# SOME LACUNARY SEQUENCE SPACES OF GENERALIZED $B^{\mu}-$ DIFFERENCE OPERATOR AND INVARIANT MEANS DEFINED BY MUSIELAK- ORLICZ FUNCTIONS ON $n-$ NORMED SPACE.

**By**
**Balili, Ado**
*Department of Mathematics, Yusuf Maitama Sule University, Kano, Nigeria*
*adobalili@gmail.com*

***Abstract:*** *The purpose of this paper is to introduce some sequence spaces which are defined by combining the concept of Musielak-Orlicz functions, Generalized$B^{\mu}$ $-$Difference operator, invariant means and Lacunary convergence on $n-$ Normed Space. We also study some algebraic and topological properties of the newly introduced spaces and a few inclusion relations*

***Keywords:*** *Generalized $B^{\mu}$ $-$Difference Operator, Invariant Means, Lacunary Sequence, Musielak-Orlicz Functions, $n-$ Normed Space,*

## 1. Introduction

The notion of difference sequence space was introduced by (Kizmaz, 1981). It was further generalized by (Et and Colak, 1995) as follows: $Z(\Delta^{\mu})=\{x=(x_k)\in \omega:(\Delta^{\mu}x_k)\in Z, for\ Z=\ell_\infty, c, and\ c_o$where $\mu$ is a non negative integer and

$$\Delta^{\mu}x_k=\Delta^{\mu-1}x_k-\Delta^{\mu-1}x_{k+1},\quad \Delta^0 x_k=x_k,\ \forall k\in\mathbb{N} \tag{1.1}$$

Or equivalent to the following binomial representation

$$\Delta^{\mu}x_k=\sum_{\nu}^{\mu}(-1)^{\nu}\binom{\mu}{\nu}x_{k+\nu} \tag{1.2}$$

These sequence spaces were generalized by (Et and Basarir, 1997)
By taking, $Z=\ell_\infty(p),\ c(p)\ and\ c_o(p)$.
Dutta (2009), introduced the following difference sequence spaces using a new difference operator

$$Z(\Delta_{(\eta)}=\{x=(x_k)\in\omega:\Delta_{(\eta)}x\in Z\}for\ Z=\ell_\infty, c, and\ c_o$$
(1.3)

Where $\Delta_{(\eta)}x=\left(\Delta_{(\eta)}x_k\right)=x_k-x_{k-\eta}\ \ \forall k,\eta\in\mathbb{N}$.

In Dutta (2010) introduced the sequence spaces $\overline{c}\left(\|.,\ \ .\|,\Delta^{\mu}_{(\nu)},\ p\right)$, $\bar{c}_o\left(\|.,\ \ .\|,\Delta^{\mu}_{(\nu)},\ p\right)$, $\ell_\infty\left(\|.,\ \ .\|,\Delta^{\mu}_{(\nu)},\ p\right)$, $m\left(\|.,\ \ .\|,\Delta^{\mu}_{(\nu)},\ p\right)$ and $m_o\left(\|.,\ \ .\|,\Delta^{\mu}_{(\nu)},\ p\right)$ where $\mu,\eta\in\mathbb{N}$ and $\Delta^{\mu}_{(\eta)}x_k=\left(\Delta^{\mu}_{(\eta)}x_k\right)=\Delta^{\mu-1}_{(\eta)}x_k-\Delta^{\mu-1}_{(\eta)}x_{k-\eta}$, and $\Delta^{0}_{(\eta)}x_k=x_k$. $\ \forall k,\eta\in\mathbb{N}$ which is equivalent to the binomial representation:

$$\Delta_{(\eta)}^{\mu} x_k = \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} x_{k-\eta\nu} \quad (1.4)$$

The difference sequence spaces have been studied by many authors (Isik, 2004); (Mursaleen, 1996) and (Raj et al, 2010) and references there in. Basar and Altay, (2003) introduced the generalized difference matrix $B = (b_{mk}), \forall k, m \in \mathbb{N}$ which is the generalization of $\Delta_{(1)}$-difference operator by

$$b_{mk} = \begin{cases} r & k = m \\ s & k = m - 1 \\ 0 & k > m (0 \le k < m - 1) \end{cases} \quad (1.5)$$

Basarir and Kayikçi (2009) defined the matrix $B^{\mu}(b_{mk}^{\mu})$ which is reduced to the difference matrix $\Delta_{(0)}^{\mu}$ in case $r = 1, s = -1$.

The generalized $B^{\mu}$-difference operator is equivalent to the binomial representation:

$$B^{\mu}(x) = B^{\mu}(x_k) = \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} r^{\mu-\nu} s^{\nu} x_{k-\nu} \quad (1.6)$$

Let$\Lambda = (\lambda_k)$ be a sequence of non zero scalars. Then for a sequence space E, the multiplier sequence $\Lambda$ is defined as

$$E_{\Lambda} = \{x = (x_k) \in \omega : (\lambda_k x_k) \in E\} \quad (1.7)$$

An Orlicz function is a function $M: [0, \infty) \to [0, \infty)$, which is continuous non decreasing and convex with $M(0) = 0, M(x) > 0 \; for \; x > 0 \; and \; M(x) \to \infty as \; x \to \infty$.

We say that an Orlicz function M satisfies the $\Delta_2$-condition if there exists K>2 and $x_0 \ge 0 \; such \; that \; M(2x) \le KM(x) forall \; x \ge x_0$. The $\Delta_2$-condition is equivalent to $M(Lx) \le KLM(x) forall \; x > x_0 > 0 \; and \; \forall L, K > 1$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to define the following sequence space:

$$l_m = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty\} \quad (1.8)$$

This is called Orlicz sequence space. The space $l_m$ is a Banach space with norm

$$\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \le 1 \quad (1.9)$$

It is shown in Lindenstrauss and Tzafriri (1971) that every Orlicz sequence space $l_m$contains a subsequence isomorphic to$l_p (p \ge 1)$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see (Maligranda, 1989) and Musielak, 1983).

A sequence $\varkappa = (N_k)$ defined by

$$\aleph_k(v) = \sup\{|v|\mu - M_k(\mu) : \mu \ge 0\}, \; k = 1,2,3 \ldots \quad (1.10)$$

Is called the complimentary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subsequence $h_{\mathcal{M}}$ are defined as follow:

$$t_{\mathcal{M}} = \{x \in \omega : I_{\mathcal{M}}(cx) < \infty, for \; some \; c > 0\} \quad (1.11)$$

$$h_{\mathcal{M}} = \{x \in \omega: I_{\mathcal{M}}(cx) < \infty, for\ all\ c > 0\}$$

Where $I_{\mathcal{M}}$ is a convex modular defined by

$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}$ (1.12)

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$\|x\| = \inf\{k > 0: I_{\mathcal{M}}(\frac{x}{k}) \leq 1\}$ (1.13)

Or equipped with the Orlicz norm (Amemiya norm)

$\|x\|^0 = \inf\{\frac{1}{k}(1 + I_{\mathcal{M}}(kx): k > 0\}$ (1.14)

By a Lacunary sequence $\theta = (i_r), \ r = 0, 1, 2, \dots, where\ i_0 = 0.$

We mean an increasing sequence of non negative integers$h_r = i_r - i_{r-1} \to \infty\ (r \to \infty).$

The intervals determined by $\theta$ are denoted by $I_r = (i_{r-1}, i_r] and\ the\ ratio\ \frac{i_r}{i_{r-1}}$ will be denoted by $q_r$. The space of Lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al, (1978) as follows:

$N_\theta = \left\{x = (x_k): \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, for\ some\ L\right\}$ (1.15)

The concept of 2-normed spaces was initially developed by Gähler in the mid of 1960s see (Gähler, 1963), while that of n-normed spaces one can see in (Misiak, 1989). Since then many others have studied this concept and obtained various results, see (Gunawan, 2001a), (Gunawan, 2001b) and Gunawan and Mashadi, 2001)

Let $n \in \mathbb{N}$ and X be linear space over the field K, where K is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$.

A real valued function $\|.,\dots,.\|\ on\ X^n$ satisfying the following four conditions:

(1) $\|(x_1, x_2, \dots, x_n)\| = 0\ if\ and\ only\ if\ x_1, x_2, \dots, x_n$ are linearly dependent in X;

(2) $\|(x_1, x_2, \dots, x_n)\|$ invariant under permutation;

(3) $\|(\alpha x_1, x_2, \dots, x_n)\| = |\alpha| \|(x_1, x_2, \dots, x_n)\|\ for\ any\ \alpha \in K;$

(4) $\|(x + x^1, x_2, \dots, x_n)\| \leq \|(x, x_2, \dots, x_n)\| + \|(x^1, x_2, \dots, x_n)\|$

Is called an n-norm on X and the pair (X, $\|.,\dots,.\|$) is called an n-normed space over the field K. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n- norm$\|(x_1, x_2, \dots, x_n)\|_E$ = the volume of n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \dots, x_n$ which may be given explicitly by the formular

$\|(x_1, x_2, \dots, x_n)\|_E = \left|\det(x_{ij})\right|$ (1.16)

Where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n for\ each\ i = 1,2,3, \dots, n\ and\ \|.\|_E$ denotes the Euclidean norm.

Let (X, $\|.,\dots,.\|$) be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ linearly independent set in X. Then the following function $\|(.,\dots,.)\|_\infty\ on\ X^{n-1}$

Defined by $\|(x_1, x_2, \dots, x_n)\| = \max\{\|(x_1, x_2, \dots, x_{n-1}, a_i)\| : i = 1,2,3, \dots, n\}$ defines an (n-1) norm on X with respect to $\{(a_1, a_2, \dots, a_n)\}$ (1.17)

A sequence $(x_k)$ in an n-normed space (X, $\|.,\dots,.\|$) is said to converge to some L$\in X$ if

$$\lim_{k\to\infty} \|(x_k - L, z_1, \dots, z_{n-1})\| = 0 \quad (1.18)$$

For every $z_1, z_2, \dots, z_n \in X$.

A sequence $(x_k)$ in a normed space (X, $\|.,\dots,.\|$) is said to be Cauchy if

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \left\|(x_k - x_p, z_1, \dots, z_{n-1})\right\| = 0 \quad (1.19)$$

For every $z_1, z_2, \dots, z_{n-1} \in X$.

Every Cauchy sequence in X converges to some $L \in X$ then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. The n-normed space has been studied in (Mursaleen et al, 2014)

Let $\sigma$ be a mapping of the positive integers into itself. A continuous linear functional $\varphi$ on $\ell_\infty$ is said to be an invariant mean or $\sigma - mean$ if and only if

(1) $\varphi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_n \geq 0, for\ all\ n.$

(2) $\varphi(e) = 1, e = (1,1,1, \dots,)$

(3) $\varphi(x_{\sigma(n)}) = \varphi(x) for\ all\ x \in \ell_\infty$

If $x = (x_k) where\ Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \{x \in \ell_\infty : \lim_k t_{kn}(x) = l, uniformly\ in\ n\}$$

Where $l = \sigma - lim x$ and

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + x_{\sigma^2(n)} + \dots + x_{\sigma^k(n)}}{k+1} \quad (1.20)$$

In the case $\sigma$ is the translation mapping $n \to n+1, \sigma - mean$ is often called a Banach limit and $V_\sigma$ the set of bounded sequences of all whose invariant means are equal. This is called the set of almost convergent sequence (Schaefer, 1972).

## 2. Definitions and Preliminaries

**Definition 2. 1:** A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence of scalars $(\alpha_k)\ with\ |\alpha_k| \leq 1.$ (Kamthan and Gupta, 1981)

**Definition 2.2:** A sequence space E is said to be monotone if it contains the canonical pre-image of all its step spaces. (Kamthan and Gupta, 1981)

**Definition 2.3** Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if the following conditions are satisfied

(1) $p(x) \geq 0\ for\ all\ x \in X,$

(2) $p(-x) = p(x), for\ all\ x \in X,$

(3) $p(x+y) \leq p(x) + p(y), for\ all\ x \in X,$

(4) If $\lfloor \alpha_n - \alpha \rfloor \to 0, and\ p(x_n - x) \to 0, imply\ p(\alpha_n x_n - \alpha x) \to 0, as\ n \to \infty for\ all\ \alpha \in \mathbb{R}\ and\ x \in X$

A paranorm p for which $p(x) = 0\ implies\ x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space.

The main purpose of this paper is to introduce the following sequence spaces and examine some of their properties.

**Definition 2.4:** Let (X, $\|.,\ldots,.\|$) be n- normed space and let $\omega(n - X)$ denotes the space of X-valued sequences. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be any sequence of positive real numbers for all $k \in \mathbb{N}$ and $u = (u_k)$ such that $u_k \neq 0 (k = 1,2,3, \ldots)$. Let s be any real number such that $s \geq 0$. Then we define the following sequence spaces:

$$[\omega^\theta, \mathcal{M}, p, u, s, \|.,\ldots,.\|]_\sigma^\infty (B_\Lambda^\mu) =$$
$$\{x = (x_k) \in \omega(n - X): \sup_{r,\mu} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k\, [M_k(\left\| \frac{t_{nk}(B_\Lambda^\mu x_k)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\|)]^{p_k} < \infty, \rho > 0, s \geq 0\}.$$

$$[\omega^\theta, \mathcal{M}, p, u, s, \|.,\ldots,.\|]_\sigma (B_\Lambda^\mu) =$$
$$\{x = (x_k) \in \omega(n - X): \lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k\, [M_k\left(\left\| \frac{t_{nk}(B_\Lambda^\mu x_k) - L}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\|\right)]^{p_k} = 0, for\ some\ L, \rho > 0, s \geq 0\}$$

$$[\omega^\theta, \mathcal{M}, p, u, s, \|.,\ldots,.\|]_\sigma^0 (B_\Lambda^\mu) =$$
$$\{x = (x_k) \in \omega(n - X): \lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k\, [M_k\left(\left\| \frac{t_{nk}(B_\Lambda^\mu x_k)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\|\right)]^{p_k} = 0, \rho > 0, s \geq 0\}$$

**Note:** if $n = 2, (B_\Lambda^\mu) = (\Delta_v^m)$ we get

$$[\omega^\theta, \mathcal{M}, p, u, s, \|.,\ldots,.\|]_\sigma^\infty (\Delta_v^m) =$$
$$\{x = (x_k) \in \omega(2 - X): \sup_{r,\mu} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k\, [M_k(\left\| \frac{t_{nk}((\Delta_v^m x_k)}{\rho}, z_1 \right\|)]^{p_k} < \infty, \rho > 0, s \geq 0\}.$$

$$[\omega^\theta, \mathcal{M}, p, u, s, \|.,\ldots,.\|]_\sigma (\Delta_v^m) =$$
$$\{x = (x_k) \in \omega(2 - X): \lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k\, [M_k\left(\left\| \frac{t_{nk}(\Delta_v^m x_k) - L}{\rho}, z_1 \right\|\right)]^{p_k} = 0, for\ some\ L, \rho > 0, s \geq 0\}$$

$$[\omega^\theta, \mathcal{M}, p, u, s, \|.,\ldots,.\|]_\sigma^0 (\Delta_v^m) =$$

$\{x=(x_k)\in\omega(2-X):\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\left[M_k\left(\left\|\frac{t_{nk}(\Delta_v^m x_k)}{\rho},z_1\right\|\right)\right]^{p_k}=0,\rho>0,s\geq 0\right\}$ (See Aiyub, 2014)

**Remark 2.1:** The following inequality will be used throughout the paper. Let $p=(p_k)$ be a positive sequence of real numbers with $0<p_k\leq \sup_k p_k=H,\ D=Max(1,2^{H-1})$, then for all $a_k,b_k\in\mathbb{C}, for\ all\ k\in\mathbb{N}$.

We have $|a_k+b_k|^{p_k}\leq D(|a_k|^{p_k}+|b_k|^{p_k})$. Also $|\alpha|^{p_k}\leq Max\{1,|\alpha|^M\}$, $M=Max\{1,H\}$

## 3. Main Results

The following results are obtained in this work.

**Theorem 3.1:** Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $\theta=(k_r)$ be Lacunary sequence. Then $[\omega^\theta,\mathcal{M},p,u,s,\|.,\dots,.\|]_\sigma^\infty(B_\Lambda^\mu)$, $[\omega^\theta,\mathcal{M},p,u,s,\|.,\dots,.\|]_\sigma(B_\Lambda^\mu)$ and $[\omega^\theta,M,p,u,s,\|.,\dots,.\|]_\sigma^0(B_\Lambda^\mu)$ are linear spaces over the field of complex numbers.

**Proof.** Let $x=(x_k),y=(y_k)\in[\omega^\theta,\mathcal{M},p,u,s,\|.,\dots,.\|]_\sigma^0(B_\Lambda^\mu), and\ \alpha,\beta\in\mathbb{C}$. In order to prove the result we need to find some $\rho_3$ such that,

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu(\alpha x_k+\beta y_k)\right)}{\rho_3},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}$$
$$=0, uniformly\ in\ n.$$

Since $x=(x_k),y=(y_k)\in[\omega^\theta,\mathcal{M},p,u,s,\|.,\dots,.\|]_\sigma^0(B_\Lambda^\mu)$ there exists $\rho_1,\rho_2$ such that $\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}(B_\Lambda^\mu x_k)}{\rho_1},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}=0, uniformly\ in\ n.$ and

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_2},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}=0, uniformly\ in\ n.$$

Define $\rho_3=Max(2|\alpha|\rho_1,2|\beta|\rho_2)$. since $(M_k)$ is non decreasing, convex functions and $\|.,\dots,.\|$ is an n-norm on X, by using inequality in Remark 2. 1. We have

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu(\alpha x_k+\beta y_k)\right)}{\rho_3},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}\leq$$
$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu(\alpha x_k)\right)}{\rho_3},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}+$$
$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu(\beta y_k)\right)}{\rho_3},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}\leq$$

$$\mathrm{D}\lim_{r}\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho_1},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}+$$
$$D\lim_{r}\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_2},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}\le$$
$$\lim_{r}\frac{1}{h_r}\sum_{k\in I_r}k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho_1},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}+$$
$$\lim_{r}\frac{1}{h_r}\sum_{k\in I_r}k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_2},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}=0, as\ r$$
$$\to\infty, uniformly\ in\ n.$$

So that $(\alpha x_k)+(\beta y_k)\in x=(x_k),y=(y_k)\in[\omega^\theta,M,p,u,s,\|.,\dots,.\|]_\sigma^0\left(B_\Lambda^\mu\right)$. This completes the proof.
Similarly, we can prove that
$[\omega^\theta,\mathcal{M},p,u,s,\|.,\dots,.\|]_\sigma\left(B_\Lambda^\mu\right)$ and $[\omega^\theta,M,p,u,s,\|.,\dots,.\|]_\sigma^\infty\left(B_\Lambda^\mu\right)$ are also linear spaces.

**Theorem 3.2:** Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $\theta=(i_r)$ be a Lacunary sequence. Then $[\omega^\theta,\mathcal{M},p,u,s,\|.,\dots,.\|]_\sigma^0\left(B_\Lambda^\mu\right)$ is a topological linear space total paranormed by

$$g_\Delta(x)=\sum_{k=1}^{\mu}|x_k|$$
$$+inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}\right)^{1/H}\right.$$
$$\left.\le 1, for\ some\ \rho>0, r=1,2,3,\dots\right\}$$

**Proof.** Clearly $g_\Delta(x)\ge 0\ and\ g_\Delta(x)=g_\Delta(-x)$. Since $M_k(0)=0, for\ all\ n\in\mathbb{N}$, we get $g_\Delta(\bar\theta)=0, for\ x=\bar\theta$. Let $x=(x_k),y=(y_k)\in[\omega^\theta,M,p,u,s,\|.,\dots,.\|]_\sigma^0\left(B_\Lambda^\mu\right)$ and let us choose $\rho_1,\ \rho_2>0$ such that
$\sup_r\frac{1}{h_r}\sum_{k\in I_r}k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho_1},z_1,z_2,\dots,z_{n-1}\right\|\right)]^{p_k}\le 1, r=1,2,\dots$, and

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right\|\right)\right]^{p_k} \leq 1, r = 1,2, \ldots$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{nk}\left(B_\Lambda^\mu (x_k + y_k)\right)}{\rho}, z_1, z_2, \ldots, z_{n-1} \right\|\right)\right]^{p_k} \leq$$

$$\frac{\rho_1}{\rho_1 + \rho_2} \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \right\|\right)\right]^{p_k} +$$

$$\frac{\rho_1}{\rho_1 + \rho_2} \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \left[M_k \left(\left\| \frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_2}, z_1, z_2, \ldots, z_{n-1} \right\|\right)\right]^{p_k} \leq 1$$

Since $\rho > 0$, we have

$$g_\Delta(x+y) = \sum_{k=1}^{\mu} |x_k + y_k|$$

$$+ inf\left\{\rho^{p_r/_H}: (\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\, [M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k})^{1/_H}\right.$$

$$\left. \le 1, for\ some\ \rho, r = 1,2,3, \dots\right\}$$

$$\le \sum_{k=1}^{\mu} |x_k|$$

$$+ inf\left\{{\rho_1}^{p_r/_H}: (\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\, [M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k})^{1/_H}\right.$$

$$\left. \le 1, for\ some\ \rho_1 > 0, r = 1,2,3, \dots\right\}$$

$$+ \sum_{k=1}^{\mu} |y_k|$$

$$+ inf\left\{{\rho_2}^{p_r/_H}: (\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\, [M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho_2}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k})^{1/_H}\right.$$

$$\left. \le 1, for\ some\ \rho_2, r = 1,2,3, \dots\right\}$$

Therefore $g_\Delta(x+y) \le g_\Delta(x) + g_\Delta(y)$.
Finally, we prove that the scalar multiplication is continuous. Let $\lambda$ be a given non zero scalar in $\mathbb{C}$. then the continuity of the product follows from the following expression.

$$g_\Delta(\lambda x) = \sum_{k=1}^{\mu} |\lambda x_k| + \inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} u_k \left[M_k\left(\left\|\frac{t_{nk}(B_\Lambda^\mu \lambda x_k)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq 1, for\ some\ \rho, r = 1,2,3,\dots\right\} =$$

$$\lambda \sum_{k=1}^{\mu} |x_k| + \inf\left\{(|\lambda|\zeta)^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} u_k \left[M_k\left(\left\|\frac{t_{nk}(B_\Lambda^\mu x_k)}{\zeta}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq 1, for\ some\ \zeta > 0, r = 1,2,3,\dots\right\}$$

Where $\zeta = \frac{\rho}{|\lambda|} > 0. since\ |\lambda|^{p_r} \leq Max(1, |\lambda|)^H$ , sup $p_r = H$
$g_\Delta(\lambda x) = Max(1, |\lambda)^H +$
$\inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} u_k \left[M_k\left(\left\|\frac{t_{nk}(B_\Lambda^\mu x_k)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{1/H} \leq 1, for\ some\ \rho, r = 1,2,3,\dots\right\}$.

This completes the proof of this theorem.

**Theorem 3.3:** Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function $p = (p_k)$ be a bounded sequence of positive real numbers and $\theta = (i_r)$ be a Lacunary sequence. If $\sup_k (M_k(x))^{p_k} < \infty, for\ all\ fixed\ x > 0$. Then

$$[\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma(B_\Lambda^\mu) \subset [\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma^\infty(B_\Lambda^\mu).$$

**Proof.** Let $x = (x_k) \in [\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma(B_\Lambda^\mu)$. Then there exists some positive number $\rho_1$ such that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r} k^{-s} u_k \left[M_k\left(\left\|\frac{t_{nk}(B_\Lambda^\mu x_k) - Le}{\rho_1}, z_1, z_2, \dots, z_{n-1}\right\|\right)\right]^{p_k} = 0$$

Define $\rho = 2\rho_1$ since $\mathcal{M} = (M_k)$ is non decreasing, convex and by using inequality in Remark 2.1, we have

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \, [M_k \left( \left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k} =$$

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \, [M_k \left( \left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k - L + L\right)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k} \leq$$

$$D\{\lim_r \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} k^{-s} u_k \, [M_k \left( \left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k - Le\right)}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k} +$$

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} k^{-s} u_k \, [M_k \left( \left\| \frac{Le}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k}\} <$$

$$D\{\lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \, [M_k \left( \left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k - L\right)}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k} +$$

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \, [M_k \left( \left\| \frac{Le}{\rho_1}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k}\}$$

Since $\sup_r [M_k(z)]^{p_k} < \infty$, we can take the $\sup_r [M_k(z)]^{p_k} = D$,

Hence we get $(x_k) \in [\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma^\infty\left(B_\Lambda^\mu\right)$

This completes the proof

**Theorem 3.4:** Let $\mathcal{M} = (M_k)$ and $T = (t_k)$ be two Musielak-Orlicz functions. Then we have

(i) $[\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma^\infty\left(B_\Lambda^\mu\right) \cap [\omega^\theta, T, p, u, s, \|., \dots, .\|]_\sigma^\infty\left(B_\Lambda^\mu\right) \subset [\omega^\theta, \mathcal{M} + T, p, u, s, \|., \dots, .\|]_\sigma^\infty\left(B_\Lambda^\mu\right)$

(ii) $[\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma\left(B_\Lambda^\mu\right) \cap [\omega^\theta, T, p, u, s, \|., \dots, .\|]_\sigma\left(B_\Lambda^\mu\right) \subset [\omega^\theta, \mathcal{M} + T, p, u, s, \|., \dots, .\|]_\sigma\left(B_\Lambda^\mu\right)$

(iii) $[\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma^0\left(B_\Lambda^\mu\right) \cap [\omega^\theta, T, p, u, s, \|., \dots, .\|]_\sigma^0\left(B_\Lambda^\mu\right) \subset [\omega^\theta, \mathcal{M} + T, p, u, s, \|., \dots, .\|]_\sigma^0\left(B_\Lambda^\mu\right)$

**Proof 1.**Let $(x_k) \in [\omega^\theta, \mathcal{M}, p, u, s, \|., \dots, .\|]_\sigma^\infty\left(B_\Lambda^\mu\right) \cap [\omega^\theta, T, p, u, s, \|., \dots, .\|]_\sigma^\infty\left(B_\Lambda^\mu\right)$

Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \, [M_k \left( \left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k} < \infty$$

and

$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} u_k \, [T_k \left( \left\| \frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1} \right\| \right)]^{p_k} < \infty$, uniformly in $n, \rho > 0$.

We have

$$[(M_k + T_k)\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k}$$
$$\leq D[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k}$$
$$+ D[T_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k}$$

By Remark 2.1. Applying $\sum_{k\in I_r}$. and multiplying by $u_k, \frac{1}{h_r}$ $and$ $k^{-s}$ both side of the inequality, we get

$$.\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[(M_k + T_k)\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k} \leq$$
$$D\,\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k}$$
$$+ D\,\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[T_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu x_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k}$$

This completes the proof. Similarly (2) and (3) can be proved.

## 4. Conclusion

We investigated the algebraic and topological properties of the newly introduced spaces. It was discovered that the spaces are topologically linear and they are also total paranormed spaces with paranorm defined by

$$g_\Delta(x) = \sum_{k=1}^{\mu}|x_k|$$
$$+ inf\left\{\rho^{p_r/_H} : \left(\frac{1}{h_r}\sum_{k\in I_r} k^{-s}u_k\,[M_k\left(\left\|\frac{t_{nk}\left(B_\Lambda^\mu y_k\right)}{\rho}, z_1, z_2, \dots, z_{n-1}\right\|\right)]^{p_k}\right)^{1/_H}\right.$$
$$\left.\leq 1, for\ some\ \rho > 0, r = 1,2,3, \dots\right\}$$

The results could be extended to the space of double sequences and some geometric properties could as well be investigated for these sequence spaces as a recommendations to fill in the gap in the existing literature.

**References**

Aiyub, M. (2014). Some Lacunary Sequence spaces of invariant means Defined by Musielak –Orlicz function on 2-normed space.*Applied Mathematics, 5,* 2602-2611.
Basar, F. and Altay, B. (2003). On the space of sequence of p-bounded variation and related matrix mappings.*Ukrainian Mathematical Journal,55(1),* 108-118.
Basarir, M. and Kayikci, M. (2009). On the generalized $B^m$ −Riez difference sequence space and $\beta -$ property.*Journal of Inequalities and Applications,* vol. 2009, Article ID 385029, 18pages.
Dutta, H. (2009). On some difference sequence spaces.*Pacific Journal of Science and Technology,10,* 243-247.
Dutta, H. (2010). Some statistically convergent difference sequence spaces defined over real 2-normed linear spaces.*Applied Sciences,12,* 37-47.
Et, M. and Çolak, R. (1995). On Some generalized difference sequence spaces.*Soochow Journal of Mathematics,21(4),* 377-386.
Et, M. and Basarir, M. (1997). On some new generalized difference sequence spaces.*Periodica Mathematica Hungarica,35(2),* 169-175.
Freedman, A.R., Sember J. J. and Raphael, M. (1978). Some Cesaro-type summability spaces.*Proceeding of the London Mathematical Society,37(3),* 508-520.
Gähler, S. (1963). 2-Metrsche Räume und three topologische struktur.*Mathematische Nachrichten,26,* 115-148.
Gunawan, H. (2001a). On n-inner product, n-norms and the Cauchy Schwartz inequality.*Scientiac Mathematicae Japanicae,5,* 47-54.
Gunawan, H. (2001b). The space of p-summable sequences and its natural n-norm.*Bulletin of the Australian Mathematical Society,64(1),* 137-147.
Gunawan, H. and Mashadi, M. (2001). n-normed spaces. *International Journal of Mathematical Sciences,27(10),* 631-639.
Isik, M. (2004). On statistical convergence of generalized difference sequence spaces.*Soochow Journal of Mathematics, 30(2),* 197-205.
Kamthan, K. P. and Gupta, M. (1981). *Sequence spaces and series,* Marcel Dekker, Inc., New York.
Kizmaz, H, (1981).On Certain Sequence spaces.*Canadian Mathematical Bulletin,24(2),* 169-176.
Lindenstrauss, J. and Tzafriri, L. (1971). On Orlicz sequence spaces.*Israel Journal of Mathemaics,10,* 379-390.
Maligranda, L. (1989). *Orlicz spaces and Interpolation,* Seminar in Mathematics, Polish Academy of science, Warsaw, Poland.
Misiak, A. (1989). n-inner product spaces.*Mathematische Nachrichten,140,* 299-319.

Mursaleen, M. (1996). Generalized space of difference sequences.*Journal of Mathematical Analysis and Applications,203(3),*738-745.
Mursaleen, M., Sharma, S. K. Mohiuddine, S. A. and Kilicman, A. (2014). New difference sequence spaces Defined by Musielak-Orlicz function.*Applied Analysis, Volume 2014,* 9pages.
Musielak, J. (1983). *Orlicz spaces and Modular spaces,* 1034 of lecture Notes in Mathematics, Springer, Berlin, Germany,
Raj, K., Sharma S. K. and Sharma, A. K.(2010). Some difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function.*Armenian Journal of Mathematics,3*, 127-141.
Schaefer, P. (1972). Infinite Matrices and Invariant means.*Proceeding of the American Mathematical Society,36,*104-110.