# CLOSED FORM SOLUTION OF A DIRICHLET HARMONIC PROBLEMS USING COMPLEX VARIABLE TECHNIQUES

By

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## Abstract

*One of the most interesting examples of applying the theory of harmonic functions to physical modelling is the study of two-dimensional steady state temperature fields. When steady state prevails, the temperature function inside a two-dimensional body* $\vartheta(x,y)$ *is readily seen to be harmonic as we explore the intimate connection between complex analysis and solutions to the Laplace equation in solving harmonic Dirichlet problems of heat flow in two- dimensional solids whose boundaries are maintained at prescribed values. The problem is formulated by complex variable techniques and solved by conformal mapping method. The method uses the appropriate mapping function to transform the domain* $D$ *and boundary* $\Gamma$ *of the given problem in the* $z$*-plane onto one in the upper half of the* $w$*-plane and the appropriate portions of the* $y-$*axis where its solutions for steady state temperature is easily identified as the imaginary part of some branch of the logarithmic function.*

*Key words: Closed form solution, steady state temperature, analytic functions, conformal mapping and Dirichlet problem.*

## 1.0 Introduction

Most problems of science and engineering when formulated mathematically lead to differential equations and the associated conditions called boundary conditions. The problem of determining solutions to the partial differential equation which satisfy the boundary condition is called a boundary value problem. Boundary value problem of complex geometry has been carried out by many researchers using different methods while problems of simple configuration can be handled by the intimate connection between complex analysis and solutions to the Laplace equation. Spiegel (1981), Churchill (2009) , Markushevich (1967).

From the theory of complex variables, we know that complex functions provide an almost inexhaustible supply of harmonic functions, that is solutions to the two-dimensional Laplace equations. Thus, to solve an associated boundary value problem, we merely find the complex function whose real part matches the prescribed boundary conditions. Unfortunately, even for relatively simple domains, this remains a daunting task because of identifying the mapping function to use for a particular problem. Actually, there is no systematic way of knowing this function rather than one's years of experience and familiarity with the manner in which curves and regions are mapped by most analytic functions. With the mapping function at our disposal, we map the original problem from the domain with a complex domain in the $z$ plane into a domain with a simpler domain in the $w$ plane. This mapping preserves the magnitude of the angles between curves as well as their orientation and it is said to be conformal.

In heat conduction, energy is involved because heat is a form of energy in transit and a good knowledge of its mechanism is very pertinent. In this paper we shall be focusing on the quantity of heat conducted per unit area per unit time across a surface located in the solid. This quantity is called the heat flux across the surface and is denoted by

$$\mathfrak{I} = -k\nabla\vartheta \tag{1}$$

where $\vartheta$ is the heat potential and $k$ is called the thermal conductivity which depends on the material composition. In two -dimensional analysis, equation (1) becomes

$$\mathfrak{I} = -k\left(\frac{\partial\vartheta}{\partial x} + i\frac{\partial\vartheta}{\partial y}\right) = Q_x + iQ_y \tag{2}$$

where

$$Q_x = -k\frac{\partial\vartheta}{\partial x}, \quad Q_y = -k\frac{\partial\vartheta}{\partial y}$$

If steady-state condition prevails, the heat potential function satisfies the Laplace's equation.

$$\nabla^2\vartheta = \frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2} = 0 \tag{3}$$

and its boundary condition. George et al (2005), Kwok (2010)

The function $\vartheta(x, y)$ which satisfies eqn. (3) is said to be harmonic and has a corresponding conjugate function $\psi(x, y)$ such that the complex heat potential

$$F(z) = \vartheta(x, y) + i\psi(x, y) \tag{4}$$

is analytic.

Definition 1.1 (Analytic function)

If the derivative $f'(z)$ exists at all points $z$ of a region $\Re$, then $f(z)$ is said to be analytic in $\Re$ and is referred to as an analytic function in $\Re$.

Definition 1.2 (Cauchy-Riemann equations)

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region $\Re$ is that, in $\Re$, $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(5)

If the partial derivatives in eqn. (5) are continuous in $\Re$, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in $\Re$.

Definition 1.3 (Harmonic functions) Francis (1983)

If the second partial derivatives of $u(x,y)$ and $v(x,y)$ exist and are continuous in a region $\Re$, then we find from (5) that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

(6)

Hence under these conditions, the real and imaginary parts of an analytic function satisfy Laplace's equation denoted by eqn. (3).

Real -valued Functions of two real variables $x$ and $y$ such as $u(x,y)$ and $v(x,y)$ which satisfy Laplace's equation in a region $\Re$ are called harmonic functions are said to be harmonic in the region $\Re$.

Definition 1.4

In polar form, the Laplace's equation can be written as

$$\frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vartheta}{\partial \theta^2} = 0$$

(7)

While the Cauchy-Riemann equation in eqn. (5) can be written thus

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \ \ldots \ (a) \qquad , \quad \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \ \ldots\text{(b)}$$

(8)

Theorem 1

The real and imaginary parts of an analytic function $w = f(z) = u(r,\theta) + iv(r,\theta)$ in polar form satisfy the Laplace's equation $\frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vartheta}{\partial \theta^2} = 0$

Proof

From eqn. (8), to eliminate $v$, differentiate (8a) partially with respect to $r$ and (8b) with respect to $\theta$. Then

$$\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}$$

(9)

$$\frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial \theta}\left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

(10)

But $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$ assuming the second partial derivatives are continuous. Hence from eqn.(9)

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \quad or \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

(11)

Similarly, by elimination of $u$, we find

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

(12)

so that the required result is proved

## 2.0 Statement of the problem

Problem 1

Consider a semi-infinite isotropic material of width $a$ whose boundaries are maintained at the indicated temperatures $\vartheta$ where $T$ is constant. The problem is to find the steady-state temperature in the shaded region.

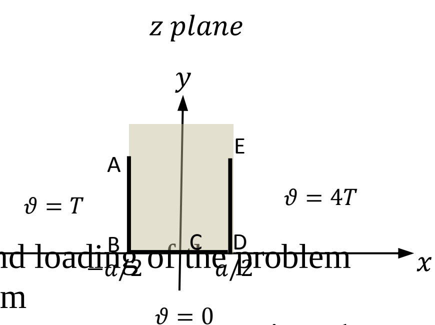


Fig.1 Geometry and loading of the problem

Solution of problem

Under the given temperature distribution, the governing equation is the Laplace's equation given in eqn.(3)

Subject to the boundary condition

$$\vartheta\left(-\frac{a}{2}, y\right) = T \quad , 0 < y < \infty$$

(13)

$$\vartheta\left(\frac{a}{2}, y\right) = 4T \quad , 0 < y < \infty$$

(14)

$$\vartheta(x, 0) = 0 \quad , \quad -\frac{a}{2} < x < \frac{a}{2}$$

(15)

Holomorphic transformation of the problem

To make the problem analyzable in polar form, we transform the original $z$-plane of analysis into an upper-half $w$ plane using the holomorphic function

$$w = \sin\left(\frac{\pi z}{a}\right) \qquad , \; z = x + iy$$

(16)

Let $w = u + iv$

Hence writing

$$u + iv = w(z) = \sin\left(\frac{\pi z}{a}\right) = \sin\frac{\pi}{a}(x+iy) = \sin\frac{\pi}{a}x\cos i\frac{\pi}{a}y + \cos\frac{\pi}{a}x\sin i\frac{\pi}{a}y$$

$$= \sin\frac{\pi}{a}x\cosh\frac{\pi}{a}y + i\cos\frac{\pi}{a}x\sinh\frac{\pi}{a}y$$

Therefore

$$u(x, y) = \sin\frac{\pi}{a}x\cosh\frac{\pi}{a}y \qquad and \qquad v(x, y) = \cos\frac{\pi}{a}x\sinh\frac{\pi}{a}y$$

(17)

Now mapping each point of the slap in the $z$-plane unto $w$-plane we have

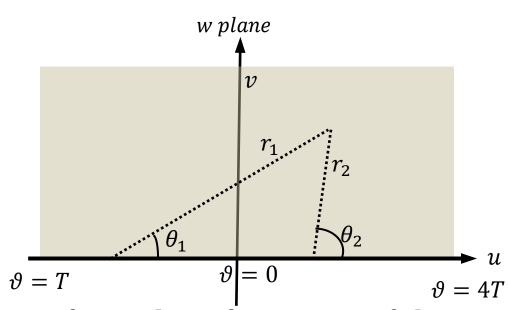


Fig. 2 The transformed configuration of the original problem

Under the transformation, the Laplace's equation and the boundary conditions becomes

$$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=0$$
(18)
Subject to
$$w(u,v)=\begin{cases} T & if \quad u<-1 \\ 0 & if \quad -1<u<1 \\ 4T & if \quad u>1 \end{cases}$$
(19)
Consider the analytic function
$$w= f\,(z)=A_1\ln(z+1)+A_2\ln(z-1)+B \qquad , \quad z=re^{i\theta}$$
(20)
Re-writing eqn. (14) in polar form, we have
$$u+iv=w=A_1\ln\left(re^{i\theta_1}+1\right)+A_2\ln\left(re^{i\theta_2}-1\right)+B$$
$$=A_1\ln \mathrm{r}+\mathrm{i}\mathrm{A}_1\,\theta_1+A_2\ln r+iA_2\theta_2+B$$
$$=A_1\ln \mathrm{r}+A_2\ln \mathrm{r}+i\left(A_1\theta_i+A_2\theta_2\right)+B$$
Therefore
$$u(r,\theta)=A_1\ln \mathrm{r}+A_2\ln \mathrm{r}$$
(21)
$$\mathrm{v}(r,\theta)=A_1\theta_1+A_2\theta_2+B$$
(22)
Next, since the transformed Dirichlet problem lie in the upper half plane, i.e. $Im(z)$ the function $\mathrm{v}(r,\theta)=A_1\theta_1+A_2\theta_2+B$ where $A_1$ , $A_2$ *and* $B$ are real constants is harmonic ( a solution of eqn. 20 ) since it is the imaginary part of $A_1\ln(z+1)+A_2\ln(z-1)+B$ .
To determine $A_1$ , $A_2$ , $B$ we use the boundary conditions
$$w=4T \quad for\ u>1,\ \ i.e \quad \theta_1=\theta_2=0$$
$$w=0 \quad for\ -1<u<1,\ \ i.e \quad \theta_1=0,\ \theta_2=\pi$$
$$w=T \quad for\ u<-1,\ \ i.e \quad \theta_1=\pi,\theta_2=\pi$$
(23)
Thus
$$4T=A_1(0)+A_2(0)+B \qquad \text{(i)}$$
$$0=A_1(0)+A_2(\pi)+B \qquad \text{(ii)}$$
(24)

$T = A_1(\pi) + A_2(\pi) + B$ (iii)

Solving eqns. (i)-(iii) , we have

$A_1 = \dfrac{T}{\pi}$ , $A_2 = -\dfrac{4T}{\pi}, B = 4T$

(25)

Therefore, the required solution in transformed plane is

$$w = A_1\theta_1 + A_2\theta_2 + B = \frac{T}{\pi}\tan^{-1}\left(\frac{v}{u+1}\right) - \frac{4T}{\pi}\tan^{-1}\left(\frac{v}{u-1}\right) + 4T$$

(26)

While the required solution to the problem in the original $z$ -plane is

$$\vartheta = \frac{T}{\pi}\tan^{-1}\left\{\frac{\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right)}{\sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right)+1}\right\} - \frac{4T}{\pi}\tan^{-1}\left\{\frac{\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right)}{\sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right)-1}\right\} + 4T$$

(27)

Problem 2

Consider an iron plate with a circular stop hole of radius 1 mm introduced at the center of the plate. If the temperatures are maintained as indicated in Fig 2., find the steady temperature at any point of the shaded region.

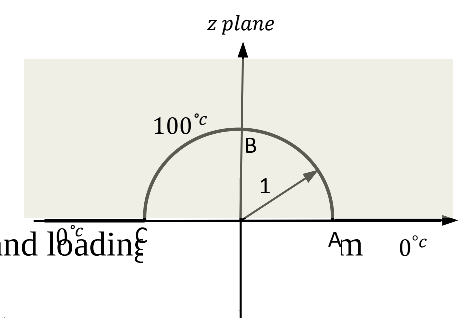


Fig.3 Geometry and loading m

## 3.0 Methodology

Under this configuration, the governing equation is the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

(28)

Subject to the boundary conditions

$$T(-\infty,0)=0^{\circ c}$$
$$T(\infty,0)=0^{\circ c}$$
$$T(0,1)=100^{\circ c}$$

(29)

Transformation of the problem

We map the inconvenient shaded region of the $z$ -plane onto the upper half of the $w$ -plane by means of the mapping function

$$w(z)=z+\frac{1}{z} \quad , \quad z=x+iy$$

(30)

Let

$$u+iv=x+iy+\frac{1}{x+iy}=x+iy+\frac{1}{x+iy}\times\frac{x-iy}{x-iy}$$
$$=x+iy+\frac{x-iy}{x^2+y^2}$$
$$=x+iy+\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$$
$$=x+\frac{x}{x^2+y^2}+i\left(y-\frac{y}{x^2+y^2}\right)$$

(31)

Equating real and imaginary parts of the complex numbers, we have

$$u=x+\frac{x}{x^2+y^2} \qquad , \quad v=y-\frac{y}{x^2+y^2}$$

(32)

Substituting the coordinate points into eqn. (32) , we have the transformed plane as

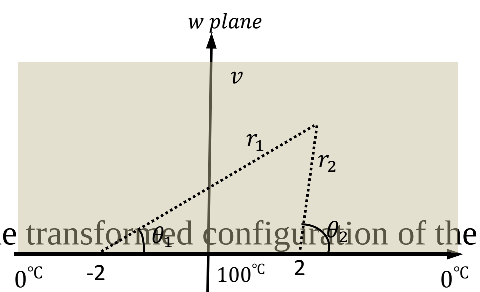


Fig .4 The transformed configuration of the original problem

The transformed equation with the boundary condition becomes

$$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=0$$
(33)

$$w(u,0)=\begin{cases}0^{\circ c} & if \quad u<-2\\ 100^{\circ c} & if \quad -2<u<2\\ 0^{\circ c} & if \quad u>2\end{cases}$$
(34)

The function $B_1\theta_1+B_2\theta_2+C$ where $B_1,\ B_2\ and\ C$ are real constants, is harmonic since it is the imaginary part of $B_1\ln(z+1)+B_2\ln(z-1)+C$

To determine $B_1,\ B_2\ ,C$ , we make use of the boundary conditions

$w=0^{\circ c}$ $\quad$ $for\ \ u>2\ \ i.e.\ \theta_1=\theta_2=0$

$w=100^{\circ c}$ $\quad$ $for\ \ -2<u<2\ \ i.e.\ \theta_1=\theta_2=\pi$

$w=0^{\circ c}$ $\quad$ $for\ \ u<-2\ \ i.e.\ \theta_1=\pi,\ \ \theta_2=\pi$

Thus

$0=B_1(0)+B_2(0)+C$ …… (i)

$100=B_1(0)+B_2(\pi)+C$ ….. (ii)

$0=B_1(\pi)+B_2(\pi)+C$ …… (iii)

Solving eqns. (i)-(iii), we have

$$B_1=-\frac{100}{\pi},\ \ B_2=\frac{100}{\pi},\ \ C=0$$
(35)

Thus, the solution to the problem in the $w$ plane is

$$w(u,v)=B_1\theta_1+B_2\theta_2+C$$
$$=-\frac{100}{\pi}\tan^{-1}\left(\frac{v}{u+2}\right)+\frac{100}{\pi}\tan^{-1}\left(\frac{v}{u-2}\right) \qquad (36)$$

Then substituting the values of $u\ and\ v$ into eqn.(36) , we have

$$w(z)=\frac{100}{\pi}\tan^{-1}\left(\frac{y\left(x^2+y^2-1\right)}{\left(x^2+y^2+1\right)x-2\left(x^2+y^2\right)}\right)-\frac{100}{\pi}\tan^{-1}\left(\frac{y\left(x^2+y^2-1\right)}{\left(x^2+y^2+1\right)x+2\left(x^2+y^2\right)}\right)$$

or, in polar coordinates,

$$\frac{100}{\pi}\tan^{-1}\left\{\frac{\left(r^2-1\right)\sin\theta}{\left(r^2+1\right)\cos\theta-2r}\right\}-\frac{100}{\pi}\tan^{-1}\left\{\frac{\left(r^2-1\right)\sin\theta}{\left(r^2+1\right)\cos\theta+2r}\right\}$$

## 4.0 Conclusion

In this paper, we have shown that a harmonic function defined in a domain is the same thing as a steady-state temperature in the domain. By exploring the application of conformal mapping and the analytic function $f(z) = A\ln(z+1) + B\ln(z-1) + C$ whose imaginary part is harmonic, we obtain the steady-state temperature. This method of analysis portrays the beauty of the theory of complex variable method in the solution of two dimensional harmonic Dirichlet problems in the theory of heat flows which is also possible in ideal fluid flows, thermal physics and electromagnetism as demonstrated in the works of Wesley et al (2010), Weimin et al. (2016).

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