Amenability of Pseudo Complete Locally Convex Algebras

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Abstract
We extend the notion of amenable Banach algebras to pseudo-complete locally convex algebra A, where A is considered as a direct limit of inductive system of Banach algebras. We endowed A with a strict inductive limit topology. In line with this, derivation on A is shown to be continuous. Consequent upon this and also base on the existence of both the locally bounded approximate identity $(lbai)$ and locally bounded approximate diagonal $(lbad)$ in A, we show that A is amenable.

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1. Introduction

The classic memoir of Johnson [9] in 1972 changed the face of amenability from amenable locally compact groups to amenable Banach algebras. In the light of the above, various notions of amenability have been introduced over the years. A typical notion introduced was in the definition of amenability of Banach algebra which shows that in a Banach algebra A, A is amenable when every continuous derivation into its dual bimodule is inner. Another milestone of Johnson's memoir is how it connects bounded approximate identity (bai) and that of bounded approximate diagonal (bad) for Banach algebra to amenability of Banach algebras. Meanwhile, as a result of this memoir, a considerable amount of work has been done on amenable Banach algebras by various authors like Helemskii and Sheinberg [see 12], Gronbeck $etal$ [7], Ghahranmani and Loy[6], e.t.c. Pirkovskii [12] in 2009 generalized the results of Johnson [9] and Helemskii and Sheinberg [see 12] from Banach algebras to Frechet algebras. He realized the Frechet algebras from the inverse limit of sequence of Banach algebras. In his work he introduced two new notions, namely, locally bounded approximate identity ($lbai$) and locally bounded approximate diagonal ($lbad$) for Frechet algebras. He used these notions and other in this direction to cover amenability in Frechet algebra. Other works in this line include Fatemeh $etal$ [5] on weak amenable Frechet algebras and Ranjbaril $etal$ [13] on ideal amenable Frechet algebras. In this paper, we consider the work of Johnson and that of Pirkovskii to extend the notion of amenability of Banach algebras to strict direct limit of inductive system of Banach algebras or pseudo-complete locally convex algebras (pseudo-complete lca). This is done by considering the continuous derivations on the pseudo-complete lca. Closed graph theory is used to show that every derivation is continuous on the pseudo-complete lca. Duality approach is also used to show that every derivation on pseudo-complete lca into its dual is continuous. In the same vain, by using the new notions introduced by Pirkovskii, it is also shown that a pseudo-complete lca admits $lbai$ and $lbad$. In line with these notions and the continuity of every derivation on the pseudo-complete lca, the notion of its amenability is proved. Section 2 of this work is devoted to definitions, notions and results that have direct bearing to our work. Section 3 contains the main results.

2. Preliminary

Definitions, notions and notations that are used in the work are discussed here. For details (see [1, 2, 3, 14 and 16])

A locally convex space(lcs) X is a topological linear space that has a base of 0-neighborhood comprising absolutely convex sets.

A subspace S of a vector space X over IK is called balanced if $\lambda S \subset S$ when $\lambda \in IK$, with $|\lambda| \leq 1$ and absolutely convex iff it is both balanced and convex.

Given two subsets S and T of a linear space X over IK, T absorbs S iff $\exists\ \alpha \in \mathbb{R}$ such that $S \subset \lambda T, \forall\ \lambda \in IK$ with $|\lambda| \geq \alpha$

However, an absolutely convex, absorbing and closed subset S is called a barrel. Hence, a lcs is referred to as barreled if every barrel is a neighbourhood.

A family U of 0-neighborhood in a lcs X is referred to as the fundamental system of 0-neighborhooh if there is a $V \subset U$ for every 0-neighborhood U and $\epsilon > 0$ with $\epsilon V \subset U$.

A collection $P = (p_i)_{i\in I}$ of continuous seminorms in X is called a fundamental system of seminorms if $U_i = \{s \in X \mid p_i(s) < i\}$, $i \in I$ forms a fundamental system of 0-neighborhood.

A metrizable lcs is a lcs whereby its topology is given by countable system of continuous seminorms. A Frechet space is the completion of a metrizable lcs.

A normed linear space (nls) X is a linear space whose topology is determined by the norm $||\cdot||$. A Banach space is the completion of a nls.

A locally convex algebra (lca) A over a field $IK(:=\mathbb{C})$ is an algebra endowed with a structure of lcs with respect to the product being separately continuous.

Let A be a Banach space. A is called a Banach algebra if A is an algebra such that the algebra multiplication satisfies

$$\| st \| \leq \| s \| \| t \| \qquad (s, t \in A).$$

Let A be an algebra that is commutative with identity. A non-empty collection $\mathcal{B}$ of subsets of A is referred to as a bound structure for A if

(i) B is absolutely convex, $B^2 \subset \mathcal{B}$, for each B in $\mathcal{B}$.

(ii) given B_1, B_2 in $\mathcal{B}$, there exists B_3 in $\mathcal{B}$ and $\lambda > 0$ such that $B_1 \cup B_2 \subset \lambda B_3$.

$(A, \mathcal{B})$ is referred to as a bound algebra. For every $B_n \in \mathcal{B}$, $A(B_n) = A_n = \{\lambda b : \lambda \in \mathbb{C},\ b \in B_n\}$ = Span(B_n) for $n \in \mathbb{N}$.

A_n is the subalgebra of A generated by B_n. A_n as defined above is a Banach algebra. The gauge p_{B_n} of B_n is given as

$$p_{B_n}(s) = \inf\{|\lambda| > 0 |\ s \in \lambda B_n\} \qquad (s \in B_n, \qquad B_n \in B).$$

Let $\wedge$ be an index set directed upward by the relation $\leq$ defined by $n \leq m$ iff $B_n \subset \lambda B_m$ for some $|\lambda| > 0$. With $\| \ \|_n = p_{B_n}$ as the norm on A_n for $n \leq m$, $A_n \subset A_m$ and the inclusion map $f_{mn}: A_n \longrightarrow A_m$ is a continuous unital monomorphism. Then $\{A_n; f_{mn}\}$ is an inductive system.

Let B_n be an absolutely convex 0-neighbourhood defined by the norm topology p_{B_n} in A_n. We define a map $f_n: A_n \longrightarrow A$, since f_n is continuous, $f(B_n) = B$, then, a fundamental system of 0-neighbourhood for the topology of A is given by the set B of all absolutely convex subsets B of A, such that for each n $f_n^{-1}(B) = B_n$ is a neighbourhood in A_n. This is an algebraic inductive topology and it coincides with the structure of lcs for A.

Proposition 2.1 [1]. An algebra A is a pseudo-complete lca with respect to some bounded structure if and only if A is isomorphic with the inductive limit of an inductive system $\{A_n; f_{mn}: m, n \in I, n \leq m\}$ of Banach algebras with identity and continuous unital monomorphisms.

We further require that the inductive limit described above be strict. That is, suppose that each A_n has a topology τ_n defined by the norm $\| \ \|_n$ under which is a Banach algebra so that for each n the induced topology on A_n by the norm $\| \ \|_{n+1}$ on A_{n+1} is the norm topology $\| \ \|_n$. This implies that each A_n is embedded algebraically and topologically in A_{n+1}. Hence, suppose τ is the inductive limit topology on $A = \lim_{\rightarrow} A_n$ so that τ induces $\tau_n = \| \ \|_n$ on each A_n, then $A = \lim_{\rightarrow} A_n$ with τ being its topology is referred to as the strict inductive limit of Banach algebras $\{A_n\}$.

A non-zero complex-valued homomorphism $\phi: A \longrightarrow \mathbb{C}$ on a topological algebra A is called a character (multiplication linear functional).
The set of characters of a unital topological algebra A is called the spectrum (or maximal ideal space), (i.e. Spec A) (See [4]).

If A is a lca and X a lcs. The bilinear map $l: A \times X \longrightarrow X$ is referred to as a left A-module (or a left action of A on X) (i.e. $l(s, r) = s \cdot r$) together with $(st) \cdot r = s \cdot (tr)$ which is an extension to a continuous bilinear map $A \times X \longrightarrow X$. Analogously definition of right modules holds. An A-bimodule is an lcs X which is both left and right A-modules such that $s(rt) = (sr)t$ holds for $s, t \in A$, $r \in X$.
Suppose L, M and N are topological linear spaces. A transformation $G: L \times M \rightarrow N$ is referred to as hypo continuous if given a bounded set $S \subset L$ (resp. $S \subset M$) and 0-neighbourhood $W \subset N$ there is a 0-neighbourhood $T \subset M$ (resp. $T \subset L$) such that $G(S, T) \subset W$ (resp. $G(T, S) \subset W$). If L and M are barreled then a bilinear map $G: L \times M \rightarrow N$ which is separately continuous is hypo continuous.
Given a lca A and a left (resp. right) A-module X. X is called a hypo module if the map $(s, r) \longmapsto s \cdot r: A \times X \longrightarrow X$ [resp. $(r, s) \longmapsto rs: X \times A \longrightarrow X$] is a hypo continous map. An A-bi module whose left and right module operations are both hypo continuous is referred to as a bi-hypo module. see [16].

Let a sequence of Banach spaces $X_1 \supset X_2 \supset X_3 \supset \cdots$ be given , then there is a projective limit $\lim_{\leftarrow} X_k = X \quad (k = 1,2, \dots)$ and also a space of linear form X'_k for each Banach space X_k. Hence $X = \bigcap X_k$ is a Frechet space and $X' = \bigcup X'_k$ is a pseudo-complete lcs endowed with the strong topology.

Given an A-bimodule X. Then we define a derivation from A to X as a linear map $D: A \longrightarrow X$ where $D(st) = D(s)t + sD(t)$ for $s, t \in A$. With $s \in X$, the derivation $\delta_r: A \longrightarrow X$ defined by $\delta_r(s) = s \cdot r - r \cdot s \quad (s \in A)$ is referred to as inner.

Let A be a lca. A net $(t_i)_{\in I}$ in A is referred to as a right (left) approximate identity (ai) if $s = \lim_i st_i$ $(s = \lim_i t_i s)$ for $s \in A$. A admits an ai if a net $(t_i)_{\in I}$ is both a left and a right ai in A. A bounded set $(t_i)_{\in I}$ is then referred to as a bai.
We call t a right (resp. left) $lbai$ for a lca A if given a 0-neighbourhood $B \subset A$ there is $c > 0$ where for each finite subset $R \subset A$ there is a $t \in cB$ with $s - st \in B$ $(s - ts \in B)$ for all $s \in R$. A is said to admits a $lbai$ if it has both right and left $lbai$. (see [12], p.96).
Given an algebra A, I is a left (right) ideal of A if $I \subset A$ such that $ax \in I (xa \in I)$ whenever $a \in A$ and $x \in I$.

Definition 2.2 [10]. The inductive tensor product $L \overline{\otimes} M$ of two topological vector spaces L and M completes the algebraic tensor product $L \otimes M$ for which there is a finest compatible tensor product topology on it.
The inductive tensor product has the following properties.
(i) Let N be a complete topological linear space, then there is an isomorphism between $L(L \overline{\otimes} M, N)$, the space of continuous linear transformations from $L \overline{\otimes} M$ to M and the space of separately continuous bilinear tranformations from $L \times M$ to N; hence there is an isomorphsim between the dual $(L \overline{\otimes} M)'$ and the space of separately continuous bilinear functionals on $L \times M$.
(ii) If L and M are Frechet spaces, separate continuity and joint continuity of bilinear forms coincide, therefore $L \overline{\otimes} M$ and the projective tensor product $L \widehat{\otimes} M$ are equal.
(iii) Given that $A = \lim_{\rightarrow} A_i$ and $B = \lim_{\rightarrow} B_J$ are inductive limits on which inductive limit topology are defined, then $A \overline{\otimes} B = \lim_{\rightarrow}(A_i \overline{\otimes} B_j)$ is also endowed with the inductive limit topology where A_i and B_j are sequences of Banach spaces.
By property (ii) of 2.2, each Banach space L_i and M_j are Frechet spaces, hence,

$$L_i \overline{\otimes} M_j = L_i \widehat{\otimes} M_j.$$

For each strict inductive limit $L = \lim_{\rightarrow} L_i$ and $M = \lim_{\rightarrow} M_j$, we have by property (iii) of 2.2 that

$$L \overline{\otimes} M = \lim_{\rightarrow}(L_i \overline{\otimes} M_j) = \lim_{\rightarrow}(L_i \widehat{\otimes} M_j).$$

Hence, $L \overline{\otimes} M$ is a complete lcs and is the strict inductive limit of Banach spaces $(L_i \widehat{\otimes} M_j)$. We also note here that, if each L_i and M_j are Banach modules, then each L and M are inductive limit hypomodules. So also $L \overline{\otimes} M$ is an inductive limit hypomodule if $L_i \widehat{\otimes} M_j$ is a Banach module.
Let A be a topological algebra. The complete inductive tensor product of A and A is denoted by $A \overline{\otimes} A$. Suppose B is a neighbourhood base in A. Given $B \in B$. We denote the closure of the absolutely convex hull of

$$B \otimes B = \{s \otimes t : s, t \in B\} \subset A \otimes A$$

by $\overline{\Gamma(B \otimes B)}$ then $\overline{\Gamma(B \otimes B)}$ is a neighbourhood base at 0 in $A \overline{\otimes} A$ (see [11], p.378).
Let A be a lca and $\pi: A \overline{\otimes} A \longrightarrow A$ a product map. A net $(w_\alpha)_{\alpha \in I}$ in $A \overline{\otimes} A$ is called an approximate diagonal (ad) for A if $s \cdot w_\alpha - w_\alpha \cdot s \longrightarrow 0$ and $\pi(w_\alpha)s \longrightarrow s$ for each $s \in A$. We say $(w_\alpha)_{\alpha \in I}$ is a bad if it is bounded as a set.
We call w a $lbad$ for a lca A if given a 0-neighbourhood $B \subset A$ there is $c > 0$ where for a finite subset $R \subset A$ there is a $w \in \overline{c\Gamma(B \otimes B)}$ with $sw - ws \in \overline{\Gamma(B \otimes B)}$ and $\pi(w)s - s \in B$ for all $s \in R$. (see [12], p.96).

3. Main Results

We first consider continuous derivation on a pseudo complete locally convex algebra A in subsection 3.1, locally bounded approximate identity and diagonal is discussed in subsection 3.2, amenability of A is considered in subsection 3.3, subsection 3.4 contains hereditary properties and examples of amenable pseudo complete locally convex algebras are constructed in subsection 3.5.

3.1 Continuous Derivation

We consider here the continuity of derivation on the pseudo-complete lca A. In Theorem 3.1.5 we used the closed graph theorem to show continuity of derivation on A.

Theorem 3.1.1. If A is a (strict inductive limit algebra) pseudo-complete lca with identity and let D be a derivation of A into A, then D is continuous.

Proof. We shall show the continuity of D by using the closed graph theorem (see [7], Theorem 25 and Remark 26). To show the continuity of D, we only need to show that the graph of D is closed. Let $\{s_n\} \subset A$ with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} D(s_n) = t$. We consider a multiplicative linear functional ϕ with $\phi(Ds) = \phi(t)$. That is $\phi(\lim(Ds_n)) = \phi(Ds) = \phi(t)$.

Let $P = \{\phi \in SpecA: s \longmapsto \phi(Ds)\}$ which is a continuous composition map from A into the complex number. If we can show that there is an isomorphism between P and the subspace of $\prod_n SpecA_n$ for each Banach algebra A_n, we are through.

By definition $SpecA = Spec\lim_{\rightarrow} A_n = \lim_{\leftarrow} SpecA_n$. $\{i.e.\ SpecA = Spec \bigcup \ A_n = \bigcap \ SpecA_n\}$. We only need to show that $\lim_{\leftarrow} SpecA_n$ is isomorphic to a subspace of $\prod_n SpecA_n$. See ([11] p.156 [Theorem 3.1]) and ([2] p.334).

Let $SpecA_i = \{\phi_i | \phi_i: A_i \longrightarrow \mathbb{C}\}$, we also have $f_{ji}: A_i \longrightarrow A_j$ with $A_i \overset{f_{ji}}{\longrightarrow} A_j \overset{\phi_j}{\longrightarrow} \mathbb{C}$ $\quad (i \leq j \in \mathbb{N})$ such that $\phi_j \circ f_{ji} = \phi_i$. Since $SpecA_j \subset SpecA_i$, we define $g_{ji}: SpecA_i \longrightarrow SpecA_j$. Let r_i be a projection from $\prod_n SpecA_n$ onto $SpecA_i$. Then,

$$\lim_{\leftarrow} SpecA_n = \{\phi \in \prod_n SpecA_n : g_{ji}(r_i(\phi_i)) = r_j(\phi) \quad whenever \quad j \geq i\}$$

It is clear that $\lim_{\leftarrow} SpecA_n$ is a subalgebra of $\prod_n SpecA_n$. We look at the map

$$f: \lim_{\leftarrow} SpecA_n \longrightarrow \prod_n SpecA_n$$

defined by

$$f(\phi(Da)) = \{\phi(D(a))\}.$$

Hence this shows that there is an isomorphism f between $\lim_{\leftarrow} SpecA_n$ and the subspace of $\prod_n SpecA_n$.

The next result establishes the continuity of derivation on A into its dual A'.

Theorem 3.1.2. If A is a pseudo-complete lca and D a derivation on A into A', its A-bimodule, then D is continuous.

Proof. We need to show first that $D: A \longrightarrow A'$ is continuous. We note that $A = \lim_{\rightarrow} A_i$, $A' = \lim_{\leftarrow} A_i$ and $\{A_i\}_i$ is the sequence of Banach algebras. This implies that $A' = \lim_{\leftarrow} A_i$ is a Frechet algebra. Hence, we consider A' as a Frechet space and an A-bimodule with module multiplication.
We define $D: A \longrightarrow A'$ by $D(s) =< s, s\alpha >$ where $s \in A$, $\alpha \in A'$.
Let $s \in A$ and $\{s_n\}_n \subset B \subset A$ where B is a bounded net in A. Hence, $\lim_{n\to\infty} s_n = s$ in the weak star topology of A.
With $r \in B$ a bounded set in A, this gives
$|\langle r, \alpha\rangle| \leq cq_{r\in B}(r)$ with $c > 0$
where $q \in \{q_B\}_B$ determine the topology of A.
Since $r \in B$ we can have $M > 0$ where $Sup_{r\in B} q(r) \leq M$
Hence, with each $r \in B$, we have

$$\begin{aligned} |\langle r, D(s_n - s)\rangle| &= |\langle r, \langle s_n - s, (s_n - s)\alpha\rangle\rangle| \\ &= |\langle r, \alpha(s_n - s)s_n - s\rangle| \\ &= |\langle r, \alpha\rangle||s_n - s| \end{aligned}$$

Therefore,

$$\begin{aligned} Sup_{r\in B}|\langle r, D(s_n - s)\rangle| &= |s_n - s| Sup_{r\in B}|\langle r, \alpha\rangle| \\ &\leq |s_n - s| c Sup_{r\in B} q(r). \\ |\langle r, D(s_n - s)\rangle| &\leq |s_n - s| cM \end{aligned}$$

In the inequality above the expression on the right side approaches zero. Hence, continuity of D follows.
We now show next that D is a derivation. Right and left module actions will now be considered since A' is an A-bimodule.
Let $A.A' = \{0\}$ then $D(s) = \langle s, \alpha s\rangle$
Hence $\langle r, D(s)\rangle = \langle r, \langle s, \alpha s\rangle\rangle = \langle r, \alpha(s)s\rangle$
$\langle r, D(s)\rangle = \langle r, \alpha\rangle s$

So also, let $A'.A = \{0\}$, then
$D(s) = \langle r, s\alpha \rangle$
Hence, $\langle r, D(s) \rangle = \langle r, \langle s, s\alpha \rangle \rangle = \langle r, s\alpha(s) \rangle$.
Therefore,
$\langle r, D(s) \rangle = s\langle r, \alpha \rangle$
Putting both modules together gives

$$\langle r, D(s) \rangle = \langle r, \alpha \rangle s - s\langle r, \alpha \rangle = 0.$$

We have shown that D is a derivation.

3.2. Locally Bounded Approximate Identity and Diagonal
Results here are based on the definitions by Pirkovskii for locally bounded approximate identity and diagonal. We shall state them without proofs.

Proposition 3.2.1 [3]. Let a Banach algebra A contain a bounded set B_n such that, given $s \in A$, $c > 0$ there is a $t \in B_n$ with $\| s - ts \| < c$. Then A admits a left bai which is t.

Moreover, since the inductive limit topology on pseudo complete lca A is strict, Theorems 3.2.2 and 3.2.3 hold. Hence, they are stated here without proofs, because they follow the same pattern of proofs in [12, (Proposition 6.2 and Proposition 6.5)].

Theorem 3.2.2. Let A be a pseudo complete lca $(i.e., \ A = \lim_{\rightarrow} A_n)$ of sequence of Banach algebras $\{A_n\}$. Then A has a right $lbai$ if each Banach algebra A_n has a right bai.

Theorem 3.2.3. Let A be a pseudo-complete lca $(i.e., A = \lim_{\rightarrow} A_n)$ of Banach algebras $\{A_n\}$. Then A has a $lbad$ if each Banach algebra A_n has a bad.

3.3 Amenability of Inductive Limit Algebras
In this subsection, we use results developed in subsections 3.1 and 3.2 to discuss the amenability of pseudo-complete lca.

Definition 3.3.1. Given a Banach algebra A, A is called amenable if every continuous derivation from A into X' is inner, where X' is the dual space of an A-bimodule X.

Remark 3.3.2. We state the following definition of amenability for pseudo complete locally convex algebra A as inspired by ([5], [12 Theorem 9.6]) and definition 3.3.1.

Definition 3.3.3. The pseudo-complete lca A is called amenable, if every continuous derivation from A into X' is inner with X' as the dual of A-bimodule X.

Theorem 3.3.4. [15]. Let A be an amenable Banach algebra. Then A has a bounded approximate identity.

Theorem 3.3.5 emphasizes amenability of pseudo-complete locally convex algebra A vis-a-vis amenability of each Banach algebra A_n.

Theorem 3.3.5. If A is a pseudo-complete lca ($i.e.$ $A = \lim_{\rightarrow} A_n$) of sequence of Banach algebras $\{A_n\}$. A is amenable if and only if A_n is amenable for each n.

Proof. Suppose each Banach algebra A_n is amenable for each n. Let X_n be an A_n-bimodule. Since A_n is amenable for each n, it implies that $D: A_n \longrightarrow X'_n$ is an inner derivation for a continuous derivation D. Consider the following map

$$A_n \overset{G_n}{\longrightarrow} A \overset{d}{\longrightarrow} X'_n$$

$D = d \circ G_n$, hence d is a derivation by Theorem 3.1.2.
By the universal property of inductive limit, continuity of d holds if and only if $D = d \circ G_n$ is continuous. Since $D = d \circ G_n$ is continuous and also an inner derivation. We have $D^{-1}(s \cdot \phi - \phi \cdot s) \in A_n$ for $s \in A_n$, $\phi \in X'_n$.

$$\begin{aligned} D^{-1}(s \cdot \phi - \phi \cdot s) &= (d \circ G_n)^{-1}(s \cdot \phi - \phi \cdot s) \\ &= G_n^{-1} d^{-1}(s \cdot \phi - \phi \cdot s) \end{aligned}$$

which implies that

$$d^{-1}(G_n(s) \cdot \phi - \phi \cdot G_n(s)) \in A.$$

Hence d is continuous.
Let $A_n \cdot X_n = \{0\}$, this implies that $X'_n \cdot A_n = \{0\}$. Then there is a net $\{t_\alpha\}$ in A_n such that $D(t_\alpha)$ converges in the w^*-topology in X'_n to ϕ.
Let $s \in A_n$

$$\begin{aligned} D(s) = D\left(\lim_\alpha s t_\alpha\right) = d \circ G_n\left(\lim_\alpha s t_\alpha\right) &= d\left(G_n\left(\lim_\alpha s t_\alpha\right)\right) \\ = d\left(G_n\left(s \lim_\alpha t_\alpha\right)\right) &= d\left(G_n(s) G_n\left(\lim_\alpha t_\alpha\right)\right) \\ &= d\left(G_n(s) \lim_\alpha G_n(t_\alpha)\right). \end{aligned}$$

By theorem (3.2.2) $G_n(t_\alpha)$ is a $lbai$ in A and $d(G_n(t_\alpha))$ converges in the weak star topology in X'_n say to t. So also let $G_n(s) = s' \in A$.
Therefore,

$$d\left(G_n(s)\lim_{\alpha} G_n(t_\alpha)\right) = d\left(s'\lim_{\alpha} G_n(t_\alpha)\right)$$

$$= d\left(\lim_{\alpha} s' G_n(t_\alpha)\right) = s' \cdot t.$$

Hence, d is inner and therefore A is amenable.
Conversely, suppose A is amenable, then there is an A-hypo bimodule X such that $\delta: A \longrightarrow X'$ is inner for a continuous derivation δ, since X' is a Frechet space, hence it is an A-bimodule. Given a map

$$\partial: A_n \xrightarrow{G_n} A \xrightarrow{\delta} X'.$$

Since δ is a derivation $\partial = \delta \circ g_n$ is also a derivation. Since G_n is continuous and 1-to-1 (injection), and also since δ is inner $\delta \circ G_n = \partial$ is inner, hence A_n is amenable for each n.

3.4 Hereditary Properties of Amenable Pseudo-Complete Locally Convex Algebras
We shall consider some results on the hereditary properties of amenable pseudo-complete locally convex algebras.

Proposition 3.4.1. Let A be an amenable pseudo-complete lca. Let be B a pseudo-complete lca and let a continuous onto map $g: A \to B$ be defined. Then B is amenable.

Proof. Let A be amenable.Then, Banach algebra A_n is amenable for each n from Theorem 3.3.5 . So also from Theorem 3.3.4, A_n has a bai e_{α_n} say for each n . Hence by ([12], Prop. 6.2), we can have an onto map $\theta_n: A_n \to B_n$ such that $\theta_n(e_{\alpha_n}) = e_{\beta_n}$ which implies that e_{β_n} is a bai for B_n. By Theorem 3.2.2, e_β is a $lbai$ for B.
We define a product map $\pi_n: (B_n \widehat{\otimes} B_n) \to B_n$ and that $\pi_n(w_{\beta_n}) = e_{\beta_n}$, this implies that w_{β_n} is a bad for B_n for each n, so also by Theorem 3.2.3, w_β is $lbad$ for B. Therefore, by ([15], Theorem 2.2.4), B_n is amenable for each n. Hence, by Theorem 3.3.5, B is amenable.

Corollary 3.4.2. Let A be an amenable pseudo-complete lca and I a closed ideal of A, then A/I is also amenable.

Proposition 3.4.3. Let A be an amenable pseudo-complete lca and I a closed ideal of A, then the following are equivalent.
(i) I is amenable
(ii) I has a $lbai$.

Proof.
(i) $\Longrightarrow$ (ii) Suppose I_n is a closed ideal of a Banach algebra A_n. Define $I = \lim_{\rightarrow} I_n$, then by Theorem 3.3.5, I_n is amenable for each n, hence I_n has a bai for each n. From Theorem 3.2.2, I has a $lbai$.
(ii) $\Longrightarrow$ (i) Since A is amenable, it implies that A_n is amenable for each n by theorem 3.3.5.
By ([12], Proposition 2.3.3) I_n, a closed ideal for A_n is amenable and has a bai. Since I also has a $lbai$, it implies that from Theorem 3.3.5, I is amenable.

3.5 Examples

(i) Given that a Hausdorff lcs X has a fundamental sequence of compact subset $(K_m)_{m\in\mathbb{N}}$. Then, the $C_{k_m}(X)$ with compact supports on $K_m,\ m \in \mathbb{N}$ is the space of continuous complex valued functions on X . This represents an increasing sequence for every compact subset $K \subset X$ contained in K_m for $m \in \mathbb{N}$.
Hence, $C_{k_m}(X)$ a Banach algebra is amenable. Therefore by ([11], IV.4(1)) the strict inductive limit algebra $C_c(X)$ defined by $C_c(X) = \lim_{\rightarrow} C_{k_m}(X)$ is amenable by Theorem 3.3.5.

(ii) For a locally compact group G, $L^1(G)$ is an amenable Banach algebra. For each group homomorphism $\mu: H \longrightarrow G$ (where $H \subset G$) we can have a Banach algebra morphism $\mu^*: L^1(H) \longrightarrow L^1(G)$ which is uniquely determined by $\mu^*(\delta_h) = \delta_{\mu(h)} \quad (h \in H)$, where δ_h denotes the function on H defined by

$$\delta(h) = \begin{cases} 1 \ \ at \ \ h \\ \\ 0 \ \ elsewhere \end{cases}$$

If μ is injective, so also is μ^*. Set $G = (G_n, \mu_{m,n}), n \in \mathbb{N}$ such that G defines a direct sequence of groups with the linking maps $\mu_{m,n}$ being injective. That is, $\mu_{m,n}: G_n \longrightarrow G_m$ for $n \leq m$, $G_n \subset G_m$. We can as well set
$L^1(G) = \lim_{\rightarrow}(L^1(G_n), \mu^*_{m,n})$. We note that $L^1(G)$ is a pseudo-complete lca of sequence of Banach algebra $L^1(G_n)$. Since $L^1(G_n)$ is amenable. Therefore, by Theorem 3.3.5, $L^1(G)$ is amenable if and only if $L^1(G_n)$ is amenable for each n.

3.6 Conclusion

We extend the notion of amenability available for Banach algebras and Frechet algebras to pseudo complete locally convex algebras. This is done from the view point of continuity of derivation as developed by Johnson [9] for amenable algebras. Hereditary properties of amenability are also considered. Examples of amenable pseudo complete locally convex algebras are constructed as well to complete the work.

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