STABILITY ANALYSIS OF QUADRUPLED FIXED POINT THEOREMS FOR MIXED MONOTONE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract
The quadrupled fixed point is a generalization of the idea of tripled fixed point. This notion of fixed point has played a very important role in the field of mathematics, especially in the aspect of nonlinear analysis. In this paper, we present the notion of stability analysis of quadrupled fixed point iterative procedures via the use of mathematical analysis properties and establish results for mixed monotone mappings in partially ordered metric space, satisfying the new modified type of contractive conditions. Our results complement and summarize some of the results in the literature.
Keywords: *stability, metric space, quadrupled fixed point, mixed monotone, iteration procedure.*

1. Introduction

The notion of fixed point is of great interest in mathematics as well as in numerous areas of applied sciences. In the case of fixed points of an operator $T: X^2 \to X$, the stability of a fixed point iterative procedures was first studied by Ostrowski (1967) in the case of Banach contraction mappings and this subject was later developed for certain contractive definitions by several authors, Rhoades (1990, 1993), Osilike (1995, 1996), Jachymski (1997), Berinde (2002, 2003), Imoru and Olatinwo (2003), Imoru, Olatinwo, Owojori (2006), Olatinwo, Owojori and Imoru (2006).

Banach-Caccioppoli-Picard Principle was applied on partially ordered complete metric spaces and starting from the results, Bhaskar and Laksmikantham (2006) extend this theory to partially ordered metric spaces and introduce the concept of coupled fixed point for mixed-monotone operators of Picard type, obtaining results involving the existence, the existence and the uniqueness of the coincidence points for mixed monotone operators $T: X^2 \to X$ in the presence of a contraction type condition.

This concept of coupled fixed points in partially ordered metric and cone metric spaces have been studied by several authors, including Ciric and Lakshmikantham (2009), Lakshmikantham and Ciric (2009), and Sabetghadam, Masiha and Sanatpour (2009), Karapinar (2010), Choudhury and Kundu (2010), Aniki and Rauf (2019).

Recently, Berinde and Borcut (2011) obtained extensions to the concept of tripled fixed points and tripled coincidence fixed points and also obtained tripled fixed points theorems and tripled coincidence fixed points theorems for contractive type mappings in partially ordered metric spaces. Research on tripled fixed point was continued by Abbas, Aydi and Karapinar (2011), Amini-Harandi (2012) and Kishore (2011).

Very recently, Rauf and Aniki, (2020) introduced quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. Also, following the series, Aniki and Rauf, (2021) established the stability theorem and results for quadrupled fixed point of contractive type single valued operators.

On the other hand, the concept of stability is adapted from the iterative fixed point method, Olatinwo (2012) studied the stability of the coupled fixed point iterative procedures using some contractive conditions for which the existence of a unique coupled fixed point has been established in literature.

2. Methodology

Firstly, we consider some notations that will be relevant in the proof of our main results. If $(X, \leq)$ is a partially ordered set and d be a metric on X such that the pair (X, d) is a complete metric space. Then, X^4 is a product space with the following partial order

$$(p, q, r, s) \leq (u, v, w, x) \Leftrightarrow u \geq p, v \leq q, w \geq r, x \leq s$$
$$\forall (p, q, r, s), (u, v, w, x) \in X^4.$$

Definition 1. Let $(\boldsymbol{X}, \leq)$ be a partially ordered set and $\boldsymbol{T}: \boldsymbol{X}^4 \to \boldsymbol{X}$ be a mapping. We say that $\boldsymbol{T}$ has the mixed monotone property if $\boldsymbol{T}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x})$ is monotone nondecreasing in $\boldsymbol{u}$ and $\boldsymbol{w}$, and monotone nonincreasing in $\boldsymbol{v}$ and $\boldsymbol{x}$, that is, for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in \boldsymbol{X}$,

$$u_1, u_2 \in X, u_1 \leq u_2 \Rightarrow T(u_1, v, w, x) \leq T(u_2, v, w, x),$$
$$v_1, v_2 \in X, v_1 \leq v_2 \Rightarrow T(u, v_1, w, x) \geq T(u, v_2, w, x),$$
$$w_1, w_2 \in X, w_1 \leq w_2 \Rightarrow T(u, v, w_1, x) \leq T(u, v, w_2, x),$$

and

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow T(u, v, w, x_1) \geq T(u, v, w, x_2).$$

Definition 2. An element $(u, v, w, x) \in X^4$ is called a quadrupled fixed point of the mapping $T: X^4 \to X$, if

$T(u,v,w,x) = u, T(v,u,v,x) = v, T(w,u,v,w) = w$, and $T(x,w,v,u) = x$.

Definition 3. The mapping $T: X^4 \to X$is said to be $(\vartheta, \kappa, \lambda, \mu)$ −contraction if and only if there exists four constants $\vartheta \geq 0, \kappa \geq 0, \lambda \geq 0, \mu \geq 0, \vartheta + \kappa + \lambda + \mu < 1$, such that for all $u, v, w, x, p, q, r, s \in X$,

$$d[T(u,v,w,x), T(p,q,r,s)] = \vartheta d(u,p) + \kappa d(v,q) + \lambda d(w,r) + \mu d(x,s), \quad (1)$$

In relation to (1), some new contractive conditions are introduced.
Let (X,d) be a metric space. For a map $T: X^4 \to X$, there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1, \beta_1 + \beta_2 + \beta_3 + \beta_4 < 1$, such that $\forall u, v, w, x, p, q, r, s \in X$, the following definitions of contractive conditions are introduced:

i. $$d\big(T(u,v,w,x), T(p,q,r,s)\big) \leq \alpha_1 d(T(u,v,w,x), u) + \beta_1 d(T(p,q,r,s), p), \quad (2)$$

$$d\big(T(v,u,v,x), T(q,p,q,s)\big) \leq \alpha_2 d(T(v,u,v,x), v) + \beta_2 d(T(q,p,q,s), q), \quad (3)$$

$$d\big(T(w,u,v,w), T(r,p,q,r)\big) \leq \alpha_3 d(T(w,u,v,w), w) + \beta_3 d(T(r,p,q,r), r), \quad (4)$$

$$d\big(T(x,w,v,u), T(s,r,q,p)\big) \leq \alpha_4 d(T(x,w,v,u), x) + \beta_4 d(T(s,r,q,p), s), \quad (5)$$

ii. $$d\big(T(u,v,w,x), T(p,q,r,s)\big) \leq \alpha_1 d(T(u,v,w,x), p) + \beta_1 d(T(p,q,r,s), u), \quad (6)$$

$$d\big(T(v,u,v,x), T(q,p,q,s)\big) \leq \alpha_2 d(T(v,u,v,x), q) + \beta_2 d(T(q,p,q,s), v), \quad (7)$$

$$d\big(T(w,u,v,w), T(r,p,q,r)\big) \leq \alpha_3 d(T(w,u,v,w), r) + \beta_3 d(T(r,p,q,r), w), \quad (8)$$

$$d\big(T(x,w,v,u), T(s,r,q,p)\big) \leq \alpha_4 d(T(x,w,v,u), s) + \beta_4 d(T(s,r,q,p), x), \quad (9)$$

Let $A, B \in M_{(m,n)}(\mathbb{R})$ be two matrices. We write $A \leq B$;if $\alpha_{ij} \leq \beta_{ij}$ for all $i = \overline{1,m}$, $j = \overline{1,n}$.
In order to prove the main stability result in this research, the next are given

Lemma 1. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of nonnegative numbers and h be a constant, such that $0 \leq h \leq 1$ and

$$\alpha_{n+1} \leq h\alpha_n + \beta_n, \quad n \geq 0,$$

If $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.
Also, given in the next result is the extension of Lemma 1 to vector sequences where inequalities between vectors means inequalities on its elements.

Lemma 2. Let $\{p_n\}, \{q_n\}, \{r_n\}, \{s_n\}$ be sequences of nonnegative real numbers. Consider a matrix $A \in M_{(4,4)}(\mathbb{R})$ with nonnegative elements, such that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \\ s_{n+1} \end{pmatrix} \le A. \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}, \quad n \ge 0, \tag{10}$$

With
i. $\lim_{n\to\infty} A^n = 0_4$,
ii. $\sum_{k=0}^{\infty} \eta_k < \infty, \sum_{k=0}^{\infty} \varepsilon_k < \infty, \sum_{k=0}^{\infty} \delta_k < \infty, and\ \sum_{k=0}^{\infty} \gamma_k < \infty$.

If $\lim_{n\to\infty} \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Proof
For $A = 0 \in M_{(4,4)}$, (10) is rewritten with $n = k$ and summing the inequalities obtained for $k = 0,1,2, \dots, n$. Then, the following is obtained for $k = 0,1,2, \dots, n$.

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \\ s_{n+1} \end{pmatrix} \le A^{n+1}. \begin{pmatrix} p_0 \\ q_0 \\ r_0 \\ s_0 \end{pmatrix} + \sum_{k=0}^{n} A^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix}, \quad n \ge 0, \tag{11}$$

From condition (ii) of Lemma 2, it follows that the sequences of partial sums$\{\mathrm{H}_n\}, \{\mathrm{E}_n\}, \{\Delta_n\}, \{\Gamma_n\}$, are given respectively by $\mathrm{H}_n = \eta_0 + \eta_1 + \dots + \eta_n, \mathrm{E}_n = \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_n, \Delta_n = \delta_0 + \delta_1 + \dots + \delta_n$, and $\Gamma_n = \gamma_0 + \gamma_1 + \dots + \gamma_n$, for $n \ge 0$, converge respectively to some $\mathrm{H}_n \ge 0, \mathrm{E}_n \ge 0, \Delta_n \ge 0$, and $\Gamma_n \ge 0$ and hence, they are bounded.
Let $M > 0$ be such that

$$\begin{pmatrix} \mathrm{H}_n \\ \mathrm{E}_n \\ \Delta_n \\ \Gamma_n \end{pmatrix} \le M. \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall\ n \ge 0.$$

By condition (i) of Lemma 2, then $\forall e > 0$, there exists $N = N(e)$ such that $A^n \le \frac{e}{2M}. I_4, \forall n \ge N,\ M > 0$.

Write

$$\sum_{k=0}^{n} A^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = A^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + A^N \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} + A^{N-1} \begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \cdots + I_4 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}$$

But

$$A^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + A^N \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \leq \frac{e}{2M} . I_4 \left[\begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \right]$$

$$= \frac{e}{2M} . I_4 \begin{pmatrix} \mathrm{H}_{n-N} \\ \mathrm{E}_{n-N} \\ \Delta_{n-N} \\ \Gamma_{n-N} \end{pmatrix} \leq \frac{e}{2M} . I_4 . M \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{e}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \forall\, n \geq N.$$

On the other hand, denote $A' = max\{I_4, A, \ldots, A^{N-1}\}$, the following is obtained

$$A^{N-1} \begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \cdots + I_4 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} \leq A' \left[\begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \cdots + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} \right]$$

$$= A' \begin{pmatrix} \mathrm{H}_n - \mathrm{H}_{n-N} \\ \mathrm{E}_n - \mathrm{E}_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix}.$$

Since N is fixed, then $\lim_{n\to\infty} \mathrm{H}_n = \lim_{n\to\infty} \mathrm{H}_{n-N} = \mathrm{H}$, $\lim_{n\to\infty} \mathrm{E}_n = \lim_{n\to\infty} \mathrm{E}_{n-N} = \mathrm{E}$, $\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \Delta_{n-N} = \Delta$, $\lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \Gamma_{n-N} = \Gamma$, which shows that there exists a positive integer k such that

$$A' \begin{pmatrix} \mathrm{H}_n - \mathrm{H}_{n-N} \\ \mathrm{E}_n - \mathrm{E}_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix} < \frac{e}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \forall\, n \geq k.$$

Now, for $m = max\{k, N\}$, the following is gotten

$$A^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \cdots + I_4 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} < e \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \forall\, n \geq m,$$

and therefore, $\lim_{n\to\infty} \sum_{k=0}^{n} A^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = 0.$

Now, letting limit in (11), as $\lim_{n\to\infty} A^n = 0$, then

$$\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

as required.

3. Main Results

Let (X, d) be a metric space and $T: X^4 \to X$ be a mapping. For $(u_0, v_0, w_0, x_0) \in X^4$ the sequence $\{(u_n, v_n, w_n, x_n)\} \subset X^4$ defined by

$$u_{n+1} = T(u_n, v_n, w_n, x_n), \qquad v_{n+1} = T(v_n, u_n, v_n, x_n),$$
$$w_{n+1} = T(w_n, u_n, v_n, w_n),\ x_{n+1} = T(x_n, w_n, v_n, u_n) \tag{12}$$

with $n = 0,1,2, \ldots$, is the quadrupled fixed point iterative procedure.

Definition 4. Let (X, d) be a complete metric space and

$$Fix_t(T) = \{(u^*, v^*, w^*, x^*) \in X^4 / \, T(u^*, v^*, w^*, x^*) = u^*, T(v^*, u^*, v^*, x^*) = v^*, T(w^*, u^*, v^*, w^*) = w^*, T(x^*, w^*, v^*, u^*) = x^*\}$$

is the set of quadrupled fixed point of T.

Let $\{(u_n, v_n, w_n, x_n)\} \subset X^4$ be the sequence generated by the iterative procedure defined by (12), where the initial value which converges to a quadrupled fixed point (u^*, v^*, w^*, x^*) of T is $(u_0, v_0, w_0, x_0) \in X^4$.

Let $\{(p_n, q_n, r_n, s_n)\} \subset X^4$ be an arbitrary sequence. For all $n = 0,1,2, \ldots$ setting $\eta_n = d\big(p_{n+1}, T(p_n, q_n, r_n, s_n)\big), \varepsilon_n = d\big(q_{n+1}, T(q_n, p_n, q_n, s_n)\big), \delta_n = d\big(r_{n+1}, T(r_n, p_n, q_n, r_n)\big), \gamma_n = d\big(s_{n+1}, T(s_n, r_n, q_n, p_n)\big)$.

Then, the quadrupled fixed point iterative procedure defined by (12) is T −stable or stable with respect to T, if and only if $\lim_{n\to\infty} (\eta_n, \varepsilon_n, \delta_n, \gamma_n) = 0_{\mathbb{R}^4}$ implies that $\lim_{n\to\infty} (p_n, q_n, r_n, s_n) = (u^*, v^*, w^*, x^*)$.

Theorem 1. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that the pair (X, d) is a complete metric space. Let $T: X^4 \to X$ be a continuous mapping having the mixed monotone property on X and satisfies the contraction conditions (2)-(5).

If there exist $u_0, v_0, w_0, x_0 \in X$ such that $u_0 \leq T(u_0, v_0, w_0, x_0), v_0 \geq T(v_0, u_0, v_0, x_0), w_0 \leq T(w_0, u_0, v_0, w_0)$, and $x_0 \geq T(x_0, w_0, v_0, u_0)$,
then there exist $u^*, v^*, w^*, x^* \in X$ such that $u^* = T(u^*, v^*, w^*, x^*), v^* = T(v^*, u^*, v^*, x^*), w^* = T(w^*, u^*, v^*, w^*)$, and $x^* = T(x^*, w^*, v^*, u^*)$.
Assuming that for every $(u, v, w, x), (u_1, v_1, w_1, x_1) \in X^4$, there exist $(p, q, r, s) \in X^4$ that can be comparable to (u, v, w, x) and (u_1, v_1, w_1, x_1). For $(u_0, v_0, w_0, x_0) \in X^4$, let $\{(u_n, v_n, w_n, x_n)\} \subset X^4$ be the quadrupled fixed point iterative procedure defined by (12). Then, the quadrupled fixed point iterative procedure is stable with respect to T.

Proof

Let $\{(u_n, v_n, w_n, x_n)\} \subset X^4$,$\eta_n = d\big(p_{n+1}, T(p_n, q_n, r_n, s_n)\big)$, $\varepsilon_n = d\big(q_{n+1}, T(q_n, p_n, q_n, s_n)\big), \delta_n = d\big(r_{n+1}, T(r_n, p_n, q_n, r_n)\big)$, and $\gamma_n = d\big(s_{n+1}, T(s_n, r_n, q_n, p_n)\big)$.
Assume also that $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$, in order to establish that $\lim_{n\to\infty} p_n = u^*, \lim_{n\to\infty} q_n = v^*, \lim_{n\to\infty} r_n = w^*$, and $\lim_{n\to\infty} s_n = x^*$.
Therefore, on using the contraction condition (2), the following is obtained

$$\begin{aligned}
d(p_{n+1}, u^*) &\leq d\big(p_{n+1}, T(p_n, q_n, r_n, s_n)\big) + d(T(p_n, q_n, r_n, s_n), u^*) \\
&= d\big(T(p_n, q_n, r_n, s_n), T(u^*, v^*, w^*, x^*)\big) + \eta_n \\
&= d\big(T(u^*, v^*, w^*, x^*), T(p_n, q_n, r_n, s_n)\big) + \eta_n \\
&\leq \alpha_1 d(T(u^*, v^*, w^*, x^*), u^*) + \beta_1 d(T(p_n, q_n, r_n, s_n), p_n) + \eta_n \\
&\leq \alpha_1 d(u^*, u^*) + \beta_1 d(T(p_n, q_n, r_n, s_n), p_{n+1}) + \beta_1 d(p_{n+1}, u^*) + \beta_1 d(u^*, p_n) + \eta_n \\
&= \alpha_1 d(u^*, u^*) + \beta_1 d(p_{n+1}, u^*) + \beta_1 d(u^*, p_n) + (\beta_1 + 1)\eta_n.
\end{aligned}$$

Hence,

$$d(p_{n+1}, u^*) - \beta_1 d(p_{n+1}, u^*) \leq \beta_1 d(u^*, p_n) + (\beta_1 + 1)\eta_n + \alpha_1 d(u^*, u^*)$$
$$(1 - \beta_1) d(p_{n+1}, u^*) \leq \beta_1 d(u^*, p_n) + \eta_n{}',$$

where $\eta_n{}' = (\beta_1 + 1)\eta_n + \alpha_1 d(u^*, u^*)$.
Applying Lemma 1 for $\frac{\boldsymbol{\beta_1}}{(\mathbf{1}-\boldsymbol{\beta_1})} \in [\mathbf{0}, \mathbf{1})$, then $\lim_{n\to\infty} \boldsymbol{p_n} = \boldsymbol{u}^*$.
Now, making use of the contraction condition (3), the following is obtained

$$\begin{aligned}
d(q_{n+1}, v^*) &\leq d\big(q_{n+1}, T(q_n, p_n, q_n, s_n)\big) + d(T(q_n, p_n, q_n, s_n), v^*) \\
&= d\big(T(q_n, p_n, q_n, s_n), T(v^*, u^*, v^*, x^*)\big) + \varepsilon_n \\
&= d\big(T(v^*, u^*, v^*, x^*), T(q_n, p_n, q_n, s_n)\big) + \varepsilon_n \\
&\leq \alpha_2 d(T(v^*, u^*, v^*, x^*), v^*) + \beta_2 d(T(q_n, p_n, q_n, s_n), q_n) + \varepsilon_n \\
&\leq \alpha_2 d(v^*, v^*) + \beta_2 d(T(q_n, p_n, q_n, s_n), q_{n+1}) + \beta_2 d(q_{n+1}, v^*) + \beta_2 d(v^*, q_n) + \varepsilon_n \\
&= \alpha_2 d(v^*, v^*) + \beta_2 d(q_{n+1}, v^*) + \beta_2 d(v^*, q_n) + (\beta_2 + 1)\varepsilon_n.
\end{aligned}$$

Hence,

$$d(q_{n+1},v^*)-\beta_2 d(q_{n+1},v^*)\le \beta_2 d(v^*,q_n)+(\beta_2+1)\varepsilon_n+\alpha_2 d(v^*,v^*)$$
$$(1-\beta_2)d(q_{n+1},v^*)\le \beta_2 d(v^*,q_n)+\varepsilon_n{}',$$

where $\varepsilon_n{}'=(\beta_2+1)\varepsilon_n+\alpha_2 d(v^*,v^*)$.

Applying Lemma 1, for $\frac{\beta_2}{(1-\beta_2)}\in[0,1)$, then $\lim_{n\to\infty} q_n=v^*$.

Now, making use of the contraction condition (4), the following is obtained

$$\begin{aligned}d(r_{n+1},w^*)&\le d\big(r_{n+1},T(r_n,p_n,q_n,r_n)\big)+d(T(r_n,p_n,q_n,r_n),w^*)\\&=d\big(T(r_n,p_n,q_n,r_n),T(w^*,u^*,v^*,w^*)\big)+\delta_n\\&=d\big(T(w^*,u^*,v^*,w^*),T(r_n,p_n,q_n,r_n)\big)+\delta_n\\&\le \alpha_3 d(T(w^*,u^*,v^*,w^*),w^*)+\beta_3 d(T(r_n,p_n,q_n,r_n),r_n)+\delta_n\\&\le \alpha_3 d(w^*,w^*)+\beta_3 d(T(r_n,p_n,q_n,r_n),r_{n+1})+\beta_3 d(r_{n+1},w^*)+\beta_3 d(w^*,r_n)+\delta_n\\&=\alpha_3 d(w^*,w^*)+\beta_3 d(r_{n+1},w^*)+\beta_3 d(w^*,r_n)+(\beta_3+1)\delta_n.\end{aligned}$$

Hence,

$$d(r_{n+1},w^*)-\beta_3 d(r_{n+1},w^*)\le \beta_3 d(w^*,r_n)+(\beta_3+1)\delta_n+\alpha_3 d(w^*,w^*)$$
$$(1-\beta_3)d(r_{n+1},w^*)\le \beta_3 d(w^*,r_n)+\delta_n{}',$$

where $\delta_n{}'=(\beta_3+1)\delta_n+\alpha_3 d(w^*,w^*)$.

Also applying Lemma 1, for $\frac{\beta_3}{(1-\beta_3)}\in[0,1)$, then $\lim_{n\to\infty} r_n=w^*$.

Similarly, making use of the contraction condition (5), the following is obtained

$$\begin{aligned}d(s_{n+1},x^*)&\le d\big(s_{n+1},T(s_n,r_n,q_n,p_n)\big)+d(T(s_n,r_n,q_n,p_n),x^*)\\&=d\big(T(s_n,r_n,q_n,p_n),T(x^*,w^*,v^*,u^*)\big)+\gamma_n\\&=d\big(T(x^*,w^*,v^*,u^*),T(s_n,r_n,q_n,p_n)\big)+\gamma_n\\&\le \alpha_4 d(T(x^*,w^*,v^*,u^*),x^*)+\beta_4 d(T(s_n,r_n,q_n,p_n),s_n)+\gamma_n\\&\le \alpha_4 d(x^*,x^*)+\beta_4 d(T(s_n,r_n,q_n,p_n),s_{n+1})+\beta_4 d(s_{n+1},x^*)+\beta_4 d(x^*,s_n)+\gamma_n\\&=\alpha_4 d(x^*,x^*)+\beta_4 d(s_{n+1},x^*)+\beta_4 d(x^*,s_n)+(\beta_4+1)\gamma_n.\end{aligned}$$

Hence,

$$d(s_{n+1},x^*)-\beta_4 d(s_{n+1},x^*)\le \beta_4 d(x^*,s_n)+(\beta_4+1)\gamma_n+\alpha_3 d(x^*,x^*)$$
$$(1-\beta_4)d(s_{n+1},x^*)\le \beta_4 d(x^*,s_n)+\gamma_n{}',$$

where $\gamma_n{}'=(\beta_4+1)\delta_n+\alpha_4 d(x^*,x^*)$.

Also applying Lemma 1, for $\frac{\beta_4}{(1-\beta_4)}\in[0,1)$, then $\lim_{n\to\infty} s_n=x^*$,

as required.

Theorem 2. Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that the pair (X,d) is a complete metric space. Let $T: X^4\to X$ be a continuous mapping having the mixed monotone property on X and satisfies the contraction conditions (6)-(9).

If there exist $u_0, v_0, w_0, x_0 \in X$ such that $u_0 \leq T(u_0, v_0, w_0, x_0), v_0 \geq T(v_0, u_0, v_0, x_0), w_0 \leq T(w_0, u_0, v_0, w_0)$, and $x_0 \geq T(x_0, w_0, v_0, u_0)$,
then there exist $u^*, v^*, w^*, x^* \in X$ such that$u^* = T(u^*, v^*, w^*, x^*), v^* = T(v^*, u^*, v^*, x^*), w^* = T(w^*, u^*, v^*, w^*)$, and $x^* = T(x^*, w^*, v^*, u^*)$.
Assuming that for every $(u, v, w, x), (u_1, v_1, w_1, x_1) \in X^4$, there exist $(p, q, r, s) \in X^4$ that can be comparable to (u, v, w, x) and (u_1, v_1, w_1, x_1). For $(u_0, v_0, w_0, x_0) \in X^4$, let $\{(u_n, v_n, w_n, x_n)\} \subset X^4$ be the quadrupled fixed point iterative procedure defined by (12). Then, the quadrupled fixed point iterative procedure is stable with respect to T.

Proof

Let $\{(u_n, v_n, w_n, x_n)\}_{n=0}^{\infty} \subset X^4$, $\eta_n = d\big(p_{n+1}, T(p_n, q_n, r_n, s_n)\big)$, $\varepsilon_n = d\big(q_{n+1}, T(q_n, p_n, q_n, s_n)\big), \delta_n = d\big(r_{n+1}, T(r_n, p_n, q_n, r_n)\big)$, *and* $\gamma_n = d\big(s_{n+1}, T(s_n, r_n, q_n, p_n)\big)$.
Assume also that $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$, in order to establish that $\lim_{n\to\infty} p_n = u^*$, $\lim_{n\to\infty} q_n = v^*$, $\lim_{n\to\infty} r_n = w^*$, and $\lim_{n\to\infty} s_n = x^*$.
Therefore, on using the contraction condition (6), the following is obtained

$$\begin{aligned}
d(p_{n+1}, u^*) &\leq d\big(p_{n+1}, T(p_n, q_n, r_n, s_n)\big) + d(T(p_n, q_n, r_n, s_n), u^*) \\
&= d\big(T(p_n, q_n, r_n, s_n), T(u^*, v^*, w^*, x^*)\big) + \eta_n \\
&= d\big(T(u^*, v^*, w^*, x^*), T(p_n, q_n, r_n, s_n)\big) + \eta_n \\
&\leq \alpha_1 d(T(u^*, v^*, w^*, x^*), p_n) + \beta_1 d(T(p_n, q_n, r_n, s_n), u^*) + \eta_n \\
&\leq \alpha_1 d(u^*, p_n) + \beta_1 d(T(p_n, q_n, r_n, s_n), p_n) + \beta_1 d(p_n, u^*) + \eta_n \\
&= (\alpha_1 + \beta_1) d(p_n, u^*) + \beta_1 d(T(p_n, q_n, r_n, s_n), p_n) + \eta_n \\
&= (\alpha_1 + \beta_1) d(p_n, u^*) + \beta_1 \eta_{n-1} + \eta_n
\end{aligned}$$

Hence, on taking limit and applying Lemma 1 for $h = \alpha_1 + \beta_1 \in [0,1)$and where $\eta_n{}' = \eta_n + \beta_1 \eta_{n-1} \to 0$, then $\lim_{n\to\infty} p_n = u^*$.
Now, on making use of the contraction condition (7), the following is obtained

$$\begin{aligned}
d(q_{n+1}, v^*) &\leq d\big(q_{n+1}, T(q_n, p_n, q_n, s_n)\big) + d(T(q_n, p_n, q_n, s_n), v^*) \\
&= d\big(T(q_n, p_n, q_n, s_n), T(v^*, u^*, v^*, x^*)\big) + \varepsilon_n \\
&= d\big(T(v^*, u^*, v^*, x^*), T(q_n, p_n, q_n, s_n)\big) + \varepsilon_n \\
&\leq \alpha_2 d(T(v^*, u^*, v^*, x^*), q_n) + \beta_2 d(T(q_n, p_n, q_n, s_n), v^*) + \varepsilon_n \\
&\leq \alpha_2 d(v^*, q_n) + \beta_2 d(T(q_n, p_n, q_n, s_n), q_n) + \beta_2 d(q_n, v^*) + \varepsilon_n \\
&= (\alpha_2 + \beta_2) d(q_n, v^*) + \beta_2 d(T(q_n, p_n, q_n, s_n), q_n) + \varepsilon_n \\
&= (\alpha_2 + \beta_2) d(q_n, v^*) + \beta_2 \varepsilon_{n-1} + \varepsilon_n
\end{aligned}$$

Hence, on taking limit and applying Lemma 1 for $h = \alpha_2 + \beta_2 \in [0,1)and\varepsilon_n{}' = \varepsilon_n + \beta_2\varepsilon_{n-1} \to 0$,then $\lim_{n\to\infty} q_n = v^*$.

Now, on making use of the contraction condition (8), the following is obtained

$$\begin{aligned} d(r_{n+1}, w^*) &\le d\big(r_{n+1}, T(r_n, p_n, q_n, r_n)\big) + d(T(r_n, p_n, q_n, r_n), w^*) \\ &= d\big(T(r_n, p_n, q_n, r_n), T(w^*, u^*, v^*, w^*)\big) + \delta_n \\ &= d\big(T(w^*, u^*, v^*, w^*), T(r_n, p_n, q_n, r_n)\big) + \delta_n \\ &\le \alpha_3 d(T(w^*, u^*, v^*, w^*), r_n) + \beta_3 d(T(r_n, p_n, q_n, r_n), w^*) + \delta_n \\ &\le \alpha_3 d(w^*, r_n) + \beta_3 d(T(r_n, p_n, q_n, r_n), r_n) + \beta_3 d(r_n, w^*) + \delta_n \\ &= (\alpha_3 + \beta_3) d(r_n, w^*) + \beta_3 d(T(r_n, p_n, q_n, r_n), r_n) + \delta_n \\ &= (\alpha_3 + \beta_3) d(r_n, w^*) + \beta_3 \delta_{n-1} + \delta_n \end{aligned}$$

Hence, on taking limit and applying Lemma 1 for $h = \alpha_3 + \beta_3 \in [0,1)$ and $\delta_n{}' = \delta_n + \beta_3\delta_{n-1} \to 0$, then $\lim_{n\to\infty} r_n = w^*$.

Similarly, on using the contraction condition (9), the following is obtained

$$\begin{aligned} d(s_{n+1}, x^*) &\le d\big(s_{n+1}, T(s_n, r_n, q_n, p_n)\big) + d(T(s_n, r_n, q_n, p_n), x^*) \\ &= d\big(T(s_n, r_n, q_n, p_n), T(x^*, w^*, v^*, u^*)\big) + \gamma_n \\ &= d\big(T(x^*, w^*, v^*, u^*), T(s_n, r_n, q_n, p_n)\big) + \gamma_n \\ &\le \alpha_4 d(T(x^*, w^*, v^*, u^*), s_n) + \beta_4 d(T(s_n, r_n, q_n, p_n), x^*) + \gamma_n \\ &\le \alpha_4 d(x^*, s_n) + \beta_4 d(T(s_n, r_n, q_n, p_n), s_n) + \beta_4 d(s_n, x^*) + \gamma_n \\ &= (\alpha_4 + \beta_4) d(s_n, x^*) + \beta_4 d(T(s_n, r_n, q_n, p_n), s_n) + \gamma_n \\ &= (\alpha_4 + \beta_4) d(s_n, x^*) \beta_4 \gamma_{n-1} + \gamma_n \end{aligned}$$

Now, on taking limit and applying Lemma 1 for $h = \alpha_4 + \beta_4 \in [0,1)$ and $\gamma_n{}' = \gamma_n + \beta_4\gamma_{n-1} \to 0$, then $\lim_{n\to\infty} s_n = x^*$.

4. Conclusion

In this research, it has been clearly shown that stability of quadrupled fixed point iterative procedure, for some modified contractive type mappings in partially ordered metric spaces with mixed monotone property exists. This is the stability analysis for the iterative procedure of quadrupled fixed point as established in Rauf and Aniki (2020).

References

Amini-Harandi, A. (2012). Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem. *Mathematical and Computer Modeling*, 57, 2343-2348.

Aniki, S.A. & Rauf, K. (2019). Stability results on coupled fixed point iterative procedures in complete metric spaces. *Islamic University Multidisciplinary Journal*, **6**(3)**,** 175-186. ISSN 2617-6513.

Aniki, S.A. & Rauf, K. (2021). Stability theorem and results for quadrupled fixed point of contractive type single valued operators. *Iranian Journal of Optimization.* (Accepted)

Abbas, M., Aydi H. & Karapinar E. (2011). Tripled fixed point of multivalued nonlinear contraction mappings in partially ordered metric spaces. *Abstract and Applied Analysis*, (2), 1-12.

Bhaskar, T.G. & Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis*, **65**(7), 1379-1393.

Berinde V. (2003). Summable almost stability of fixed point iteration procedures. *Carpathian Journal of Mathematics*, **19**(2), 81–88.

Berinde, V. & Borcut, M. (2011). Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Analysis*, **74**(15), 4889-4897.

Ciri´c L.B. & Lakshmikantham V. (2009). Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Stochastic Analysis and Applications*, **27**(6), 1246–1259.

Choudhury, B. S. & Kundu, A. A. (2010) Coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Analysis*, **73**(8), 2524-2531.

Imoru, C. O. & Olatinwo M. O. (2003). On the stability of Picard andMann iteration processes. *Carpathian Journal of Mathematics*, **19**(2), 155–160.

Jachymski J. R. (1997). An extension of A. Ostrowski's theorem on the round-off stability of iterations. *Aequationes Mathematicae*, **53**(3), 242–253.

Karapinar, E. (2010). Coupled foxed point theorems for nonlinear contractions in cone metric spaces. *Computer & Mathematics with Applications*, **59**(12), 3656-3668.

Lakshmikantham, V. & Ciri´c, L. B. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Analysis*, 70(12), 4341-4349.

Olatinwo, M. O., Owojori, O.O. & Imoru, C.O. (2006). Some stability results for fixed point iteration processes. *Australian Journal of Mathematical Analysis and Applications*, 2(1), 1–7.

Osilike, M. O. (1996). A stable iteration procedure for quasi-contractive maps. *Indian Journal of Pure and Applied Mathematics*, **27**(1), 25–34.

Osilike, M. O. (1995). Stability results for fixed point iteration procedure. *Journal of the Nigerian Mathematical Society*, **14**,17–29.

Ostrowski, A. M. (1967). The round-off stability of iterations. *Journal of Applied Mathematics and Mechanics*, **47**(1), 77–81.

Rauf, K. & Aniki, S. A. (2020). Quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. *Confluence Journal of Pure and Applied Sciences***.** (Accepted)

Rao, K. P. R. & Kishore, G. N. V. (2011). A Unique Common tripled fixed point theorem in partially ordered cone metric spaces. *Bulletin of Mathematical Analysis and Applications,* 3(4), 213-222.

Rhoades, B. E. (1990). Fixed point theorems and stability results for fixed point iteration procedures. *Indian Journal of Pure and Applied Mathematics,* **21**(1), 1–9.

Rhoades, B.E. (1993) Fixed point theorems and stability results for fixed point iteration procedures II. *Indian Journal of Pure and Applied Mathematics*, **24**(11), 691–703.

Sabetghadam, F., Masiha, H. P. & Sanatpour, A. H. (2009). Some coupled fixed point theorems in cone metric spaces. *Fixed Point Theory and Applications*, Article ID 125426. doi:10.1155/2009/125426