| Vol. 2.

GAP 4.11.1 (2021). The GAP Group, GAP -- Groups, Algorithms, and Programming, *Version 4.11.1; 2021*. (https://www.gap-system.org).

Hamma, S. and Haruna, Y. (2009). Transitive and primitive dihedral groups of degree pq. Icastor journal of Mathematical sciences. Vol. 3, No. 1, 101-111.

Hamma, S. and Aliyu, S. O. (2010). On transitive and primitive dihedral groups of degree $at\ most\ 2^r\ (r \geq 2)$. *Scholars research library, archive of applied science research.* 2(5),152-160.

Hamma, S. and Mohammed, M. A. (2012). On Transitive and Primitive Dihedral Groups of degree p^3. *The Journal of the mathematical Association of Nigeria,* 39 (2), 65-74.

Passman, D. (1968). Permutation Groups. *Yale University*, U.S.A.

Sylow, M. L. (1872). Theorems of Group Substitutions of Group. *Annals of mathematics Journal.* Volume 5, pages 584 –594. Link: https://doi.org/10.1007/BF01442913

Thanos, G. (2006). Solvable Groups – A Numerial Approach. Mathematics Department, *University of florida*, Gainesville, USA