DIAGONAL MULTIVARIATE GENERALISED AUTOREGRESSIVE CONDITIONAL HETEROSKEDASTICITY MODELS AND THEIR APPLICATIONS TO REAL-TIME VOLATILITY SERIES.

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Abstract
The goal of this work was to create new multivariate time series models for the volatility series. Existing multivariate time series models for volatility series, such as Multivariate Generalized Autoregressive Conditional Heteroskedasticity (MGARCH) models, are used to identify new classes of models under certain conditions. The parameters for the UDMGARCH and LDMGARCH models are limited to the upper and lower diagonals of the coefficient matrices, respectively. Using empirical evidence from Nigerian crude oil quantity and price volatility series, the novel models are found to be adequate and have the same comparative advantage as the existing general MGARCH. As a result, UDMGARCH and LDMGARCH are established as new MGARCH model classes.

***Keywords**: UDMGARCH, LDMGARCH, MGARCH, Crude Oil Quantity Volatility and Crude Oil Price Volatility.*

1. Introduction

The Multivariate Generalized Autoregressive Conditional Heteroskedasticity Model is a multivariate expansion of the univariate GARCH model, with response variance determined by the conditional variance lag term and squared error. Multiple response conditional variance, which is a linear combination of autoregressive and moving average processes, is used in the multivariate model. The autoregressive component is represented by the lag terms of the variances, whereas the moving average aspect of the MGARCH model is represented by the lag terms of the squared error. The Pure Diagonal Model, the Upper Diagonal Model, and the Lower Diagonal Model are the three types of diagonal models used in multivariate time series.

The parameters of the pure diagonal model are restricted to the principal diagonal of the coefficient matrices. The predictor variable parameters are not included in the model. This is a multivariate representation of a univariate time series in which the parameters in the coefficient matrices are restricted to the principal diagonal and are associated with the lag terms of the response variance at different orders. The upper and lower diagonal models restrict the coefficient matrices' parameters to the upper and lower diagonals, respectively.

The upper and lower diagonal models have an advantage over the pure diagonal model in that they allow for interactive effects.

This allows for interdependence and evaluation of the feed forward and feedback mechanism between response and predictor variance. Upper and Lower Diagonal Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models are components of the Multivariate GARCH model. According to [1], MGARCH model for volatility series predicted by the conditional variances' distributed lags and squared errors. A set of multiple variances in the MGARCH models are linear combinations of the distributed lag of the squared error and the response and predictor variances expressed as autoregressive and moving average processes with the orders "p" and "q." MGARCH models establish the interactions and interdependence between variances with their respective lag terms and lag terms of their respective squared errors. Multivariate Simultaneous GARCH models were developed by [2]. Conditional CAPM using MGARCH-RM has been tested by [4]. Review on MGARCH models, which are flexible with heavy parameterization have been carried out by [7], [9]. The introduction of Diagonal VEC and BEKK is a way to reduce the number of parameters in the MGARCH models. The performance of the nonparametric and semi-parametric models was compared using empirical data. Lower diagonal bilinear moving average vector model with parameter restrictions to the lower diagonal in the coefficient matrices was proposed by [13]. The conditions for identifying lower diagonal bilinear moving average models were not specified, and empirical evidence results were not compared to existing multivariate bilinear moving average models. Relationship between the VEC and BEKK multivariate GARCH models have been investigated by [8]. On investigation of stationarity and ergodicity of BEKK multivariate GARCH models. A long-memory process in asset returns with multivariate GARCH innovations, [5]. The general form of multivariate generalised autoregressive conditional heteroskedasticity model for volatility series is presented by [11]. A review on Multivariate GARCH models with time varying variance-covariance for the exchange rate is carried out by [10]. In this paper, we propose Upper and Lower Diagonal Multivariate Generalised Autoregressive Conditional Heteroskedasticity Models.

2. Methodology

2.1 Diagonal MGARCH Models

The section presents two classes of Multivariate Generalized Autoregressive Heteroskedasticity Models and their conditions for identification.

Proposition 1
Given $Y_{it(i=1,\dots,m)}$ a multivariable time processes with conditional variances $\sigma^2_{it(i=1,\dots,m)}$, squared error terms $\epsilon^2_{vt(v=1,\dots,n)}$ and constants $\gamma_{i(i=1,\dots,m)}$, σ^2_{jt-k} *and* ϵ^2_{vt-s} represent the lag terms of the autoregressive and moving average components of the volatility measure such that $\sigma^2_{it(i=1,\dots,m)}$ are expressed as a function of σ^2_{jt-k} and ϵ^2_{vt-s} with the respective matrices of coefficients $\varphi_{ij.k(j=1,\dots,n)}$ and $\theta_{iv.s(v=1,\dots,n)}$. If the number of $\varphi_{1j.k} > \varphi_{2j.k} > \cdots > \varphi_{mj.k}$ and $\theta_{1v.s} > \theta_{2v.s} > \cdots > \theta_{mv.s}$, then we have Upper Diagonal Multivariate Generalised Autoregressive Conditional Heteroskedasticity (UDMGARCH) Models. If the number of $\varphi_{1j.k} < \varphi_{2j.k} < \cdots < \varphi_{mj.k}$ and $\theta_{1v.s} < \theta_{2v.s} < \cdots < \theta_{mv.s}$, then we have Lower Diagonal Multivariate Generalised Autoregressive Conditional Heteroskedasticity (LDMGARCH).

Case 1: Upper Diagonal MGARCH Models
Given σ^2_{it}; $if\ i = 1;\ j = 1, 2, 3, \dots, n\ ; k = 1, 2, 3, \dots, p; v = 1, 2, 3, \dots, n; s = 1, 2, \dots, q$

$if\ i = 2; j = 2, 3, \dots, n; k = 1, 2, 3, \dots, p;\ v = 2, 3, \dots, n; s = 1, 2, \dots, q$

$if\ i = 3;\ j = 3, \dots, n; k = 1, 2, 3, \dots, p;\ v = 3, \dots, n; s = 1, 2, \dots, q$

$if\ i = m;\ j = n; k = 1, 2, 3, \dots, p;\ v = n; s = 1, 2, \dots, q$

σ^2_{it} is a compendium of Upper Diagonal MGARCH models with sequential coefficients $\varphi_{1j.k}, \varphi_{2j.k}, \dots, \varphi_{mj.k}$ *and* $\theta_{1v.s}, \theta_{2v.s}, \dots, \theta_{mv.s}$ presented in the form,

$$\sigma^2_{it} = \begin{cases} \gamma_1 + \varphi_{1j.k}\sigma^2_{jt-k} + \theta_{1v.s}\epsilon^2_{vt-s}, j = 1, 2, 3, \dots, n; k = 1, 2, 3, \dots, p; \\ \qquad v = 1,2,3, \dots, n;\ s = 1,2, \dots, q \\ \gamma_2 + \varphi_{2j.k}\sigma^2_{jt-k} + \theta_{2v.s}\epsilon^2_{vt-s}, j = 2, 3, \dots, n; k = 1, 2, 3, \dots, p; \\ \qquad v = 2,3, \dots, n; s = 1,2, \dots, q \\ \gamma_3 + \varphi_{3j.k}\sigma^2_{jt-k} + \theta_{3v.s}\epsilon^2_{vt-s}, j = 3, \dots, n; k = 1, 2, 3, \dots, p; \\ \qquad v = 3, \dots, n; s = 1,2, \dots, q \\ \vdots \\ \gamma_m + \varphi_{mj.k}\sigma^2_{jt-k} + \theta_{mv.s}\epsilon^2_{vt-s}, j = n; k = 1,2, 3, \dots, p; \\ \qquad v = n; s = 1,2, \dots, q \end{cases} \tag{1}$$

Equation (1) remains valid for $\varphi_{1j.k} > \varphi_{2j.k} > \cdots > \varphi_{mj.k}$ *and* $\theta_{1v.s} > \theta_{2v.s} > \cdots > \theta_{mv.s}$

Hence, from equation “1”, Upper Diagonal MGARCH model is,

$$\sigma_{it}^2 = \gamma_i + \sum_{j=1}^{n}\sum_{k=1}^{p} \varphi_{ij.k}\sigma_{jt-k}^2 + \sum_{v=1}^{n}\sum_{s=1}^{q} \theta_{iv.s}\epsilon_{vt-s}^2 \, , i = 1, \dots , m \qquad (2)$$

for $\varphi_{1j.k} > \varphi_{2j.k} > \cdots > \varphi_{mj.k}$ and $\theta_{1v.s} > \theta_{2v.s} > \cdots > \theta_{mv.s}$

Proof:
Let the general Multivariate Generalised Autoregressive Conditional Heteroskedasticity (MGARCH) model be presented as

$$\sigma_{it}^2 = \gamma_i + \sum_{j=1}^{n}\sum_{k=1}^{p} \varphi_{ij.k}\sigma_{jt-k}^2 + \sum_{v=1}^{n}\sum_{s=1}^{q} \theta_{iv.s}\epsilon_{vt-s}^2 \, , i = 1, \dots , m \qquad (3)$$

By expansion, we have

$$\begin{aligned}\sigma_{it}^2 = \gamma_i &+ \sum_{j=1}^{n}\left[\varphi_{ij.1}\sigma_{jt-1}^2 + \varphi_{ij.2}\sigma_{jt-2}^2 + \cdots + \varphi_{ij.p}\sigma_{jt-p}^2\right] \\ &+ \sum_{v=1}^{n}\left[\theta_{iv.1}\epsilon_{vt-1}^2 + \theta_{iv.2}\epsilon_{vt-2}^2 + \cdots + \theta_{iv.q}\epsilon_{vt-p}^2\right]\end{aligned}$$

$$\begin{aligned}= \gamma_i + \big[\big(&\varphi_{i1.1}\sigma_{1t-1}^2 + \varphi_{i2.2}\sigma_{2t-1}^2 + \cdots + \varphi_{in.1}\sigma_{nt-p}^2\big) \\ &+ (\varphi_{i1.2}\sigma_{1t-2}^2 + \varphi_{i2.2}\sigma_{2t-2}^2 + \cdots + \varphi_{in.2}\sigma_{nt-2}^2) + \cdots \\ &+ \big(\varphi_{i1.p}\sigma_{1t-p}^2 + \varphi_{i2.p}\sigma_{2t-p}^2 + \cdots + \varphi_{in.p}\sigma_{nt-p}^2\big)\big]\end{aligned}$$

$$\begin{aligned}+ \ \big[(&\theta_{i1.1}\epsilon_{1t-1}^2 + \theta_{i2.2}\epsilon_{2t-1}^2 + \cdots + \theta_{in.1}\epsilon_{nt-1}^2) \\ &+ (\theta_{i1.2}\epsilon_{1t-2}^2 + \theta_{i2.2}\epsilon_{2t-2}^2 + \cdots + \theta_{in.2}\epsilon_{nt-2}^2) + \cdots \\ &+ \big(\theta_{i1.q}\epsilon_{1t-q}^2 + \theta_{i2.q}\epsilon_{2t-q}^2 + \cdots + \theta_{in.q}\epsilon_{nt-q}^2\big)\big] \qquad (4)\end{aligned}$$

From (4), for $i = 1;\ j = 1,2, \dots , n; k = 1,2, \dots , p; v = 1,2, \dots , n; s = 1,2, \dots , q$, we have

$$\sigma_{1t}^2 = \gamma_1 + \varphi_{11.1}\sigma_{1t-1}^2 + \varphi_{12.1}\sigma_{2t-1}^2 + \cdots + \varphi_{1n.1}\sigma_{nt-1}^2 + \varphi_{11.2}\sigma_{1t-2}^2 + \varphi_{12.2}\sigma_{2t-2}^2 + \cdots + \varphi_{1n.2}\sigma_{nt-2}^2 + \varphi_{11.p}\sigma_{1t-p}^2 + \varphi_{12.p}\sigma_{2t-p}^2 + \cdots + \varphi_{1n.p}\sigma_{nt-p}^2 + \theta_{11.1}\epsilon_{1t-1}^2 + \theta_{12.1}\epsilon_{2t-1}^2 + \cdots + \theta_{1n.1}\epsilon_{nt-1}^2 + \theta_{11.2}\epsilon_{1t-2}^2 + \theta_{12.2}\epsilon_{2t-2}^2 + \cdots + \theta_{1n.2}\epsilon_{nt-2}^2 + \theta_{11.q}\epsilon_{1t-q}^2 + \theta_{12.q}\epsilon_{2t-q}^2 + \cdots + \theta_{1n.q}\epsilon_{nt-q}^2 \quad (5)$$

Equation (5) reduces to,

$$\sigma_{1t}^2 = \gamma_1 + \varphi_{1j.k}\sigma_{jt-k}^2 + \theta_{1v.s}\epsilon_{vt-s}^2 \quad (6)$$

For $i = 2;\ j = 2,3, \ldots, n; k = 1,2, \ldots, p; v = 2,3, \ldots, n; s = 1,2, \ldots, q$, we have,

$$\sigma_{2t}^2 = \gamma_2 + \varphi_{22.1}\sigma_{2t-1}^2 + \varphi_{23.1}\sigma_{3t-1}^2 + \cdots + \varphi_{2n.1}\sigma_{nt-1}^2 + \varphi_{22.2}\sigma_{2t-2}^2 + \varphi_{23.2}\sigma_{3t-2}^2 + \cdots + \varphi_{2n.2}\sigma_{nt-2}^2 + \varphi_{22.p}\sigma_{2t-p}^2 + \varphi_{23.p}\sigma_{3t-p}^2 + \cdots + \varphi_{2n.p}\sigma_{nt-p}^2 + \theta_{22.1}\epsilon_{2t-1}^2 + \theta_{23.1}\epsilon_{3t-1}^2 + \cdots + \theta_{2n.1}\epsilon_{nt-1}^2 + \theta_{22.2}\epsilon_{2t-2}^2 + \theta_{23.2}\epsilon_{3t-2}^2 + \cdots + \theta_{2n.2}\epsilon_{nt-2}^2 + \theta_{22.q}\epsilon_{2t-q}^2 + \theta_{23.q}\epsilon_{3t-q}^2 + \cdots + \theta_{2n.q}\epsilon_{nt-q}^2 \quad (7)$$

Equation (7) reduces to,

$$\sigma_{2t}^2 = \gamma_2 + \varphi_{2j.k}\sigma_{jt-k}^2 + \theta_{2v.s}\epsilon_{vt-s}^2 \quad (8)$$

For $i = 3;\ j = 3,4, \ldots, n; k = 1,2, \ldots, p; v = 3,4, \ldots, n; s = 1,2, \ldots, q$, we have,

$$\sigma_{3t}^2 = \gamma_3 + \varphi_{33.1}\sigma_{3t-1}^2 + \varphi_{34.1}\sigma_{4t-1}^2 + \cdots + \varphi_{3n.1}\sigma_{nt-1}^2 + \varphi_{33.2}\sigma_{3t-2}^2 + \varphi_{34.2}\sigma_{4t-2}^2 + \cdots + \varphi_{3n.2}\sigma_{nt-2}^2 + \varphi_{33.p}\sigma_{3t-p}^2 + \varphi_{34.p}\sigma_{4t-p}^2 + \cdots + \varphi_{3n.p}\sigma_{nt-p}^2 + \theta_{33.1}\epsilon_{3t-1}^2 + \theta_{34.1}\epsilon_{4t-1}^2 + \cdots + \theta_{3n.1}\epsilon_{nt-1}^2 + \theta_{33.2}\epsilon_{3t-2}^2 + \theta_{34.2}\epsilon_{4t-2}^2 + \cdots + \theta_{3n.2}\epsilon_{nt-2}^2 + \theta_{33.q}\epsilon_{3t-q}^2 + \theta_{34.q}\epsilon_{4t-q}^2 + \cdots + \theta_{3n.q}\epsilon_{nt-q}^2 \quad (9)$$

Equation reduces to,

$$\sigma_{3t}^2 = \gamma_3 + \varphi_{3j.k}\sigma_{jt-k}^2 + \theta_{3v.s}\epsilon_{vt-s}^2 \quad (10)$$

For $i = m;\ j = n; k = 1,2, \ldots, p; v = n; s = 1,2, \ldots, q$, we have,

$$\sigma_{mt}^2 = \gamma_m + \varphi_{mn.1}\sigma_{nt-1}^2 + \varphi_{mn.2}\sigma_{nt-2}^2 + \cdots + \varphi_{mn.p}\sigma_{nt-p}^2 + \theta_{mn.1}\epsilon_{nt-1}^2 + \theta_{mn.2}\epsilon_{nt-2}^2 + \cdots + \theta_{mn.q}\epsilon_{nt-q}^2 \quad (11)$$

Equation (11) reduces to,

$$\sigma_{mt}^2 = \gamma_m + \varphi_{mj.k}\sigma_{jt-k}^2 + \theta_{mv.s}\epsilon_{vt-s}^2 \tag{12}$$

Therefore, (6), (8), (10) and (12) are a compendium of UDMGARCH models, which complete the proof.

Case 2: Lower Diagonal MGARCH Models

Given σ_{it}^2; $if\ i = 1; j = 1; k = 1, 2, 3, \dots, p;\ v = 1; s = 1, 2, \dots, q$
$if\ i = 2;\ j = 1, 2; k = 1, 2, 3, \dots, p;\ v = 1, 2; s = 1, 2, \dots, q$
$if\ i = 3;\ j = 1, 2, 3; k = 1, 2, 3, \dots, p;\ v = 1, 2, 3; s = 1, 2, \dots, q$
$if\ i = m; j = 1, 2, 3, \dots, n; k = 1, 2, 3, \dots, p;\ v = 1, 2, 3, \dots, n; s = 1, 2, \dots, q$

σ_{it}^2 is a compendium of Upper Diagonal MGARCH models with sequential coefficients $\varphi_{1j.k}, \varphi_{2j.k}, \dots, \varphi_{mj.k}\ and\ \theta_{1v.s}, \theta_{2v.s}, \dots, \theta_{mv.s}$ presented in the form,

$$\sigma_{it}^2 = \begin{cases} \gamma_1 + \varphi_{1j.k}\sigma_{jt-k}^2 + \theta_{1v.s}\epsilon_{vt-s}^2, j = 1; k = 1, 2, 3, \dots, p; \\ \qquad v = 1;\ s = 1,2, \dots, q \\ \gamma_2 + \varphi_{2j.k}\sigma_{jt-k}^2 + \theta_{2v.s}\epsilon_{vt-s}^2, j = 1, 2; k = 1, 2, 3, \dots, p; \\ \qquad v = 1,2; s = 1,2, \dots, q \\ \gamma_3 + \varphi_{3j.k}\sigma_{jt-k}^2 + \theta_{3v.s}\epsilon_{vt-s}^2, j = 1, 2, 3; k = 1, 2, 3, \dots, p; \\ \qquad v = 1, 2, 3; s = 1,2, \dots, q \\ \vdots \\ \gamma_m + \varphi_{mj.k}\sigma_{jt-k}^2 + \theta_{mv.s}\epsilon_{vt-s}^2, j = 1, 2, 3, \dots, n; k = 1,2, 3, \dots, p; \\ \qquad v = 1, 2, 3, \dots, n; s = 1,2, \dots, q \end{cases} \tag{13}$$

Equation (13) is valid for $\varphi_{1j.k} < \varphi_{2j.k} < \cdots < \varphi_{mj.k}\ and\ \theta_{1v.s} < \theta_{2v.s} < \cdots < \theta_{mv.s}$

Hence, from equation “13”, Lower Diagonal MGARCH model is,

$$\sigma_{it}^2 = \gamma_i + \sum_{j=1}^{n}\sum_{k=1}^{p}\varphi_{ij.k}\sigma_{jt-k}^2 + \sum_{v=1}^{n}\sum_{s=1}^{q}\theta_{iv.s}\epsilon_{vt-s,}^2\, i = 1, \dots, m \tag{14}$$

for $\varphi_{1j.k} < \varphi_{2j.k} < \cdots < \varphi_{mj.k}\ and\ \theta_{1v.s} < \theta_{2v.s} < \cdots < \theta_{mv.s}$

Proof:
The proof of Lower Diagonal Multivariate Autoregressive Conditional Heteroskedasticity (LDMGARCH) Models applies same expansion of (3) to derive (4).
Therefore, from (4), the following conditions are set;

For $i = 1;\ j = 1; k = 1,2, \dots, p; v = 1; s = 1,2, \dots, q$, we have

$$\sigma_{1t}^2 = \gamma_1 + \varphi_{11.1}\sigma_{1t-1}^2 + \varphi_{11.2}\sigma_{1t-2}^2 + \varphi_{11.p}\sigma_{1t-p}^2 + \theta_{11.1}\epsilon_{1t-1}^2 + \theta_{11.2}\epsilon_{1t-2}^2 + \theta_{11.q}\epsilon_{1t-q}^2 \quad (15)$$

Equation (15) reduces to

$$\sigma_{1t}^2 = \gamma_1 + \varphi_{1j.k}\sigma_{jt-k}^2 + \theta_{1v.s}\epsilon_{vt-s}^2 \quad (16)$$

For $i = 2;\ j = 1, 2; k = 1,2, \dots, p; v = 1, 2; s = 1,2, \dots, q$, we have

$$\sigma_{2t}^2 = \gamma_2 + \varphi_{21.1}\sigma_{1t-1}^2 + \varphi_{22.1}\sigma_{2t-1}^2 + \varphi_{21.2}\sigma_{1t-2}^2 + \varphi_{22.2}\sigma_{2t-2}^2 + \cdots + \varphi_{21.p}\sigma_{1t-p}^2 + \varphi_{22.p}\sigma_{2t-p}^2 + \theta_{21.1}\epsilon_{1t-1}^2 + \theta_{22.1}\epsilon_{2t-1}^2 + \theta_{21.2}\epsilon_{1t-2}^2 + \theta_{22.2}\epsilon_{2t-2}^2 + \cdots + \theta_{21.q}\epsilon_{1t-q}^2 + \theta_{22.q}\epsilon_{2t-q}^2 \quad (17)$$

Equation (17) reduces to

$$\sigma_{2t}^2 = \gamma_2 + \varphi_{2j.k}\sigma_{jt-k}^2 + \theta_{2v.s}\epsilon_{vt-s}^2 \quad (18)$$

For $i = 3;\ j = 1,2\ 3; k = 1,2, \dots, p; v = 1,2\ 3; s = 1,2, \dots, q$, we have

$$\sigma_{3t}^2 = \gamma_3 + \varphi_{31.1}\sigma_{1t-1}^2 + \varphi_{32.1}\sigma_{2t-1}^2 + \varphi_{33.1}\sigma_{3t-1}^2 + \varphi_{31.2}\sigma_{1t-2}^2 + \varphi_{32.2}\sigma_{2t-2}^2 + \varphi_{33.2}\sigma_{3t-2}^2 + \cdots + \varphi_{31.p}\sigma_{1t-p}^2 + \varphi_{32.p}\sigma_{2t-p}^2 + \varphi_{33.p}\sigma_{3t-p}^2 + \theta_{31.1}\epsilon_{1t-1}^2 + \theta_{32.1}\epsilon_{2t-1}^2 + \theta_{33.1}\epsilon_{3t-1}^2 + \theta_{31.2}\epsilon_{1t-2}^2 + \theta_{32.2}\epsilon_{2t-2}^2 + \theta_{33.2}\epsilon_{3t-2}^2 + \cdots + \theta_{31.q}\epsilon_{1t-q}^2 + \theta_{32.q}\epsilon_{2t-q}^2 + \theta_{33.q}\epsilon_{3t-q}^2 \quad (19)$$

Equation (19) reduces to

$$\sigma_{3t}^2 = \gamma_3 + \varphi_{3j.k}\sigma_{jt-k}^2 + \theta_{3v.s}\epsilon_{vt-s}^2 \quad (20)$$

For $i = m;\ j = 1, 2, \dots, n; k = 1,2, \dots, p; v = 1, 2, \dots, n; s = 1,2, \dots, q$, we have,

$$\sigma_{mt}^2 = \gamma_m + \varphi_{m1.1}\sigma_{1t-1}^2 + \varphi_{m2.1}\sigma_{2t-1}^2 + \cdots + \varphi_{mn.1}\sigma_{nt-1}^2 + \varphi_{m1.2}\sigma_{1t-2}^2 + \varphi_{m2.2}\sigma_{2t-2}^2 + \cdots + \varphi_{mn.2}\sigma_{nt-2}^2 + \ldots + \varphi_{m1.p}\sigma_{1t-p}^2 + \varphi_{m2.p}\sigma_{2t-p}^2 + \cdots + \varphi_{mn.p}\sigma_{nt-p}^2 + \theta_{m1.1}\epsilon_{1t-1}^2 + \theta_{m2.1}\epsilon_{2t-1}^2 + \cdots + \theta_{mn.1}\epsilon_{nt-1}^2 + \theta_{m1.2}\epsilon_{1t-2}^2 + \theta_{m2.2}\epsilon_{2t-2}^2 + \cdots + \theta_{mn.2}\epsilon_{nt-2}^2 + \ldots + \theta_{m1.q}\epsilon_{1t-q}^2 + \theta_{m2.q}\epsilon_{2t-q}^2 + \theta_{mn.q}\epsilon_{nt-q}^2 \quad (21)$$

Equation (21) reduces to,

$$\sigma_{mt}^2 = \gamma_m + \varphi_{mj.k}\sigma_{jt-k}^2 + \theta_{mv.s}\epsilon_{vt-s}^2 \quad (22)$$

Therefore, (16), (18), (20) and (22) are a compendium of LDMGARCH models, if $\varphi_{1j.k} < \varphi_{2j.k} < \cdots < \varphi_{mj.k}$ *and* $\theta_{1v.s} < \theta_{2v.s} < \cdots < \theta_{mv.s}$. These complete the proof.

The parameters of equations "2" and "4" are restricted to the upper and lower diagonal elements of the coefficient matrices. As parsimonious models, the UDMGARCH and LDMGARCH models modify [1],[11] and [12].

2. 2 Graphical Analyses

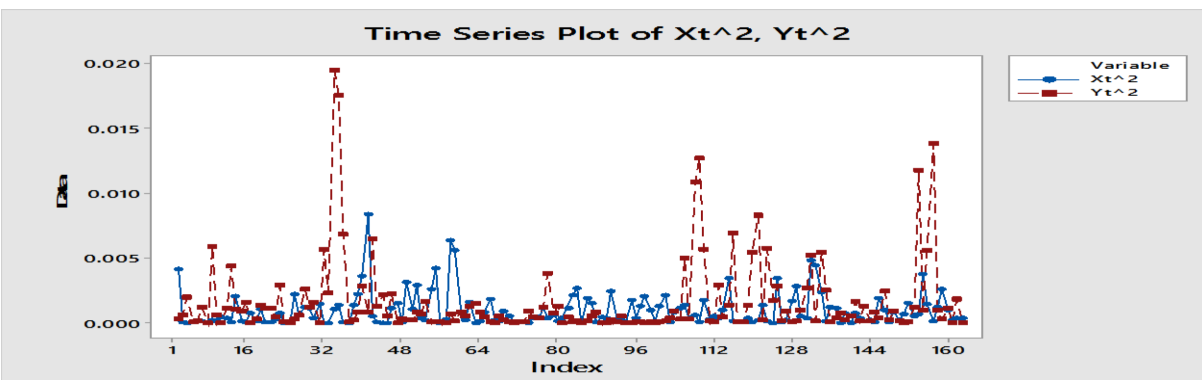

Figure 1: Volatility plots of Crude Oil Quantity and Price

The above figure represents the volatility plots of the Nigeria Crude Oil Quantity and Price as empirical example to explain the behaviours of the two series. The outliers indicate the presence of volatilities in the crude oil quantity and price. Furthermore, this paper considers the usefulness of the autocorrelation and partial autocorrelation functions of the series as tools to suggest appropriate models for the series. See figures 2, 3, 4 and 5 for the ACFs and PACFs of the volatility series.

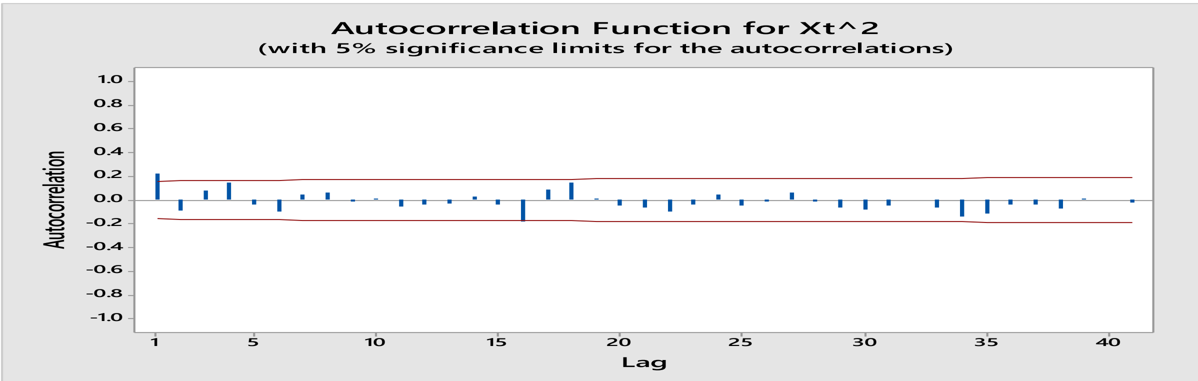

Figure 2: Autocorrelation Functions of Crude Oil Quantity Volatility

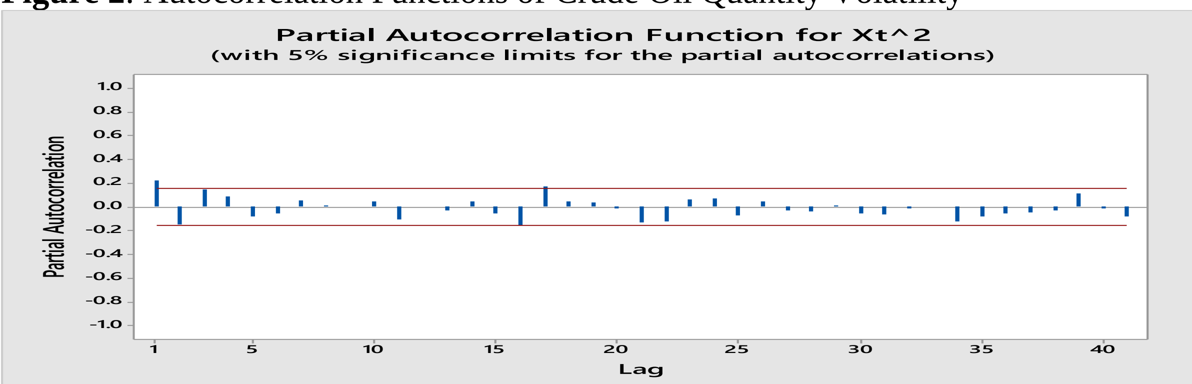

Figure 3: Partial Autocorrelation Function of Crude Oil Quantity Volatility

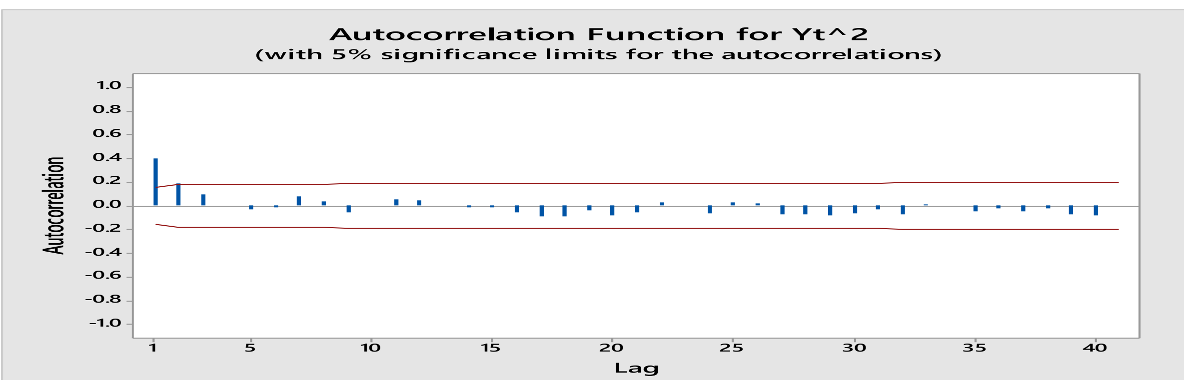

Figure 4: Autocorrelation Function of Crude Oil Price Volatility

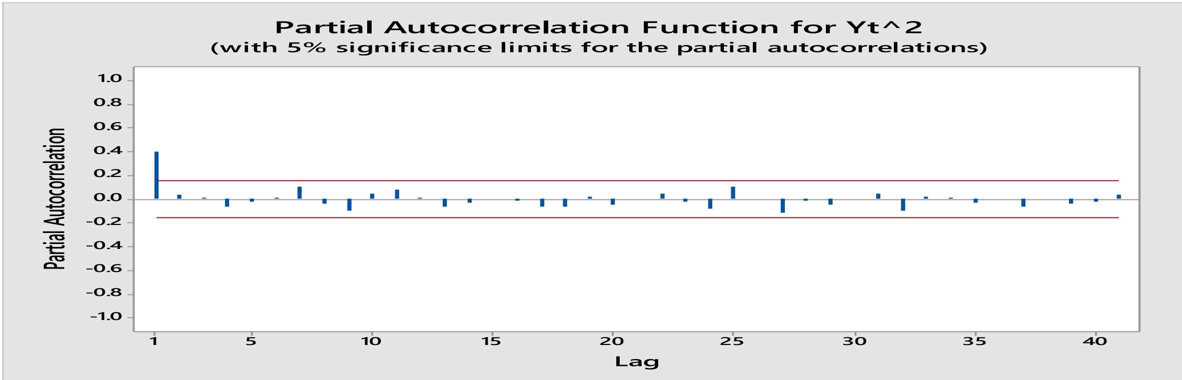

Figure 5: Partial Autocorrelation Function of Crude Oil Price Volatility

GARCH (p,q) is represented in this work as components of autoregressive and moving average processes. Autocorrelation and partial autocorrelation functions are utilized in the standard Autoregressive Moving Average Process to choose and rank the ARMA model. Because the GARCH (p,q) model is expressed as a mixture of the two processes, this technique is used here. The autocorrelation and partial autocorrelation functions of crude oil quantity and price volatilities are shown in Figures 2, 3, 4, and 5. In Multivariate Generalized Autoregressive Heteroskedasticity Models, the correlogram aids in establishing the maximum lag length that accounts for the order of the ARMA components. Figures 2, 3, 4, and 5 in the plots show large spikes in the autocorrelation and partial autocorrelation functions of crude oil quantity and price volatilities at the first time lag. The data in the two series' ACFs and PACFs points to the MGARCH [p(1,1), q(1,1)] model. For both quantity and price volatility series, this means that each of the ACFs and PACFs spikes at lag1. The OLS regression approach is proposed for the estimation of the model parameters.

2.3 Model Selection Criteria

Upper and Lower Diagonal Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models are identified as two types of MGARCH models in this paper. We propose two model selection criteria in this section, which may be compared to the current MGARCH Models.

i. Akaike Information Criterion (AIC):

$$AIC = \ln\left(\frac{RSS}{n}\right) + \left(\frac{2k}{n}\right) \tag{23}$$

where RSS = residual sum of squares, n = number of observations, k= number of parameters in the model.

iii. Schwartz's Information Criterion (SIC)

$$SIC = ln\left(\frac{RSS}{n}\right) + \left(\frac{k}{n}\right)\ln(n) \quad (24)$$

where RSS, n and k are as defined as above.

3. Model Estimation

In this part, we use the Nigeria Crude Oil Quantity and Price Volatilities to demonstrate the models. The upper and lower diagonal models' parameters are estimated using the ordinary least squares approach. Volatility as a square of log return series, Gujarati and Porter (2009).

Given $Y_{it(i=1,2)}$ to represent the Crude Oil Quantity and Price with σ^2_{1t} and σ^2_{2t} as crude oil quantity and price volatility measures respectively.

3.1. Estimation of UDMGARCH Model Parameters

Figures 2, 3, 4 show the ACF and PACF, which recommended the MGARCH [p(1,1), q(1,1)] model for crude oil production amount and price volatilities σ^2_{1t} and σ^2_{2t}. To begin, we show a set of simplified MGARCH [p(1,1), q(1,1)] models for σ^2_{1t} and σ^2_{2t} as

$$\begin{pmatrix}\sigma^2_{1t}\\ \sigma^2_{2t}\end{pmatrix} = \begin{pmatrix}\varphi_{11.1} & \varphi_{12.1}\\ \varphi_{21.1} & \varphi_{22.1}\end{pmatrix}\begin{pmatrix}\sigma^2_{1t-1}\\ \sigma^2_{2t-1}\end{pmatrix} + \begin{pmatrix}\theta_{11.1} & \theta_{12.1}\\ \theta_{21.1} & \theta_{22.1}\end{pmatrix}\begin{pmatrix}\epsilon^2_{1t-1}\\ \epsilon^2_{2t-1}\end{pmatrix} \quad (25)$$

Equation (25) is a complete MGARCH model for the first order of σ^2_{jt-k} and ϵ^2_{vt-s}. The expansion of (25) presents σ^2_{1t} and σ^2_{2t} as linear combinations of the lag terms of the variances and squares of the errors.

From the above model, the Upper Diagonal Multivariate GARCH Model for the two volatility series is presented as,

$$\begin{pmatrix}\sigma^2_{1t}\\ \sigma^2_{2t}\end{pmatrix} = \begin{pmatrix}\varphi_{11.1} & \varphi_{12.1}\\ 0 & \varphi_{22.1}\end{pmatrix}\begin{pmatrix}\sigma^2_{1t-1}\\ \sigma^2_{2t-1}\end{pmatrix} + \begin{pmatrix}\theta_{11.1} & \theta_{12.1}\\ 0 & \theta_{22.1}\end{pmatrix}\begin{pmatrix}\epsilon^2_{1t-1}\\ \epsilon^2_{2t-1}\end{pmatrix} \quad (26)$$

The parameters of Equation (26) are restricted to the upper diagonal of the coefficients matrices. The restriction gives room for parsimonious model due to the reduction in the number of parameters. The parameter estimates using ordinary least squares regression approach for Equations (25) and (26) give the following estimated models;

$$\begin{pmatrix} \hat{\sigma}_{1t}^2 \\ \hat{\sigma}_{2t}^2 \end{pmatrix} = \begin{pmatrix} 0.3398 & 0.0461 \\ 0.314 & 0.4398 \end{pmatrix} \begin{pmatrix} \sigma_{1t-1}^2 \\ \sigma_{2t-1}^2 \end{pmatrix} + \begin{pmatrix} 0.0733 & 0.0099 \\ 0.090 & 0.0171 \end{pmatrix} \begin{pmatrix} \epsilon_{1t-1}^2 \\ \epsilon_{2t-1}^2 \end{pmatrix} \quad (27)$$

$$\begin{pmatrix} \hat{\sigma}_{1t}^2 \\ \hat{\sigma}_{2t}^2 \end{pmatrix} = \begin{pmatrix} 0.3398 & 0.0461 \\ 0 & 0.4398 \end{pmatrix} \begin{pmatrix} \sigma_{1t-1}^2 \\ \sigma_{2t-1}^2 \end{pmatrix} + \begin{pmatrix} 0.0733 & 0.0099 \\ 0 & 0.0171 \end{pmatrix} \begin{pmatrix} \epsilon_{1t-1}^2 \\ \epsilon_{2t-1}^2 \end{pmatrix} \quad (28)$$

From the result, the Upper Diagonal MGARCH reduces the number of model parameters, as $\hat{\sigma}_{2t}^2$ is a function of σ_{2t-1}^2 and ϵ_{2t-1}^2. This restriction distinguishes the Upper Diagonal MGARCH from the general MGARCH model.

3.2. Estimation of LDMGARCH Model Parameters

From the complete MGARCH [p(1,1), q(1,1)] model for the crude oil production quantity and price volatilities, the Lower Diagonal Multivariate GARCH Model for the two volatility series is presented as;

$$\begin{pmatrix} \sigma_{1t}^2 \\ \sigma_{2t}^2 \end{pmatrix} = \begin{pmatrix} \varphi_{11.1} & 0 \\ \varphi_{21.1} & \varphi_{22.1} \end{pmatrix} \begin{pmatrix} \sigma_{1t-1}^2 \\ \sigma_{2t-1}^2 \end{pmatrix} + \begin{pmatrix} \theta_{11.1} & 0 \\ \theta_{21.1} & \theta_{22.1} \end{pmatrix} \begin{pmatrix} \epsilon_{1t-1}^2 \\ \epsilon_{2t-1}^2 \end{pmatrix} \quad (29)$$

Equation (29) limits parameters as $\hat{\sigma}_{1t}^2$ to $\varphi_{11.1}$ and $\theta_{11.1}$ which are associated is a function of σ_{1t-1}^2 and ϵ_{1t-1}^2. This restriction distinguishes the Lower Diagonal MGARCH from the general MGARCH model. The ordinary least squares regression estimates produce the results,

$$\begin{pmatrix} \hat{\sigma}_{1t}^2 \\ \hat{\sigma}_{2t}^2 \end{pmatrix} = \begin{pmatrix} 0.3398 & 0 \\ 0.314 & 0.4398 \end{pmatrix} \begin{pmatrix} \sigma_{1t-1}^2 \\ \sigma_{2t-1}^2 \end{pmatrix} + \begin{pmatrix} 0.0733 & 0 \\ 0.090 & 0.0171 \end{pmatrix} \begin{pmatrix} \epsilon_{1t-1}^2 \\ \epsilon_{2t-1}^2 \end{pmatrix} \quad (30)$$

Equations (28) and (30) are estimated models for the UDMGARCH and LDMGARCH models. The parameters of the upper and lower diagonal models are restricted to the upper and lower coefficient matrices. The two multivariate models capture volatility clustering in the Crude Oil Quantity and Price.

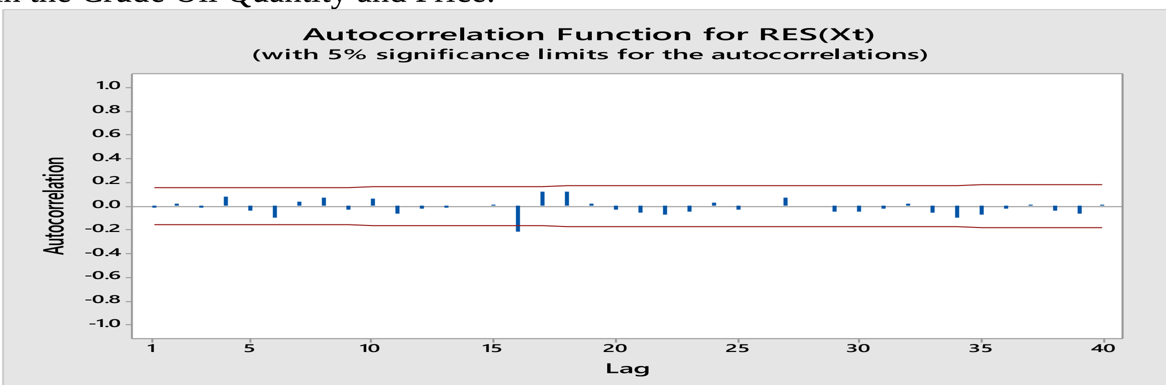

Figure 6: ACF of the Crude Oil Quantity Volatility Residual

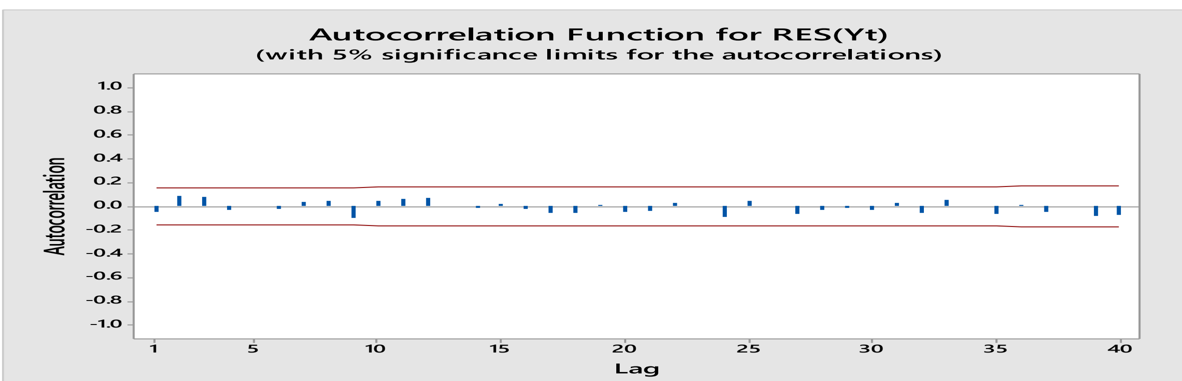

Figure 7: ACF of the Crude Oil Price Volatility Residual

Figures 6 and 7 are the residuals autocorrelation functions of the crude oil quantity and price volatilities. The behaviour of the ACF indicates a pure white noise process of the residual terms. These validate the Upper and Lower Diagonal MGARCH models for modelling multivariate volatility series of economic and financial time series.

MODEL SELECTION INFORMATION CRITERIA

Table 3: Information Selection Criteria For σ_{1t}^2 and σ_{2t}^2

S/N	Model Specification	AIC	SIC
σ_{1t}^2			
1	MGARCH$[p(1,1), q(1,1)]$	-13.11	-13.06
2	UDMGARCH$[p(0,1), q(0,1)]$	-13.12	-13.05
σ_{2t}^2			
1	MGARCH$[p(1,1), q(1,1)]$	-12.10	-12.12
2	LDMGARCH$[p(1,0), q(1,0)]$	-12.12	-12.09

Summary

The main focus of this research was to establish new classes of parsimonious volatility models from the current Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models with parameter restrictions to upper and lower diagonal of the coefficient matrices. The Upper and Lower Diagonal MGARCH models are established and compared to the MGARCH models of [11] and [12]. Autocorrelation and partial autocorrelation functions in figures 2, 3, 4 and 5 are used for the choice of the order of the MGARCH $[p(p_1, p_2), q(q_1, q_2)]$ model. From the plots, ACFs and PACFs of σ_{1t}^2 and σ_{2t}^2 were observed to have significant spike each at the first time lag. This suggested MGARCH$[p(1,1), q(1,1)]$, MGARCH$[p(0,1), q(0,1)]$ and MGARCH$[p(1,0), q(1,0)]$ for the general MGARCH, Upper Diagonal MGARCH and Lower Diagonal MGARCH models. The autocorrelation functions of the residuals indicate fitness of the new classes of the models. Further checks using model selection criteria revealed competitiveness of the new classes of the models with the already established MGARCH Models.

Conclusion

The interest in this paper was to identify Upper and Lower Diagonal Multivariate Generalised Autoregressive Conditional Heteroskedasticity Models under certain conditions. The Upper Diagonal MGARCH Models are established from the MGARCH models. UDMGARCH models are models whose parameters are restricted to the upper diagonal of the coefficient matrices. In contrast, the LDMGARCH models have parameters restriction to lower diagonal elements of the coefficient matrices. These new models are parsimonious with the reduction in the number of parameters of each model. The empirical evidence with the Nigeria Crude Oil Quantity and Price Volatilities reveals the applicability of the new set of diagonal models in capturing multivariate time volatility series.

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