A- 4 STEP HERMITE BASED MULTIDERIVATIVE BLOCK INTEGRATOR FOR SOLVING FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

By

[1]*O.M Ogunlaran and [2]M.A Kehinde

[1,2]Mathematics Programme, College of Agriculture, Engineering and Science, Bowen University, Iwo, Osun State, Nigeria matthew.ogunlaran@bowen.ed.ng

ABSTRACT

A - 4 step block method was formulated with Hermite Polynomials as basis function. Discrete schemes were developed from continuous schemes obtained using interpolation and collocation techniques to derive the block. The order, consistency, zero stability and convergence of the method were investigated. The numerical results obtained from the two test problems show that the method performs better than some existing methods in the literature in term of accuracy and efficiency.

Keywords*: Multiderivative method, Hermite Polynomial, Continuous Scheme, ordinary differential equations.*

1.0 INTRODUCTION

Consider fourth order initial value problem of the form

$$y'^{V} = f(x, y, y', y'', y'''),$$
$$y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma, y'''(a) = \delta. \qquad (1.1)$$

Equation(1.1) is traditionally handled by reducing the problem to an equivalent system of four first order ordinary differential equations in four dependent variables y_1, y_2, y_3 and y_4 in the form:

$$y_1{}' = f_1(x, y_1, y_2, y_3, y_4), \qquad y_1(a) = \mu_1$$
$$y_2{}' = f_2(x, y_1, y_2, y_3, y_4), \qquad y_2(a) = \mu_2$$
$$y_3{}' = f_3(x, y_1, y_2, y_3, y_4), \qquad y_3(a) = \mu_3$$
$$y_4{}' = f_4(x, y_1, y_2, y_3, y_4),$$
$$y_4(a) = \mu_4\,, \qquad (1.2)$$

$x, y_i \in \mathbb{R}^2$ and f_i are continuously differentiable function in the given interval $[a, b]$.
The approach of reducing nth order ordinary differential equations is extensively discussed by Awoyemi (1992), Jennings (1987), and Jator (2001, 2007) among many other

researchers. A major drawback associated with the methods that solve equivalent system of first order ordinary differential equations is the problem of writing computer subroutine-sub program within the main program in order to get starting values. According to Awoyemi (1992, 1999, 2001), the consequence of this is extra computational effort, more computer time and storage wastage. In view of these setbacks, reduction method is inefficient and not very suitable in application.

To circumvent these drawbacks, many researchers such as Awoyemi (2003), Olabode (2009),Badmus and Yaya (2009), Anake, Adesanya, Ogbhonyon and Agarana (2013),Adeniyi and Mohammed (2014), and Ademola (2017) have put in several effort into developing various methods for solving problem (1.1) without first reducing it to system (1.2). They developed high order methods to handle high order ordinary differential equations.

Block methods are derived in terms of Linear Multistep Methods. They combined the self starting and easy variation of step length advantages of one- step methods like Runge-Kutta methods (Lambert, 1973).

Kayode, Duromola and Bolarinwa (2014) developed an Implicit Hybrid Block Methods for solving fourth order Ordinary differentials Equations.Adeniran and Longe (2019), Guler et al (2019), Ramos et al (2020) and Singla et al (2021) have developed block methods for solution of second order initial value problems. Adeniran and Zurni (2019) developed a three- step implicit hybrid solver for third order initial value problems.

This paper is concerned with development of block method for the solution of fourth order initial value problems using the probabilistic Hermite polynomial as basis function.

2.0 FORMULATION OF THE METHOD

In order to solve equation (1.1), we employ the approximate solution:

$$y(x) = \sum_{r=0}^{t+s-1} a_r H_r(x - x_n) \,, \qquad (2.1)$$

where $H_r(x)$ is the probabilistic Hermite polynomial, t and s are numbers of interpolation and collocation points respectively on the partition $a = x_0 \le x_1 \le \cdots x_n = b$ of the integration interval $[a, b]$.

The recurrence relation for the Hermite polynomial is given as follows:

$$H_{r+1}(x) = xH_r(x) - H'_r(x), \quad r \ge 1 \qquad (2.2)$$

where $H_0(x) = 1$ and $H_1(x) = x$.

To interpolate at $x = x_{n+i}, i = 0,1,2,3$ and collocate at $x = x_{n+i}, i = 0(1)4$, equation (2.1) becomes:

$$\begin{aligned} y(x) = a_0 &+ a_1(x - x_n) + a_2[(x - x_n)^2 - 1] + a_3[(x - x_n)^3 - 3(x - x_n)] \\ &+ a_4[(x - x_n)^4 - 6(x - x_n)^2 + 3] \\ &+ a_5[(x - x_n)^5 - 10(x - x_n)^3 + 15(x - x_n)] \\ &+ a_6[(x - x_n)^6 - 15(x - x_n)^4 + 45(x - x_n)^2 - 15] \\ &+ a_7[(x - x_n)^7 - 21(x - x_n)^5 + 105(x - x_n)^3 - 105(x - x_n)] \\ &+ a_8[(x - x_n)^8 - 28(x - x_n)^6 + 210(x - x_n)^4 - 420(x - x_n)^2 \\ &+ 105] \qquad (2.3) \end{aligned}$$

First, second, third and fourth derivatives of (2.3) are obtained as follows:

$$\begin{aligned} 2hy'(x) = a_1 &+ 2a_2(x - x_n) + a_3[3(x - x_n)^2 - 3] + a_4[4(x - x_n)^3 - 12(x - x_n)] \\ &+ a_5[5(x - x_n)^4 - 30(x - x_n)^2 + 15] \\ &+ a_6[6(x - x_n)^5 - 60(x - x_n)^3 + 90(x - x_n)] \\ &+ a_7[7(x - x_n)^6 - 105(x - x_n)^4 + 315(x - x_n)^2 - 105] \\ &+ a_8[8(x - x_n)^7 - 168(x - x_n)^5 + 840(x - x_n)^3 \\ &- 840(x - x_n)] \qquad (2.4) \end{aligned}$$

$$\begin{aligned} 4h^2y''(x) = 2a_2 &+ 6a_3(x - x_n) + a_4[12(x - x_n)^2 - 12] \\ &+ a_5[20(x - x_n)^3 - 60(x - x_n)] \\ &+ a_6[30(x - x_n)^4 - 180(x - x_n)^2 + 90] \\ &+ a_7[42(x - x_n)^5 - 420(x - x_n)^3 + 630(x - x_n)] \\ &+ a_8[56(x - x_n)^6 - 1344(x - x_n)^4 + 2520(x - x_n)^2 \\ &- 840] \qquad (2.5) \end{aligned}$$

$$\begin{aligned} 8h^3y'''(x) = 6a_3 &+ 24a_4(x - x_n) + a_5[60(x - x_n)^2 - 60] \\ &+ a_6[120(x - x_n)^3 - 360(x - x_n)] \\ &+ a_7[210(x - x_n)^4 - 1260(x - x_n)^2 + 630] \\ &+ a_8[336(x - x_n)^5 - 5376(x - x_n)^3 + 5040(x - x_n)] \qquad (2.6) \end{aligned}$$

and

$$\begin{aligned} 16h^4y'^v(x) = 24a_4 &+ 120a_5(x - x_n) + a_6[360(x - x_n)^2 - 360] \\ &+ a_7[840(x - x_n)^3 - 2520(x - x_n)] \\ &+ a_8[1680(x - x_n)^4 - 16128(x - x_n)^2 \\ &+ 5040] \qquad (2.7) \end{aligned}$$

Interpolating (2.3) at $x = x_n, x_{n+1}, x_{n+2}$ and $x_{n+3}$and collocating (2.7) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} lead to the matrix equation:

$$\begin{pmatrix} 1 & -1 & 1 & 2 & -2 & -6 & 6 & 20 & -132 \\ 1 & -\frac{1}{2} & -\frac{3}{4} & \frac{11}{8} & \frac{25}{16} & -\frac{201}{32} & -\frac{299}{64} & \frac{5123}{128} & \frac{3249}{256} \\ -1 & 0 & -1 & 0 & 3 & 0 & -15 & 0 & 105 \\ 1 & \frac{1}{2} & -\frac{3}{4} & -\frac{11}{8} & \frac{25}{16} & \frac{201}{32} & -\frac{299}{64} & -\frac{5123}{128} & \frac{3249}{256} \\ 0 & 0 & 0 & 0 & 24 & -120 & 0 & 1680 & -9408 \\ 0 & 0 & 0 & 0 & 24 & -60 & -270 & 1155 & 113 \\ 0 & 0 & 0 & 0 & 24 & 0 & -360 & 0 & 5040 \\ 0 & 0 & 0 & 0 & 24 & 60 & -270 & -1155 & 1113 \\ 0 & 0 & 0 & 0 & 24 & 120 & 0 & -1680 & -9408 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ 16h^2 f_n \\ 16h^2 f_{n+1} \\ 16h^2 f_{n+2} \\ 16h^2 f_{n+3} \\ 16h^2 f_{n+4} \end{pmatrix} \qquad (2.8)$$

Solving (2.8) simultaneously in the unknown parameters $a_i, i = 0(1)8$produces:

$$a_0 = \frac{304247}{25200} h^4 f_{n+2} + \frac{324967}{151200} h^4 f_n - \frac{274777}{37800} h^4 f_{n+1} - \frac{274777}{37800} h^4 f_{n+3} + \frac{324967}{151200} h^4 f_{n+4} + 2 y_{n+1} - 3 y_{n+2} + 2 y_{n+3},$$

$$a_1 = \frac{2453}{720} h^4 f_{n+2} - \frac{20189}{30240} h^4 f_n - \frac{4693}{7560} h^4 f_{n+1} - \frac{12709}{7560} h^4 f_{n+3} + \frac{42211}{30240} h^4 f_{n+4} + 10 y_{n+1} - 11 y_{n+2} + \frac{14}{3} y_{n+3} - \frac{11}{3} y_n,$$

$$a_2 = \frac{1077607}{151200} h^4 f_n - \frac{926617}{37800} h^4 f_{n+1} + \frac{972887}{25200} h^4 f_{n+2} - \frac{926617}{37800} h^4 f_{n+3} + \frac{1077607}{151200} h^4 f_{n+4} + 2 y_{n+1} - 4 y_{n+2} + 2 y_{n+3},$$

$$a_3 = \frac{223}{180} h^4 f_{n+2} - \frac{1069}{1080} h^4 f_n + \frac{89}{135} h^4 f_{n+1} - \frac{202}{135} h^4 f_{n+3} + \frac{271}{216} h^4 f_{n+4} + 4 y_{n+1} - 4 y_{n+2} + \frac{4}{3} y_{n+3} - \frac{4}{3} y_n$$

$$a_4 = \frac{224}{15} h^4 f_{n+2} + \frac{127}{45} h^4 f_n + \frac{127}{45} h^4 f_{n+4} - \frac{448}{45} h^4 f_{n+1} - \frac{448}{45} h^4 f_{n+3},$$

$$a_5 = -\frac{11}{45} h^4 f_n + \frac{11}{45} h^4 f_{n+4} + \frac{16}{45} h^4 f_{n+1} - \frac{16}{45} h^4 f_{n+3},$$

$$a_6 = \frac{187}{675} h^4 f_n + \frac{334}{225} h^4 f_{n+2} + \frac{187}{675} h^4 f_{n+4} - \frac{688}{675} h^4 f_{n+1} - \frac{688}{675} h^4 f_{n+3},$$

$$a_7 = -\frac{4}{315} h^4 f_n + \frac{8}{315} h^4 f_{n+1} - \frac{8}{315} h^4 f_{n+3} + \frac{4}{315} h^4 f_{n+4},$$

and

$$a_8 = \frac{2}{315} h^4 f_n - \frac{8}{315} h^4 f_{n+1} + \frac{4}{105} h^4 f_{n+2} - \frac{8}{315} h^4 f_{n+3} + \frac{2}{315} h^4 f_{n+4} . \quad (2.9)$$

Substituting (2.9) into equations (2.3) to (2.6) and evaluating the resulting continuous schemesfor (2.3) at $x = x_{n+4}$other continuous schemesat $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$respectively lead to:

$$y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n = \frac{h^4}{720}(143f_n - 452f_{n+1} + 1338f_{n+2} - 452f_{n+3} + 143f_{n+4}) \quad (2.10)$$

$$-2y_{n+3} + 9y_{n+2} - 18y_{n+1} + 11y_n + 6hy_n' = \frac{h^4}{16800}(-14069f_n + 34316f_{n+1} - 84894f_{n+2} + 52556f_{n+3} - 13109f_{n+4}) \quad (2.11)$$

$$y_{n+3} - 6y_{n+2} + 3y_{n+1} + 2y_n + 6hy'_{n+1} = \frac{h^4}{50400}(7909f_n - 17716f_{n+1} + 60294f_{n+2} - 33556f_{n+3} + 8269f_{n+4})$$

$$-2y_{n+3} - 3y_{n+2} + 6y_{n+1} - y_n + 6hy'_{n+2} = \frac{h^4}{10080}(-953f_n + 2060f_{n+1} - 9366f_{n+2} + 4268f_{n+3} - 1049f_{n+4})$$

$$-11y_{n+3} + 18y_{n+2} - 9y_{n+1} + 2y_n + 6hy'_{n+3} = \frac{h^4}{16800}(3917f_n - 9908f_{n+1} + 42342f_{n+2} - 15188f_{n+3} + 4037f_{n+4}) \quad (2.14)$$

$$-26y_{n+3} + 57y_{n+2} - 42y_{n+1} + 11y_n + 6hy'_{n+4} = \frac{h^4}{50400}(149437f_n - 505708f_{n+1} + 1284942f_{n+2} - 450988f_{n+3} + 152317f_{n+4}) \quad (2.15)$$

$$y_{n+3} - 4y_{n+2} + 5y_{n+1} - 2y_n + h^2y''_n = \frac{h^4}{302400}(-39173f_n + 456572f_{n+1} - 329118f_{n+2} + 251612f_{n+3} - 62693f_{n+4}) \quad (2.16)$$

$$-y_{n+2} + 2y_{n+1} - y_n + h^2y''_{n+1} = \frac{h^4}{302400}(-32033f_n + 101252f_{n+1} - 187998f_{n+2} + 124772f_{n+3} - 31193f_{n+4})$$

$$-y_{n+3} + 2y_{n+2} - y_{n+1} + h^2y''_{n+2} = \frac{h^3}{302400}(-953f_n + 2972f_{n+1} - 29238f_{n+2} + 2972f_{n+3} - 953f_{n+4})$$

$$-2y_{n+3} + 5y_{n+2} - 4y_{n+1} + y_n + h^2y''_{n+3} = \frac{h^4}{302400}(28867f_n - 65068f_{n+1} + 373962f_{n+2} - 88588f_{n+3} + 28027f_{n+4}) \quad (2.19)$$

$$-3y_{n+3} + 8y_{n+2} - 7y_{n+1} + 2y_n + h^2 y''_{n+4} = \frac{h^4}{302400}(57427f_n - 128068f_{n+1} + 794802f_{n+2} + 76892f_{n+3} + 80947f_{n+4}) \quad (2.20)$$

$$-y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n + h^3 y'''_n = \frac{h^4}{160}(-37f_n - 248f_{n+1} + 106f_{n+2} - 80f_{n+3} + 19f_{n+4}) \quad (2.21)$$

$$-y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n + h^3 y'''_{n+1} = \frac{h^4}{1440}(169f_n - 940f_{n+1} + 426f_{n+2} - 508f_{n+3} + 133f_{n+4})$$

$$-y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n + h^3 y'''_{n+2} = \frac{h^4}{1440}(113f_n - 248f_{n+1} + 1338f_{n+2} - 656f_{n+3} + 155f_{n+4})$$

$$-y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n + h^3 y'''_{n+3} = \frac{h^4}{160}(17f_n - 44f_{n+1} + 250f_{n+2} + 4f_{n+3} + 13f_{n+4}) \quad (2.24)$$

$$-y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n + h^3 y'''_{n+4} = \frac{h^4}{1440}(115f_n - 184f_{n+1} + 1722f_{n+2} + 1328f_{n+3} + 619f_{n+4})$$

Equations (2.10) – (2.25) combined constitute the new block method.

3.0 ANALYSIS OF THE METHOD

The block method (2.10) – (2.25) can be written in the form:

$$AY_m = BY_{m-1} + h^4[CF_{m-1} + DF_m], \quad (3.1)$$

where

$Y_m =$
$\big(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, hy'_{n+1}, hy'_{n+2}, hy'_{n+3}, hy'_{n+4}, h^2y''_{n+1}, h^2y''_{n+2}, h^2y''_{n+3}, h^2y''_{n+4}, h^3y''$
$Y_{m-1} =$
$\Big(y_{n-3}, y_{n-2}, y_{n-1}, y_n, hy'_{n-3}, hy'_{n-2}, hy'_{n-1}, hy'_{n}, h^2y''_{n-3}, h^2y''_{n-2}, h^2y''_{n-1}, h^2y''_{n}, h^3y'''_{n-3}, h$
$F_{m-1} = (f_{n-3}, f_{n-2}, f_{n-1}, f_n)^T,$
$F_m = \ (f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4})^T \ .$
Such that:

$$A = \begin{pmatrix} 5 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & 9 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -6 & 1 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & -3 & -2 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 18 & -11 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & 57 & -26 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 5 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -7 & 8 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B=\begin{pmatrix}
0&0&0&2&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&-11&0&0&0&-6&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-11&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix},$$

$$\boldsymbol{C}=\begin{pmatrix}
\mathbf{0}&\mathbf{0}&\mathbf{0}&-\frac{39173}{302400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&-\frac{14069}{16800}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&-\frac{37}{160}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{143}{720}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{7909}{50400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&-\frac{953}{10080}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{3917}{16800}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{149437}{50400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&-\frac{32033}{302400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&-\frac{953}{302400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{28867}{302400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{57427}{302400}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{169}{1440}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{113}{1440}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{17}{1440}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\frac{115}{1440}
\end{pmatrix},$$

$$
\boldsymbol{D} = \begin{pmatrix}
-\frac{452}{720} & \frac{1338}{720} & -\frac{452}{720} & \frac{143}{720} \\
\frac{34316}{16800} & -\frac{84894}{16800} & \frac{52556}{16800} & -\frac{13109}{16800} \\
-\frac{17716}{50400} & \frac{60294}{50400} & -\frac{33556}{50400} & \frac{8269}{50400} \\
\frac{2060}{10080} & -\frac{9366}{10080} & \frac{4268}{10080} & -\frac{1049}{10080} \\
-\frac{9908}{16800} & \frac{42342}{16800} & -\frac{15188}{16800} & \frac{4037}{16800} \\
-\frac{505708}{50400} & \frac{1284942}{50400} & -\frac{450988}{50400} & \frac{152317}{50400} \\
\frac{456572}{302400} & -\frac{329118}{302400} & \frac{251612}{392400} & -\frac{62693}{392400} \\
\frac{101252}{302400} & -\frac{187998}{302400} & \frac{124772}{302400} & -\frac{31193}{302400} \\
\frac{2972}{302400} & -\frac{29238}{302400} & \frac{2972}{302400} & -\frac{953}{302400} \\
-\frac{65068}{302400} & \frac{373962}{302400} & -\frac{88588}{302400} & \frac{28027}{302400} \\
-\frac{128068}{302400} & \frac{794802}{302400} & \frac{76892}{302400} & \frac{80947}{302400} \\
-\frac{248}{160} & \frac{106}{160} & -\frac{80}{160} & \frac{19}{160} \\
-\frac{940}{1440} & \frac{426}{1440} & -\frac{508}{1440} & \frac{133}{1440} \\
-\frac{248}{1440} & \frac{1338}{1440} & -\frac{656}{1440} & \frac{155}{1440} \\
-\frac{44}{160} & \frac{250}{160} & \frac{4}{160} & \frac{13}{160} \\
-\frac{184}{1440} & \frac{1722}{1440} & \frac{1328}{1440} & \frac{619}{1440}
\end{pmatrix}.
$$

Basic properties of the block method are considered and analyzed to establish the efficiency and reliability of the method. The following properties are examined: Order, error constant, consistence and zero stability.

3.1 Local Truncation Error

FollowingFatunla (1988) and Lambert (1973), the local truncation error associated with the fourth order linear multistep method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^4 \sum_{i=0}^{k} \beta_i f_{n+i} \qquad (3.2)$$

is defined by the difference operator

$$L[y(x):h] = \sum_{i=0}^{k} [\alpha_i y(x_n + ih) - h^4 \beta_i f(x_n + ih)] , \qquad (3.3)$$

where$y(x)$ is an arbitrary function, continuously differentiable on [a, b].
Expanding (3.3) in Taylor series about point x leads to the expression:

$$L[y(x):h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_p h^p y^{(p)}(x) + \cdots + c_{p+4} h^{p+4} y^{(p+4)}(x) \quad (3.4)$$

where$c_0, c_1, c_2, \cdots, c_p, \cdots, c_{p+3}$ are obtained as follow:

$$c_0 = \sum_{i=0}^{k} \alpha_i,$$

$$c_1 = \sum_{i=1}^{k} i\alpha_i,$$

$$c_2 = \frac{1}{2!} \sum_{i=1}^{k} i^2 \alpha_i,$$

$$\vdots$$

$$c_q = \frac{1}{q!} \left[\sum_{i=1}^{k} i^q \alpha_i - q(q-1)(q-2) \sum_{i=1}^{k} \beta_i i^{q-3} \right].$$

Therefore, method (3.2) is of order p if
$c_0 = c_1 = c_2 = \cdots = c_p = c_{p+1} = c_{p+2} = c_{p+3} = 0andc_{p+4} \neq 0$.
The constant $c_{p+4} \neq 0$ is called the error constant and $c_{p+4} h^{p+4} y^{(p+4)}(x)$ is the principal local truncation error at x_n.
Using the above definition, the block method (2.10) – (2.25) is of order $p = 4$ and error constant is

$c_{p+4} = \left[-\frac{1}{5}, \frac{39}{50}, -\frac{4}{25}, \frac{1}{10}, -\frac{6}{25}, -\frac{4}{50}, \frac{61}{300}, \frac{31}{300},\right.$
$\left.\frac{1}{300}, -\frac{29}{300}, -\frac{59}{300}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{10}, -\frac{1}{10}\right]^T.$

3.2 Consistency

The block method (2.10) - (2.25) is said to be consistent since $p \geq 1$ is a sufficient condition for a block method to be consistent (see (Jator, 2007)).

3.3 Zero Stability

To analyze the block method for zero stability, system (2.10) - (2.25) written as (3.1) is normalized and yields

$$A^*Y_m = B^*Y_{m-1} + h^4[C^*F_{m-1} + D^*F_m], \qquad (3.5)$$

where A^*, B^*,$C^*$and $D^*$are normalized versions of A,B, $C$and D respectively. Hence, the zero stability of the method is determined by the expression:

$$\rho(r) = det(rA^* - B^*) \text{ as } h \to 0 \qquad (3.6)$$

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B^* = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

Solving for r in equation (3.6), $\rho(r) = r^{15}(r-1) = 0 \Rightarrow r = 0,1$. Therefore, the method is zero stable.

3.4 Convergence

According to Dahlquist (1994), the necessary and sufficient condition for a linear multistep method to be convergent is to be consistent and zero stable. Thus, the block method is convergent since it is consistent and zero stable.

4.0 NUMERICAL EXAMPLES

To demonstrate the efficiency and accuracy of the new method, we apply the method to solve two test problems.

Example 4.1

Solve the initial value problem:

$$y'^{v} + x = 0\,;\;\; y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0. \quad h = 0.1$$

Analytical Solution is $y(x) = \frac{x^5}{120} + x$

Source: Kuboye et-al(2020).

Example 4.2

Solve the nonlinear in homogeneous initial value problem:

$y'^{v} = (y')^2 - yy'' - 4x^2 + e^x(1 - 4x + x^2), 0 \leq x \leq 1. y(0) = y'(0) = y''(0) = y'''(x) = 1.$
Analytical Solution: $y(x) = x^2 + e^x$
Source: Alechenu and Oyewola (2019)

Absolute errors in the solution of Example (4.1) and Example (4.2)based on our method are respectively presented in Tables (4.1) - (4.2) in comparison with some existing methods in literature.
The following acronyms are used for the following Tables of results:

EINM- Error in New Method
EIK- Error in Kayode et-al
EIM- Error in Mumahammed
EIOK- Error in Omar and Kuboye
EIK*- Error in Kuboye et-al
EIAO- Error in Alechenu and Oyewola
EIAB – Error in Adam - Basforth

Table 4.1. Absolute Errors for Example 4.1 with h= 0.1

x	New Method	EINM	EIK	EIM	EIOK	EIK*
0.1	0.10000008333333333333	0.00	1.83E-13	7.00E-10	1.00E-12	0.00
0.2	0.20000266666666666667	0.00	4.84E-12	9.00E-10	0.00	0.00
0.3	0.30002025000000000000	0.00	7.21E-12	2.60E-09	0.00	0.00
0.4	0.40008533333333333333	0.00	6.83E-11	5.10E-09	0.00	5.55E-17
0.5	0.50026041666666666666	1.0E-20	7.42E-11	7.80E-09	1.00E-12	1.11E-16
0.6	0.60064799999999999999	1.0E-20	2.71E-11	1.18E-08	2.76E-12	1.11E-16
0.7	0.70140058333333333333	0.00	2.82E-10	1.24E-08	3.51E-12	2.22E-16
0.8	0.80273066666666666667	0.00	3.41E-10	1.41E-08	3.51E-12	0.00E-16

Table 4.2. Absolute Error for Example 4.2 with h= 0.1

x	New Method	EINM	EIAO	EIAB
0.1	1.115170921	3.00E-09	0.0	0.0
0.2	1.261402764	6.00E-09	0.0	0.0
0.3	1.439858816	8.00E-09	0.0	0.0
0.4	1.651824709	1.10E -08	0.0	0.0
0.5	1.898721292	2.10E-08	1.71E-05	2.53E-01
0.6	2.182118833	3.30E-08	9.44E-05	7.19E-01
0.7	2.503752751	4.40E-08	3.11E-04	1.44E-00
0.8	2.865540985	5.70E-08	7.94E-04	2.33E-00
0.9	3.269603223	1.12E-07	1.73E-03	3.41E-00
1.0	3.718282010	1.82E-07	3.38E-03	4.72E-00

5.0 DISCUSSION

The new scheme performed better than the other four methods in the literatures with respect to example 4.1. Approximate solutions obtained from the experiment clearly shows the efficiency of the new scheme over Kayode et-al (2014), Mumahammed (2010), Omar and Kuboye (2015) and Kuboye et-al (2020). The scheme performed favourably also in example 4.2 when compared with other methods in the literature.

6.0 CONCLUSION

In this paper, we have constructed a direct 4 –step multiderivative integrator which is efficient and suitable for solving fourth order ordinary differential equations. The method has shown acceptable solution and performed better than existing methods in the literature. The superiority of the new method is clearly seeing in its associated least absolute error in comparison with the other four methods in Table 4.1. Similarly, the other two methods compared with the new method in Table 4.2 seem to perform better at the beginning of the interval of integration of Example 4.2, however, the superiority of the new method is clearly demonstrated as from $x = 0.5$ to the end of the interval. While the two methods produced erratic solutions, the new method is consistent and reliable in its performance.

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