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AN EOQ MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS WITH TWO PHASE DEMAND RATES, LINEAR HOLDING COST AND PARTIAL BACKLOGGING RATE UNDER TRADE CREDIT POLICY

By

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Abstract

In this article, an EOQ model for non-instantaneous deteriorating items with two phase demand rates, time dependent linear holding cost and shortages under trade credit policy was developed. The demand rate before deterioration begins is assumed to be time dependent quadratic and that after deterioration begins is considered as a constant. Shortages are allowed and partially backlogged. The purpose of this work is to determine the optimal time with positive inventory, cycle length and economic order quantity simultaneously such that total variable cost has minimum value. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions have been established. Some numerical examples have been given to illustrate the theoretical results of the model. Sensitivity analysis has been carried out to see the effect of changes in some model parameters on decision variables and suggestions toward minimising the total variable cost were also given.

Keywords: *Non-instantaneous Deterioration, Two Phase Demand Rates, Trade Credit Policy, Time Dependent Linear Holding Cost, Partially Backlogged Shortages.*

Introduction

The classical EOQ model assumed implicitly that shortages are not allowed to occur. However, sometimes customers demand cannot be fulfilled by the supplier from the current stocks, this situation is known as stock out or shortage condition. In real life situations, stock out is unavoidable due to various uncertainties. Thus, it is important to consider shortage condition while developing inventory model. According to sharma (2003), allowing shortages occur increase cycle length, spread the ordering cost over a long period of time and hence reducing the total variable cost. Deb and Chaudhuri (1987) developed a heuristic approach for replenishment of trended inventories considering shortages. Goswami and Chaudhuri (1991) established an EOQ model for instantaneous deteriorating items with linear time dependent demand rate and shortages under inflation and time discounting. Roy (2008) developed an EOQ model for instantaneous deteriorating items with price dependent demand rate, where deterioration rate and holding cost are considered as linearly increasing function of time, shortages are allowed and completely backlogged. Choudhury et al. (2015) presented an inventory model for deteriorating items with stock dependent demand rate, time varying holding cost. Shortages are allowed to occur and completely backlogged. Other related studies on inventory model with shortages include Dave (1989), Goyal et al. (1992), Ghosh and Chaudhuri (2004), and so on. These researchers assumed that the shortages are completely backlogged.

However, when shortages occur, one cannot be certain that the customers are willing to wait for a backorder due to customers’ impatient and dynamic nature. When shortages occur, some customers whose needs are not critical at that time may wait for the backorders to be fulfilled, while others may opt to buy from other sellers. Consequently, the opportunity cost due to lost sales should be considered. Geetha and Uthayakumar (2010) developed an EOQ based model for non-instantaneous deteriorating items with constant demand rate under permissible delay in payments, shortages are allowed and partially backlogged; the backlogging rate is variable and dependent on the waiting time for the next replenishment. Chang and Feng (2010) presented a partially backlogged inventory model for non-instantaneous deteriorating items with stock dependent demand rate under the influence of inflation. Baraya and Sani (2013) developed an EOQ model for delayed deteriorating items with inventory level dependent demand rate and partial backlogging. Sarkar and Sarkar (2013) developed an inventory model for deteriorating items with stock dependent demand rate and time varying deterioration. Shortages are allowed and partially backlogged; the backlogging rate dependent on the waiting time for the next replenishment. Dutta and Kumar (2015) developed a partially backlogged inventory model for deteriorating items with time varying demand rate and holding cost. Moreover, some related studies on inventory models with partially backlogged shortages can be found in Wee (1995), Chang and Dye (2001), Wu et al. (2006), Yang et al. (2010) and so on, assumed that only a fraction of the demand can be backlogged and remaining fraction is lost forever.

In this paper, an EOQ model for non-instantaneous deteriorating items with two phase demand rates, time dependent linear holding cost and partial backlogging rate under trade credit policy has been developed. The demand rate before deterioration sets in is assumed to be time dependent quadratic, which is more realistic because it represents both accelerated and retarded growth in demand rate of items such as in petrochemicals, aircrafts, computers, machines, and their spare parts, seasonal product whose demand rises rapidly to a peak in the midseason and then falls rapidly as the season wanes out and seems to be a better representation of time varying market demand. The demand rate after deterioration sets reduces to a constant rate up to when the inventory is completely depleted. This is because the demand of items such as fashionable goods, android mobiles, machines, microcomputer chip of high technology products substituted by another, photographic film and so on may become obsolete as technology changes, tend to depreciate in value and become steady due to the introduction of newly launched products. The holding cost of items is assumed to linear time dependent. Shortages are allowed and partially backlogged. To the best of authors' knowledge, an EOQ model with above assumptions has not yet been discussed in inventory literature. The model could be used in inventory management and control of items such as petrochemicals, aircrafts, computers, seasonal products, fashionable goods, android mobiles, automobiles, photographic films, television, computer chips and so on. The aim of this research is to develop an EOQ model that will determine the optimal time with positive inventory; cycle length and lot size such that the total variable cost has a minimum value. The optimal solutions and conditions for its uniqueness and existence have been established. Some numerical examples have been given to illustrate the theoretical results of the model. Sensitivity analysis of some model parameters on optimal solutions has been carried out and suggestions for minimising the total variable cost of the inventory system were equally given.

Notations

A	Ordering cost per order.
C	Unit purchasing cost per unit per unit time ($/unit/ year).
S	Unit selling price per unit per unit time ($/unit/ year).
C_b	Shortage cost per unit per unit of time.
C_π	Unit cost of lost sales per unit.
I_c	Interest charged in stock by the supplier per unit cost per year ($/unit/year)$(I_c \geq I_e)$.
I_e	Interest earned per unit cost per year ($/unit/year).
M	Trade credit period (in year) for settling accounts.

θ	Deterioration rates function$(0 < \theta < 1)$.
t_d	Length of time in which the product exhibits no deterioration.
t_1	Length of time in which the inventory has no shortage.
T	Length of the replenishment cycle time (time unit).
Q_m	Maximum inventory level.
B_m	Backorder level during the shortage period.
Q	Order quantity during the cycle length,i.e., $Q = (Q_m + B_m)$.

Assumptions

This model was developed based on the assumptions below.

1. The replenishment rate is instantaneous and the lead time is zero.
2. Only one type of non-instantaneous deteriorating item is modelled.
3. During the fixed period,t_d, there is no deterioration and at the end of this period, the items deteriorate at the rate θ.
4. Deteriorated items are not replaced or repaired.
5. Time dependant quadratic demand rate is considered before deterioration begins and is given by$\alpha + \beta t + \gamma t^2$where $\alpha \geq 0, \beta \neq 0, \gamma \neq 0$.
   1. Demand rate before deterioration begins is assumed to be continuous time dependent quadratic and is given by$\alpha + \beta t + \gamma t^2$ where $\alpha \geq 0, \beta \neq 0, \gamma \neq 0$.
   Here α is the initial demand rate, β is the rate at which the demand rate changes and γ is the rate of change at which the demand rate changes itself.
6. A constant rate $\lambda$is considered after deterioration sets in,, $\lambda > 0$.
7. Holding cost $C_1(t)$per unit time is linear time dependent and is assumed to be $C_1(t) = h_1 + h_2 t$ where $h_1 > 0, h_2 > 0$.
8. During the trade credit period$M$ $(0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest bearing account. At the end of the period, the retailer pays off all units bought, and starts to pay the capital opportunity cost for the items in stock.
9. A partially backlogged shortage is allowed to occur at the rate $\delta(0 < \delta < 1)$, $\delta = 0$ is a case of no shortages and $\delta = 1$ is a case of complete backlogging.

Formulation of the model

At the beginning of each replenishment cycle (i.e., at time $t = 0$), $Q_m$units of a single product from the manufacturer arrives. During the time interval $[0, t_d]$, the stock level $I_1(t)$ is depleting gradually as a result of market demand only and the demand rate here assumed to be continuous quadratic function of time t. At time interval $[t_d, t_1]$, the inventory level$I_2(t)$ is depleting as a result of combined effects of customers demand and deterioration and the demand rate at this time is reduced toλ. At time $t = t_1$, the stock level depletes to zero. Shortage occurs at the time $t = t_1$ and are partially backlogged at the rate δ. The whole process of the inventory is repeated. The behaviour of the inventory system is described in figure below.

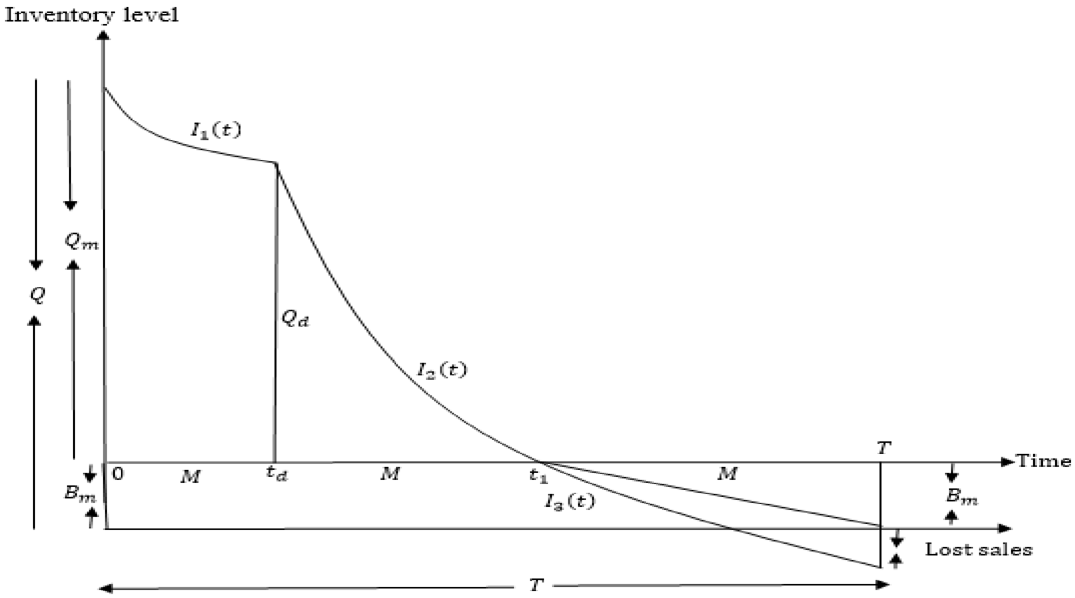

Fig. 1: Description of the Inventory system

From Fig. 1 above, the change of inventory level at any time $t \in [0, T]$ is described by the following differential equations

$$\frac{dI_1(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad 0 \le t \le t_d \qquad (1)$$

with boundary conditions $I_1(0) = Q_m$and $I_1(t_d) = Q_d$.

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -\lambda, \qquad t_d \le t \le t_1 \qquad (2)$$

with boundary conditions$I_2(t_1) = 0$ and $I_2(t_d) = Q_d$.

$$\frac{dI_3(t)}{dt} = -\delta\lambda, \qquad t_1 \le t \le T \qquad (3)$$

with condition $I_3(t_1) = 0$ at $t = t_1$.

The solution of equations (1), (2) and (3) are respectively given by

$$I_1(t) = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \alpha(t_d - t) + \frac{\beta}{2}(t_d^2 - t^2) + \frac{\gamma}{3}(t_d^3 - t^3), \quad 0 \le t \le t_d \qquad (4)$$

$$I_2(t) = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t)} - 1\right), \qquad t_d \le t \le t_1 \qquad (5)$$

and

$$I_3(t) = \lambda\delta(t_1 - t) \qquad (6)$$

From Fig.1, using the condition $I_1(0) = Q_m$ in equation (4), the maximum stock level is given by

$$Q_m = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) \qquad (7)$$

Moreover, the value of Q_d can be derived at $t = t_d$, then it follows from equation (5) that

$$Q_d = \frac{\lambda}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) \qquad (8)$$

The maximum backordered units B_m is obtained at $t = T$, and then from equation (6), it follows that

$$B_m = \lambda\delta(T - t_1) \tag{9}$$

Thus, the economic order quantity during time interval $[0,\ T]$ is

$$Q = Q_m + B_m$$

$$= \frac{\lambda}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) + \lambda\delta(T - t_1) \tag{10}$$

The total demand during the period$[t_d,\ t_1]$ is given by

$$D_M = \int_{t_d}^{t_1} \lambda\, dt$$

$$= \lambda(t_1 - t_d) \tag{11}$$

The total number of items that deteriorate per cycle is given by

$$D_P = Q_d - D_M$$

Substituting Q_d and D_M from equations (8) and (11) respectively intoD_P yields

$$D_P = \frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right] \tag{12}$$

The deterioration cost is given by

$$D_C = C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right] \tag{13}$$

The cost of placing order is given by A

The cost of holding items in the stock during the interval $[0,\ t_1]$is given by

$$C_H = \int_0^{t_d} (h_1 + h_2 t) I_1(t)\,dt + \int_{t_d}^{t_1} (h_1 + h_2 t) I_2(t)\,dt \qquad (14)$$

Substituting equations(4)and(5) into equation (14) yields

$$C_H = h_1 \left[\frac{\lambda t_d}{\theta} e^{\theta(t_1 - t_d)} + \frac{\alpha}{2} t_d^2 + \frac{\beta}{3} t_d^3 + \frac{\gamma}{4} t_d^4 + \frac{\lambda}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta} \right] + h_2 \left[\frac{\lambda t_d^2}{2\theta} e^{\theta(t_1 - t_d)} + \frac{\alpha}{6} t_d^3 + \frac{\beta}{8} t_d^4 + \frac{\gamma}{10} t_d^5 + \frac{\lambda t_d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{\lambda t_1}{\theta^2} - \frac{\lambda}{\theta^3} + \frac{\lambda}{\theta^3} e^{\theta(t_1 - t_d)} - \frac{\lambda t_1^2}{2\theta} \right] \qquad (15)$$

The backordered cost during the interval $[t_1,\ T]$ is given by

$$SC = C_b \int_{t_1}^{T} -I_3(t)\,dt$$

$$= \frac{C_b \delta \lambda}{2} (T - t_1)^2 \qquad (16)$$

The cost of lost sales during the interval $[t_1,\ T]$ is given by

$$LC = C_\pi \int_{t_1}^{T} \lambda(1 - \delta)\,dt$$

$$= C_\pi \lambda (1 - \delta)(T - t_1) \qquad (17)$$

The total variable cost per unit time for a replenishment cycle (denoted by $Z(T)$) is given by

$$Z(t_{1,}T) = \begin{cases} Z_1(t_{1,}T) & 0 < M \le t_d \\ Z_2(t_{1,}T) & t_d < M \le t_1 \\ Z_3(t_{1,}T) & M > t_1 \end{cases} \tag{18}$$

where $Z_1(t_{1,}T)$, $Z_2(t_{1,}T)$, and $Z_3(t_{1,}T)$ are discussed for three different cases follows.

Case 1$(0 < M \le t_d)$

The interest charge

This is the period before deterioration sets in, and payment for goods is settled with the capital opportunity cost rate I_c for the items in stock. Thus, the interest charge is given below.

$$I_{P1} = cI_c\left[\int_M^{t_d} I_1(t)dt + \int_{t_d}^{t_1} I_2(t)dt\right]$$

$$= cI_c\left[\frac{\lambda(t_d - M)}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2 + \frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right)\right] \tag{19}$$

The interest earned

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although, the retailer has to settle the accounts at period M, for that money has to be arranged at some specified rate of interest in order to get the remaining stocks financed for the period Mtot_d. The interest earned is

$$I_{E1} = sI_e\left[\int_0^M (\alpha + \beta t + \gamma t^2)tdt\right]$$

$$
\begin{aligned}
&= sI_e\left(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}\right. \\
&\left.+\gamma\frac{M^4}{4}\right) \qquad (20)
\end{aligned}
$$

The total variable cost per unit time $(0 < M \leq t_d)$ is

$Z_1(t_1, T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + lost sales cost+ interest charge – interest earned during the cycle}

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ & A + h_1\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)}+\frac{\alpha}{2}t_d^2+\frac{\beta}{3}t_d^3+\frac{\gamma}{4}t_d^4+\frac{\lambda}{\theta^2}e^{\theta(t_1-t_d)}-\frac{\lambda}{\theta^2}-\frac{\lambda t_1}{\theta}\right] \\
& + h_2\left[\frac{\lambda t_d^2}{2\theta}e^{\theta(t_1-t_d)}+\frac{\alpha}{6}t_d^3+\frac{\beta}{8}t_d^4+\frac{\gamma}{10}t_d^5+\frac{\lambda t_d}{\theta^2}e^{\theta(t_1-t_d)}-\frac{\lambda t_1}{\theta^2}\right. \\
& \left.-\frac{\lambda}{\theta^3}+\frac{\lambda}{\theta^3}e^{\theta(t_1-t_d)}-\frac{\lambda t_1^2}{2\theta}\right]+C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)}-1-\theta(t_1-t_d)\right] \\
& + \frac{C_b\delta\lambda}{2}(T-t_1)^2+C_\pi\lambda(1-\delta)(T-t_1) \\
& + cI_c\left[\frac{\lambda(t_d-M)}{\theta}\left(e^{\theta(t_1-t_d)}-1\right)+\frac{\alpha}{2}(t_d-M)^2\right. \\
& + \frac{\beta}{6}(2t_d+M)(t_d-M)^2+\frac{\gamma}{12}(3t_d^2+2t_dM+M^2)(t_d-M)^2 \\
& \left.+\frac{\lambda}{\theta^2}\left(e^{\theta(t_1-t_d)}-1-\theta(t_1-t_d)\right)\right] \\
& - sI_e\left(\alpha\frac{M^2}{2}+\beta\frac{M^3}{3}\right. \\
& \left.\left.+\gamma\frac{M^4}{4}\right)\right\} \qquad (21)
\end{aligned}
$$

Case 2 $(t_d < M \leq t_1)$

The interest charge

This is when the end point of credit period is greater than the period with no deterioration but shorter than or equal to the length of period with positive inventory stock of the items. The interest charge is

$$I_{P2} = cI_c\left[\int_M^{t_1} I_2(t)dt\right]$$

$$= cI_c\left[\frac{\lambda}{\theta^2}\left(e^{\theta(t_1-M)} - 1 - \theta(t_1 - M)\right)\right] \tag{22}$$

The interest earned

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although, the retailer has to settle the accounts at period M, for that money has to be arranged at some specified rate of interest in order to get the remaining stocks financed for the period M tot_1. The interest earned is

$$I_{E2} = sI_e\left[\int_0^{t_d}(\alpha + \beta t + \gamma t^2)tdt + \int_{t_d}^{M}\lambda tdt\right]$$

$$= sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right] \tag{23}$$

The total variable cost per unit time $(t_d < M \leq t_1)$ is

$Z_2(t_1, T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + lost sales cost + interest charge – interest earned during the cycle}

$$
\begin{aligned}
= \frac{1}{T}\Bigg\{ & A + h_1\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta}\right] \\
& + h_2\left[\frac{\lambda t_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda t_1}{\theta^2} - \frac{\lambda}{\theta^3}\right. \\
& \left. + \frac{\lambda}{\theta^3}e^{\theta(t_1-t_d)} - \frac{\lambda t_1^2}{2\theta}\right] + C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d)\right] \\
& + \frac{C_b\delta\lambda}{2}(T-t_1)^2 + C_\pi\lambda(1-\delta)(T-t_1) \\
& + cI_c\left[\frac{\lambda}{\theta^2}\left(e^{\theta(\mathbf{t_1}-M)} - 1 - \theta(t_1-M)\right)\right] \\
& - sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right]\Bigg\} \qquad (24)
\end{aligned}
$$

Case 3$(M > t_1)$

The interest charge

In this case, the trade credit period is greater than period with positive inventory. In this case the retailer pays no interest. Therefore,$I_{P3} = 0$.

The interest earned

In this case, the trade credit period (M) is greater than period with positive inventory(t_1). In this case the retailer earns interest on the sales revenue up to the trade credit period and no interest is payable during the period for the item kept in stock. Interest earned for the time period $[0, T]$

$$
\begin{aligned}
I_{E3} = sI_e\Bigg[& \int_0^{t_d}(\alpha+\beta t+\gamma t^2)t\,dt + (M-t_1)\int_0^{t_d}(\alpha+\beta t+\gamma t^2)dt + \int_{t_d}^{t_1}\lambda t\,dt \\
& + (M-t_1)\int_{t_d}^{t_1}\lambda dt\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= sI_e\Bigg[& \left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + (M-t_1)\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) - \frac{\lambda}{2}(t_1-t_d)^2 \\
& + M\lambda(t_1 - t_d)\Bigg] \qquad (25)
\end{aligned}
$$

The total variable cost per unit time $(M > t_1)$ is

$Z_3(t_1,T) = \frac{1}{T}${Ordering cost + inventory holding cost + deterioration cost + backordered cost + lost sales cost – interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{A + h_1\left[\frac{\lambda t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 + \frac{\lambda}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda}{\theta^2} - \frac{\lambda t_1}{\theta}\right] + h_2\left[\frac{\lambda t_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{\lambda t_1}{\theta^2} - \frac{\lambda}{\theta^3} + \frac{\lambda}{\theta^3}e^{\theta(t_1-t_d)} - \frac{\lambda t_1^2}{2\theta}\right] + C\frac{\lambda}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1-t_d)\right] + \frac{C_b\delta\lambda}{2}(T-t_1)^2 + C_\pi\lambda(1-\delta)(T-t_1) - sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + (M-t_1)\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) - \frac{\lambda}{2}(t_1-t_d)^2 + M\lambda(t_1-t_d)\right]\Bigg\} \qquad (26)$$

Since $0 < \theta < 1$, by utilizing a quadratic approximation for the exponential terms in equations (21), (24) and (26)yields

$$Z_1(t_1,T) = \frac{\lambda}{T}\left\{\frac{1}{2}A_1t_1^2 - B_1t_1 + C_1 + \frac{C_b\delta T^2}{2} - C_b\delta t_1T + C_\pi(1-\delta)T\right\} \qquad (27)$$

where

$$A_1 = \left[h_1(t_d\theta+1) + h_2\left(\frac{t_d\theta}{2}+1\right)t_d + C\theta + C_b\delta + cI_c(\theta(t_d-M)+1)\right],$$

$$B_1 = \Big[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + C_\pi(1-\delta) + cI_c(M + (t_d - M)\theta t_d)\Big]$$

and

$$C_1 = \frac{1}{\lambda}\Big[A + h_1\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + h_2\left(\frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d^4\theta}{4}\right) + \frac{C\lambda\theta t_d^2}{2} + cI_c\left(\frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2 + \lambda M t_d - \frac{\lambda t_d^2}{2} + \frac{\lambda}{2}(t_d - M)\theta t_d^2\right) - sI_e\left(\alpha\frac{M^2}{2} + \beta\frac{M^3}{3} + \gamma\frac{M^4}{4}\right)\Big].$$

Similarly

$$Z_2(t_{1,}T) = \frac{\lambda}{T}\left\{\frac{1}{2}A_2t_1^2 - B_2t_1 + C_2 + \frac{C_b\delta T^2}{2} - C_b\delta t_1 T + C_\pi(1-\delta)T\right\} \qquad (28)$$

Where

$$A_2 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + C_b\delta + cI_c\right],$$

$$B_2 = \left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + C_\pi(1-\delta) + cI_cM\right]$$

and

$$C_2 = \frac{1}{\lambda}\Bigg[A + h_1\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + h_2\left(\frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d^4\theta}{4}\right) + \frac{C\lambda\theta t_d^2}{2} + cI_c\frac{\lambda}{2}M^2 - sI_e\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4} + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right)\Bigg].$$

and

$$Z_3(t_{1,}T) = \frac{\lambda}{T}\left\{\frac{1}{2}A_3t_1^2 - B_3t_1 + C_3 + \frac{C_b\delta T^2}{2} - C_b\delta t_1T + C_\pi(1-\delta)T\right\} \qquad (29)$$

where

$$A_3 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + C_b\delta + sI_e\right],$$

$$B_3 = \left[h_1t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2Ct_d\theta + C_\pi(1-\delta) + sI_e\left[(M + t_d) - \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda}\right]\right]$$

and

$$C_3 = \frac{1}{\lambda}\left[A + h_1\left(\frac{\alpha}{2}t_d^2 + \frac{\beta}{3}t_d^3 + \frac{\gamma}{4}t_d^4 - \frac{\lambda t_d^2}{2} + \frac{\lambda t_d^3\theta}{2}\right) + h_2\left(\frac{\alpha}{6}t_d^3 + \frac{\beta}{8}t_d^4 + \frac{\gamma}{10}t_d^5 + \frac{\lambda t_d^4\theta}{4}\right) + \frac{C\lambda\theta t_d^2}{2} - sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)M - \frac{\lambda}{2}(2M + t_d)t_d\right]\right]$$

Optimal decision

This section determines optimal ordering policy that minimises the total variable cost per unit time by establishing necessary and sufficient conditions. The necessary condition for the total variable cost per unit time $Z_i(t_1,T)$ to be minimum are $\frac{\partial Z_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_i(t_1,T)}{\partial T} = 0$ for $i = 1,2,3$. The value of (t_1,T) obtained from $\frac{\partial Z_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_i(t_1,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 Z_i(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 Z_i(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 Z_i(t_1,T)}{\partial t_1\,\partial T}\right)^2\right\} > 0$ is satisfied gives a minimum for the total cost per unit time $Z_i(t_1,T)$.

For $0 < M \leq t_d$

The necessary condition for the total variable cost in equation (27) to be the minimum are$\frac{\partial Z_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_1(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_1(t_1,T)}{\partial t_1} = \frac{\lambda}{T}\{A_1 t_1 - B_1 - C_b \delta T\} = 0 \quad (30)$$

and

$$T = \frac{1}{C_b \delta}(A_1 t_1 - B_1) \quad (31)$$

Note that

$$\begin{aligned} A_1 t_1 - B_1 = \Big[& h_1(t_d \theta (t_1 - t_d) + t_1) + \frac{h_2 t_d \theta}{2}(t_1 - t_d) t_d + h_2 \left(t_1 - \frac{t_d}{2}\right) t_d \\ & + C\theta(t_1 - t_d) + C_b \delta t_1 + C_\pi \delta - C_\pi \\ & + cI_c\big((t_1 - M) + \theta(t_d - M)(t_1 - t_d)\big)\Big] > 0 \end{aligned}$$

since $(t_d - M) \geq 0, (t_1 - t_d),\ (t_1 - M) > 0$

Similarly

$$\frac{\partial Z_1(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2} A_1 t_1^2 - B_1 t_1 + C_1 - \frac{C_b \delta T^2}{2}\right\} = 0 \quad (32)$$

Substituting from equation (31) into equation (32) to obtain

$$A_1(A_1 - C_b \delta) t_1^2 - 2B_1(A_1 - C_b \delta) t_1 - (2C_b \delta C_1 - B_1^2) = 0 \quad (33)$$

Let $\Delta_1 = A_1(A_1 - C_b \delta) t_d^2 - 2B_1(A_1 - C_b \delta) t_d - (2C_b \delta C_1 - B_1^2)$, then the following result is obtained.

Lemma 1.

(i) If $\Delta_1 \leq 0$, then the solution of $t_1 \in [t_d, \infty)$ (sayt_{11}^*) which satisfies equation (33)does not only exists but is also unique.
(ii) If $\Delta_1 > 0$, then the solution of $t_1 \in [t_d, \infty)$ which satisfies equation (33) does not exist.

Proof of part (i): From equation(33), a new function $F_1(t_1)$ is definedas follows

$$F_1(t_1) = A_1(A_1 - C_b\delta)t_1^2 - 2B_1(A_1 - C_b\delta)t_1 - (2C_b\delta C_1 - B_1^2), \qquad t_1 \in [t_d, \infty) \qquad (34)$$

Taking the first order derivative of $F_1(t_1)$with respect to$t_1 \in [t_d, \infty)$ yields

$$\frac{F_1(t_1)}{dt_1} = 2(A_1t_1 - B_1)(A_1 - C_b\delta) > 0$$

Because
$(A_1t_1 - B_1) > 0$and

$$(A_1 - C_b\delta) = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + cI_c(\theta(t_d - M) + 1)\right] > 0$$

Hence $F_1(t_1)$is a strictly increasing of $t_1$in the interval $[t_d, \infty)$.Moreover, $\lim_{t_1 \to \infty} F_1(t_1) = \infty$and$F_1(t_d) = \Delta_1 \leq 0$. Therefore, by applying intermediate value theorem, there exists a uniquet_1 say$t_{11}^* \in [t_d, \infty)$ such that$F_1(t_{11}^*) = 0$. Hence t_{11}^* is the unique solution of equation (33). Thus, the value of t_1 (denoted byt_{11}^*) can be found from equation (33) and is given by

$$t_{11}^* = \frac{B_1}{A_1} + \frac{1}{A_1}\sqrt{\frac{(2A_1C_1 - B_1^2)C_b\delta}{(A_1 - C_b\delta)}} \qquad (35)$$

Oncet_{11}^* is obtained, then the value of T (denoted byT_1^*) can be found from equation (31) and is given by

$$T_1^* = \frac{1}{C_b\delta}(A_1t_{11}^* - B_1) \qquad (36)$$

Equation(35) and (36) give the optimal t_{11}^* and T_1^* respectively for the total cost in equation (27) only if B_1 satisfies the inequality given in equation (37)

$$B_1^2 < 2A_1C_1 \tag{37}$$

Proof of part (ii): If $\Delta_1 > 0$, then from equation (34), $F_1(t_1) > 0$. Since $F_1(t_1)$ is a strictly increasing function of $t_1 \in [t_d, \infty)$, then $F_1(t_1) > 0$ for all $t_1 \in [t_d, \infty)$. Thus, a value of $t_1 \in [t_d, \infty)$ cannot be found such that $F_1(t_1) = 0$. This completes the proof.

Theorem 1.

(i) If $\Delta_1 \leq 0$,then the total variable cost $Z_1(t_1, T)$ is convex and reaches its global minimum at the point (t_{11}^*, T_1^*), where (t_{11}^*, T_1^*) is the point which satisfies equations (33)and (30).

(i) If $\Delta_1 > 0$, then the total variable cost $Z_1(t_1, T)$has a minimum value at the point(t_{11}^*, T_1^*) where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b\delta}(A_1t_d - B_1)$

Proof of part (i): When $\Delta_1 \leq 0$,it is seen that t_{11}^* and T_1^* are the unique solutions of equations(33) and (30) respectively from Lemma l(i). Taking the second derivative of $Z_1(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point (t_{11}^*, T_1^*) yields

$$\left.\frac{\partial^2 Z_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)} = \frac{\lambda}{T_1^*}A_1 > 0$$

$$\left.\frac{\partial^2 Z_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)} = -\frac{\lambda}{T_1^*}C_b\delta$$

$$\left.\frac{\partial^2 Z_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)} = \frac{\lambda}{T_1^*}C_b\delta > 0$$

and

$$\left(\frac{\partial^2 Z_1(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right)\left(\frac{\partial^2 Z_1(t_1,\ T)}{\partial T^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right)-\left(\frac{\partial^2 Z_1(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*,\ T_1^*)}\right)^2$$

$$= \frac{\lambda^2 C_b \delta}{T_1^{*2}}(A_1 - C_b\delta)$$

$$= \frac{\lambda^2 C_b \delta}{T_1^{*2}}\left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + cI_c(\theta(t_d - M) + 1)\right] > 0 (38)$$

It is therefore concluded from equation (38) and Lemma 1 that $Z_1(t_{11}^*,\ T_1^*)$ is convex and $(t_{11}^*,\ T_1^*)$ is the global minimum point of $Z_1(t_1,\ T)$. Hence the values of t_1 and T in equations (35) and (36) respectively are optimal.

Proof of part (ii): When $\Delta_1 > 0,\ F_1(t_1) > 0$ for all $t_1 \in [t_d,\ \infty)$.Thus, $\frac{\partial Z_1(t_1,\ T)}{\partial T} = \frac{F_1(t_1)}{T^2} > 0$ for all $t_1 \in [t_d, \infty)$ which implies $Z_1(t_1,\ T)$ is a strictly increasing function of T. Thus $Z_1(t_1,\ T)$ has a minimum value when T is minimum. Therefore, $Z_1(t_1,\ T)$ has a minimum value at the point $(t_{11}^*,\ T_1^*)$ where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b\delta}(A_1 t_d - B_1)$. This completes the proof.

For $t_d < M \leq t_1$

The necessary condition for the total variable cost in equation (28) to be the minimum are $\frac{\partial Z_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_2(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_2(t_1, T)}{\partial t_1} = \frac{\lambda}{T}\{A_2 t_1 - B_2 - C_b\delta T\} = 0 \qquad (39)$$

and

$$T = \frac{1}{C_b\delta}(A_2 t_1 - B_2) \qquad (40)$$

Note that

$$A_2t_1 - B_2 = \Big[h_1(t_d\theta(t_1-t_d)+t_1) + \frac{h_2t_d\theta}{2}(t_1-t_d)t_d + h_2\left(t_1-\frac{t_d}{2}\right)t_d + C\theta(t_1-t_d) + C_b\delta t_1 + C_\pi\delta - C_\pi + cI_c(t_1-M)\Big] > 0$$

since $(t_1-t_d) > 0, (t_1-M) \geq 0$

Similarly

$$\frac{\partial Z_2(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}A_2t_1^2 - B_2t_1 + C_2 - \frac{C_b\delta T^2}{2}\right\} = 0 \qquad (41)$$

Substituting T from equation (40) into equation (41) to obtain

$$A_2(A_2-C_b\delta)t_1^2 - 2B_2(A_2-C_b\delta)t_1 - (2C_b\delta C_2 - B_2^2) = 0 \qquad (42)$$

Let $\Delta_2 = A_2(A_2-C_b\delta)M^2 - 2B_2(A_2-C_b\delta)M - (2C_b\delta C_2 - B_2^2)$, then the following result is obtain

Lemma 2.

(i) If $\Delta_2 \leq 0$, then the solution of $t_1 \in [M,\infty)$ (say t_{12}^*) which satisfies equation (42) does not only exists but is also unique.

(ii) If $\Delta_2 > 0$, then the solution of $t_1 \in [M,\infty)$ which satisfies equation (42) does not exist.

Proof of part (i): From equation (42), a new function $F_2(t_1)$ is defined as follows

$$F_2(t_1) = A_2(A_2-C_b\delta)t_1^2 - 2B_2(A_2-C_b\delta)t_1 - (2C_b\delta C_2 - B_2^2), \quad t_1 \in [M,\infty) \qquad (43)$$

Taking the first order derivative of $F_2(t_1)$with respect to$t_1 \in [M,\infty)$ yields

$$\frac{F_2(t_1)}{dt_1} = 2(A_2t_1 - B_2)(A_2 - C_b\delta) > 0$$

Because $(A_2t_1 - B_2) > 0$ and $(A_2 - C_b\delta) = \left[h_1(t_d\theta+1) + h_2\left(\frac{t_d\theta}{2}+1\right)t_d + C\theta + cI_c\right] > 0$

Hence $F_2(t_1)$ is a strictly increasing of t_1 in the interval $[M,\infty)$. Moreover, $\lim_{M\to\infty} F_2(t_1)=\infty$ and $F_2(M)=\Delta_2\leq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{12}^*\in[M,\infty)$ such that $F_2(t_{12}^*)=0$. Hence t_{12}^* is the unique solution of equation (42). Thus, the value of t_1 (denoted byt_{12}^*) can be found from equation (42) and is given by

$$t_{12}^* = \frac{B_2}{A_2} + \frac{1}{A_2}\sqrt{\frac{(2A_2C_2-B_2^2)C_b\delta}{(A_2-C_b\delta)}} \tag{44}$$

Once t_{12}^* is obtained, then the value of T (denoted by T_2^*) can be found from equation (40) and is given by

$$T_2^* = \frac{1}{C_b\delta}(A_2t_{12}^* - B_2) \tag{45}$$

Equation (44) and (45) give the optimal values of t_{12}^* and T_2^* respectively for the cost function in equation (28) only if B_2 satisfies the inequality given in equation (46)

$$B_2^2 < 2A_2C_2 \tag{46}$$

Proof of part (ii): If $\Delta_2 > 0$, then from equation(43), $F_2(t_1)>0$. Since $F_2(t_1)$ is a strictly increasing function of $t_1\in[M,\infty)$, then $F_2(t_1)>0$ for all $t_1\in[M,\infty)$. Thus, a value of $t_1\in[M,\infty)$ cannot be found such that $F_2(t_1)=0$. This completes the proof.

Theorem 2.

(i) If $\Delta_2\leq 0$, then the total variable cost $Z_2(t_1,T)$ is convex and reaches its global minimum at the point (t_{12}^*,T_2^*), where (t_{12}^*,T_2^*) is the point which satisfies equations(42) and (39).

(ii) If $\Delta_2 > 0$,then the total variable cost $Z_2(t_1,T)$ has a minimum value at the point (t_{12}^*,T_2^*) where $t_{12}^*=M$ and $T_2^*=\frac{1}{C_b\delta}(A_2M-B_2)$

Proof of part (i): When $\Delta_2 \leq 0$, it is seen that t_{12}^* and T_2^* are the unique solutions of equations (42) and (39) respectively from Lemma 2(i). Taking the second derivative of $Z_2(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point $(t_{12}^*,\ T_2^*)$ yields

$$\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)} = \frac{\lambda}{T_2^*} A_2 > 0$$

$$\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{12}^*,\ T_2^*)} = -\frac{\lambda}{T_2^*} C_b \delta$$

$$\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12}^*,\ T_2^*)} = \frac{\lambda}{T_2^*} C_b \delta > 0$$

and

$$\left(\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)}\right)\left(\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12}^*,\ T_2^*)}\right) - \left(\left.\frac{\partial^2 Z_2(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{12}^*,\ T_2^*)}\right)^2$$

$$= \frac{\lambda^2 C_b \delta}{T_2^{*2}} (A_2 - C_b \delta)$$

$$= \frac{\lambda^2 C_b \delta}{T_2^{*2}} \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + cI_c\right] > 0 \qquad (47)$$

It is therefore concluded from equation (47) and Lemma 2 that $Z_2(t_{12}^*,\ T_2^*)$ is convex and $(t_{12}^*,\ T_2^*)$ is the global minimum point of $Z_2(t_1,\ T)$. Hence the values of t_1 and T in equations(44) and (45) respectively are optimal.

Proof of part (ii): When $\Delta_2 > 0, F_2(t_1) > 0$ for all $t_1 \in [M,\ \infty)$. Thus, $\frac{\partial Z_2(t_1,\ T)}{\partial T} = \frac{F_2(t_1)}{T^2} > 0$ for all $t_1 \in [M, \infty)$ which implies $Z_2(t_1,\ T)$ is a strictly increasing function of T. Thus $Z_2(t_1,\ T)$ has a minimum value when T is minimum. Therefore, $Z_2(t_1,\ T)$ has a minimum value at the point $(t_{12}^*,\ T_2^*)$ where $t_{12}^* = M$ and $T_2^* = \frac{1}{C_b\delta}(A_2 M - B_2)$. This completes the proof.

For $M > t_1$

The necessary condition for the total variable cost in equation (29) to be the minimum are $\frac{\partial Z_3(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial Z_3(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_3(t_1,T)}{\partial t_1} = \frac{\lambda}{T}\{A_3 t_1 - B_3 - C_b \delta T\} = 0 \tag{48}$$

and

$$T = \frac{1}{C_b \delta}(A_3 t_1 - B_3) \tag{49}$$

Note that

$$A_3 t_1 - B_3 = \left[h_1(t_d \theta (t_1 - t_d) + t_1) + \frac{h_2 t_d \theta}{2}(t_1 - t_d)t_d + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + C\theta(t_1 - t_d) + C_b \delta t_1 + C_\pi \delta - C_\pi + sI_e\left[(t_1 - t_d) + \left(\alpha t_d + \beta \frac{t_d^2}{2} + \gamma \frac{t_d^3}{3}\right)\frac{1}{\lambda}\right] - M\right] > 0$$

since $(t_1 - t_d) > 0$

Similarly

$$\frac{\partial Z_3(t_1,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}A_3 t_1^2 - B_3 t_1 + C_3 - \frac{C_b \delta T^2}{2}\right\} = 0 \tag{50}$$

Substituting$T$ from equation(49) into equation (50) to obtain

$$A_3(A_3 - C_b\delta)t_1^2 - 2B_3(A_3 - C_b\delta)t_1 - (2C_b\delta C_3 - B_3^2) = 0 \tag{51}$$

LetΔ_{31}= $A_3(A_3 - C_b\delta)t_d^2 - 2B_3(A_3 - C_b\delta)t_d - (2C_b\delta C_3 - B_3^2)$
and
Δ_{32}= $A_3(A_3 - C_b\delta)M^2 - 2B_3(A_3 - C_b\delta)M - (2C_b\delta C_3 - B_3^2)$, then the following result is obtained.

Lemma 3.

(i) If $\Delta_{31} \le 0 \le \Delta_{32}$, then the solution of $t_1 \in [t_d, M]$ (say t_{13}^*) which satisfies equation (51) not only exists but also is unique.

(ii) If $\Delta_{32} < 0$, then the solution of $t_1 \in [t_d, M]$ which satisfies equation (51) does not exist.

Proof of part (i): From equation (51), a new function $F_3(t_1)$ is defined as follows

$$F_3(t_1) = A_3(A_3 - C_b\delta)t_1^2 - 2B_3(A_3 - C_b\delta)t_1 - (2C_b\delta C_3 - B_3^2), \qquad t_1 \in [t_d, M] \quad (52)$$

Taking the first order derivative of $F_3(t_1)$with respect to$t_1 \in [t_d, M]$ yields

$$\frac{F_3(t_1)}{dt_1} = 2(A_3 - C_b\delta)(A_3t_1 - B_3) > 0$$

Because $(A_3t_1 - B_3) > 0$ and $(A_3 - C_b\delta) = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + sI_e\right] > 0$

Hence $F_3(t_1)$ is a strictly increasing of t_1 in the interval $[t_d, M]$. Moreover, $F_3(t_d) \leq 0$ and $F_3(M) \geq 0$, that is, $F_3(t_d) \leq 0 \leq F_3(M)$. Thus, a unique value of t_1 say $t_{13}^* \in [t_d, M]$ can be found such that $F_3(t_{13}^*) = 0$. Hence t_{13}^* is the unique solution of equation (51). Thus, the value of t_1 (denoted byt_{13}^*) can be found from equation (51) is given by

$$t_{13}^* = \frac{B_3}{A_3} + \frac{1}{A_3}\sqrt{\frac{(2A_3C_3 - B_3^2)C_b\delta}{(A_3 - C_b\delta)}} \qquad (53)$$

Once t_{13}^* is obtained, then the value of T (denoted by T_3^*) can be found from equation(49) and is given by

$$T_3^* = \frac{1}{C_b\delta}(A_3t_{13}^* - B_3) \qquad (54)$$

Equations(53) and (54) give the optimal values of t_{13}^* and T_3^* for the total cost function in equation (29) only if B_3 satisfies the inequality given in equation (55)

$$B_3^2 < 2A_3C_3 \tag{55}$$

Proof of part (ii): If $\Delta_{32} < 0, F_3(M) < 0$. Since $F_3(t_1)$ is a strictly increasing function of t_1 in the interval $[t_d,\ M]$ and $M > t_1, F_3(t_1) < 0$ for all $t_1 \in [t_d,\ M]$. This implies that a value of $t_1 \in [t_d, M]$ cannot be found such that $F_3(t_1) = 0$. This completes the proof.

Theorem 3.

(i) If $\Delta_{31} \leq 0 \leq \Delta_{32}$, then the total variable cost $Z_3(t_1, T)$ is convex and reaches its global minimum at the point (t_{13}^*, T_3^*), where (t_{13}^*, T_3^*) is the point which satisfies equation (51) and (48).

(ii) If $\Delta_{32} < 0$, then the total variable cost $Z_3(t_1, T)$ has a minimum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = M$ and $T_3^* = \frac{1}{C_b\delta}(A_3M - B_3)$

(iii) If $\Delta_{31} > 0$, then the total variable cost $Z_3(t_1, T)$ has a minimum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = t_d$ and $T_3^* = \frac{1}{C_b\delta}(A_3t_d - B_3)$

Proof of part (i): When $\Delta_{32} \leq 0 \leq \Delta_{32}$, it is seen that t_{13}^* and T_3^* are the unique solutions of equations (51) and (48) respectively from Lemma 3(i). Taking the second derivative of $Z_3(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point $(t_{13}^*,\ T_3^*)$ yields

$$\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)} = \frac{\lambda}{T_3^*}A_3 > 0$$

$$\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)} = -\frac{\lambda}{T_3^*}C_b\delta$$

$$\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)} = \frac{\lambda}{T_3^*}C_b\delta > 0$$

and

$$\left(\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)}\right)\left(\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right)-\left(\left.\frac{\partial^2 Z_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)}\right)^2$$

$$= \frac{\lambda^2 C_b \delta}{T_3^{*2}}(A_3 - C_b\delta) > 0$$

$$= \frac{\lambda^2 C_b \delta}{T_3^{*2}}\left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + sI_e\right] > 0 \qquad (56)$$

It is therefore conclude from equation (56) and Lemma 3 that $Z_3(t_{13}^*,\ T_3^*)$ is convex and $(t_{13}^*,\ T_3^*)$ is the global minimum point of $Z_3(t_1,\ T)$. Hence the values of t_1 and T in equations (53) and (54) respectively are optimal.

Proof of part (ii): When $\Delta_{32} < 0, F_3(M) < 0$. Since $F_3(t_1)$ is a strictly increasing function of t_1 in the interval $[t_d,\ M]$, $F_3(t_1) < 0$ for all $t_1 \in [t_d, M]$. This implies that $\frac{\partial Z_3(t_1,\ T)}{\partial T} = \frac{F_3(t_1)}{T^2}$, for all $t_1 \in [t_d, M]$. So, $Z_3(t_1,\ T)$ is a decreasing function of T in the interval $[t_d, M]$. Thus $Z_3(t_1,\ T)$ has a minimum value at $(t_{13}^*,\ T_3^*)$ where $t_{13}^* = M$ and the corresponding minimum value of T_3^* is $T_3^* = \frac{1}{C_b\delta}(A_3 M - B_3)$.

Proof of part (iii): When $\Delta_{31} > 0,\ F_3(t_d) > 0$, then $F_3(t_1) > 0$ for all $t_1 \in [t_d, M]$, which implies $\frac{\partial Z_3(t_1,\ T)}{\partial T} = \frac{F_3(t_1)}{T^2} > 0$ for all $t_1 \in [t_d,\ M]$. So, $Z_3(t_1,\ T)$ is a strictly increasing function of T in the interval $[t_d, M]$. Thus $Z_3(t_1,\ T)$ has a minimum value at $(t_{13}^*,\ T_3^*)$ where $t_{13}^* = t_d$ and the corresponding minimum value of T_3^* is $T_3^* = \frac{1}{C_b\delta}(A_3 t_d - B_3)$.

Thus, the EOQ corresponding to the optimal cycle length T^* will be computed as follows:
EOQ^* =Total demand before deterioration sets in + total demand after deterioration sets in +total number of deteriorated items + total number of items backordered

$$= \int_0^{t_d}(\alpha + \beta t + \gamma t^2)dt + \int_{t_d}^{t_1^*}\lambda dt + \left[\frac{\lambda}{\theta}\left(e^{\theta(t_1^* - t_d)} - 1\right) - \lambda(t_1^* - t_d)\right] + \lambda\delta(T^* - t_1^*)$$

$$= \alpha t_d + \beta \frac{t_d^2}{2} + \gamma \frac{t_d^3}{3} + \frac{\lambda}{\theta}\left(e^{\theta(t_1^* - t_d)} - 1\right) + \lambda\delta(T^* - t_1^*) \qquad (57)$$

Numerical Examples

This section provides some numerical examples to illustrate the theoretical results of model developed.

Example 4.1 (for $0 < M \leq t_d$*)*

Consider an inventory system with the following input parameters: $A = \$350$/order, $C = \$45$/unit/year, $S = \$65$/unit/year, $h_1 = \$15$/unit/year, $h_2 = \$5$/unit/year, $C_b = \$20$/unit/year, $C_\pi = \$5$/unit/year, $\theta = 0.05$ units/year, $\alpha = 980$ units, $\beta = 180$ units, $\gamma = 15$ units, $\lambda = 450$ units, $t_d = 0.2136$ year (78 days), $M = 0.0684$ year (25 days), $I_c = 0.10, I_e = 0.08$ and $\delta = 0.8$. It is seen that $M \leq t_d, \Delta_1 = -16.5278 < 0$, $B_1^2 = 3.78255$, $2A_1C_1 = 102.8074$ and hence $B_1^2 < 2A_1C_1$. Substituting the above values inequations $(35), (36), (27)$ and (57), the value of optimal time with positive inventory, cycle length, total variable cost and EOQ are respectively obtained as follows: $t_{11}^* = 0.2625$ year (96 days), $T_1^* = 0.5186$ year (189 days), $Z_1(T_1^*,\ t_{11}^*) = \$2293.5980$ per year, and $EOQ_1^* = 327.6931$ units per year.

Example 4.2 (for $t_d < M \leq t_1$*)*

The data are same as in Example 4.1 except that $M = 0.2382$ year (87 days). It is seen that $M > t_d, \Delta_2 = -6.8850 < 0 B_2^2 = 7.3008$, $2A_2C_2 = 86.3460$ and hence $B_2^2 < 2A_2C_2$. Substituting the above values in equations $(44), (45), (28)$ and (57), the value of optimal time with positive inventory, cycle length, total variable cost and EOQ are respectively obtained as follows: $t_{12}^* = 0.2596$ year (95 days), $T_2^* = 0.4636$ year (169 days), $Z_2(T_2^*,\ t_{12}^*) = \$1919.0162$ per year and $EOQ_2^* = 307.6548$units per year.

Example 4.3 (for $M > t_1$*)*

The data are same as in Example 4.1 except that $t_d = 0.1254$ (46 days) and $M = 0.2378$ year (87 days). It is seen that $M > t_d, \Delta_{31} = -15.3534 < 0$, $\Delta_{32} = 1.4087 > 0$, $B_3^2 = 3.0857$, $2A_3C_3 = 55.0985$. Here $\Delta_{31} \leq 0 \leq \Delta_{32}$ and $B_3^2 < 2A_3C_3$. Substituting the above values inequations $(53), (54), (29)$ and (57), the value of optimal time with positive inventory, cycle length, total variable cost and EOQ are respectively obtained as follows: $t_{13}^* = 0.1978$ year (72 days), $T_3^* = 0.3745$ year (137 days), $Z_3(T_3^*,\ t_{13}^*) = \$1722.3973$ per year and $EOQ_3^* = 223.0945$ units per year.

Sensitivity Analysis

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from $-20\%, -10\%$, $+10\%$ to 20% taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these parameters on time with positive inventory, cycle length, total variable cost and the economic order quantity per cycle for example 4.1, 4.2 and 4.3 are summarised in Tables 2-4.

Table 2 Effect of changes of some parameters on decision variables for example 4.1

Parameters	% Change in parameter	% Change in t_{11}^*	% Change in T_1^*	% Change in EOQ_1^*	% Change in $Z_1(t_{11}^*, T_1^*)$
θ	-20	0.385	0.181	0.130	-0.023
	-10	0.191	0.090	0.064	-0.011
	$+10$	-0.187	-0.088	-0.063	0.011
	$+20$	-0.370	-0.174	-0.125	0.022
C	-20	2.452	0.547	0.491	-1.130
	-10	1.194	0.260	0.235	-0.560
	$+10$	-1.133	-0.236	-0.217	0.550
	$+20$	-2.211	-0.450	-0.418	1.090
S	-20	0.170	0.210	0.132	0.202
	-10	0.085	0.105	0.066	0.101
	$+10$	-0.085	-0.105	-0.066	-0.101
	$+20$	-0.171	-0.211	-0.133	-0.202
I_c	-20	2.046	0.357	0.353	-1.104
	-10	1.006	0.172	0.172	-0.548
	$+10$	-0.973	-0.161	-0.162	0.541
	$+20$	-1.915	-0.310	-0.316	1.074

I_e	-20	0.170	0.210	0.132	0.202
	-10	0.085	0.105	0.066	0.101
	$+10$	-0.085	-0.105	-0.066	-0.101
	$+20$	-0.171	-0.211	-0.133	-0.202
δ	-20	3.112	4.700	-3.077	3.689
	-10	1.577	2.203	-1.523	1.869
	$+10$	-1.612	-1.989	1.495	-1.911
	$+20$	-3.256	-3.818	2.966	-3.859
C_b	-20	-3.817	6.938	3.674	-4.525
	-10	-1.783	3.141	1.659	-2.114
	$+10$	1.578	-2.645	-1.392	1.871
	$+20$	2.986	-4.906	-4.906	3.540

Table 3 Effect of changes of some parameters on decision variables for example 4.2

Parameters	% Change in parameter	% Change in t_{12}^*	% Change in T_2^*	% Change in EOQ_2^*	% Change in $Z_2(t_{12}^*, T_2^*)$
θ	-20	0.361	0.187	0.128	-0.026
	-10	0.179	0.093	0.062	-0.013
	$+10$	-0.175	-0.091	-0.062	0.013
	$+20$	-0.347	-0.180	-0.123	0.025
C	-20	0.684	0.362	0.249	-0.036
	-10	0.332	0.176	0.121	-0.018
	$+10$	-0.314	-0.166	-0.114	0.017
	$+20$	-0.610	-0.323	-0.222	0.033
S	-20	2.096	2.860	1.713	2.930
	-10	1.056	1.440	0.862	1.480
	$+10$	-1.071	-1.461	-0.875	-1.500
	$+20$	-2.158	-2.944	-1.763	-3.020

I_c	−20	0.328	0.177	0.121	−0.011
	−10	0.161	0.087	0.060	−0.005
	+10	−0.155	−0.084	−0.057	0.005
	+20	−0.304	−0.164	−0.112	0.010
I_e	−20	2.096	2.860	1.713	2.930
	−10	1.056	1.440	0.862	1.480
	+10	−1.071	−1.461	−0.875	−1.500
	+20	−2.158	−2.944	−1.763	−3.020
δ	−20	3.455	2.929	−3.028	4.830
	−10	1.737	1.423	−1.506	2.430
	+10	−1.756	−1.366	1.491	−2.460
	+20	−3.532	−2.692	2.966	−4940
C_b	−20	−2.765	6.673	3.408	−3.870
	−10	−1.287	3.019	1.539	−1.800
	+10	1.131	−2.540	−1.291	1.580
	+20	2.135	−4.708	−2.390	2.990

Table 4 Effect of changes of some parameters on decision variables for example 4.3

Parameters	% Change in parameter	% Change in t_{13}^{*}	% Change in T_3^{*}	% Change in EOQ_3^{*}	% Change in $Z_3(t_{13}^{*}, T_3^{*})$
θ	-20	0.682	0.305	0.234	-0.087
	-10	0.338	0.151	0.116	-0.043
	$+10$	-0.332	-0.148	-0.114	0.043
	$+20$	-0.657	-0.293	-0.226	0.084
C	-20	0.654	0.292	0.230	-0.084
	-10	0.324	0.145	0.114	-0.042
	$+10$	-0.318	-0.142	-0.112	0.041
	$+20$	-0.631	-0.282	-0.221	0.081
S	-20	5.428	4.811	3.350	3.044
	-10	2.677	2.405	1.671	1.552
	$+10$	-2.608	-2.407	-1.667	-1.612
	$+20$	-5.154	-4.821	-3.332	-3.286
I_c	-20	0.000	0.000	0.000	0.000
	-10	0.000	0.000	0.000	0.000
	$+10$	0.000	0.000	0.000	0.000
	$+20$	0.000	0.000	0.000	0.000
I_e	-20	5.428	4.811	3.350	3.044
	-10	2.677	2.405	1.671	1.552
	$+10$	-2.608	-2.407	-1.667	-1.612
	$+20$	-5.154	-4.821	-3.332	-3.286
δ	-20	5.251	2.903	-3.537	6.288
	-10	2.643	1.469	-1.753	3.166
	$+10$	-2.681	-1.502	1.721	-3.211
	$+20$	-5.401	-3.035	3.409	-6.469
C_b	-20	-3.328	6.856	3.874	-3.985
	-10	-1.552	3.104	1.750	-1.859

$+10$	1.370	-2.613	-1.468	1.641
$+20$	2.590	-4.845	-2.718	3.102

Based on the computed results shown on Tables2, 3 and 4, the following managerial insights are obtained.

1. When the rate of deterioration (θ) increases, the optimal time with positive inventory (t_1^*), cycle length(T^*) and economic order quantity (EOQ^*) decrease while total variable cost $(Z(T^*,\ t_1^*))$ increases and vice versa. This is very obvious, because when the number of deteriorated items increases, then the total variable cost will be high. Hence the retailer shall orders less quantity to avoid the items being deteriorating when the deterioration rate increases. This decreases the inventory holding cost and hence reducing the total variable cost. The rate of deterioration can also be reduce by improving the equipment in warehouse.
2. When the unit purchasing cost (C) increases, the optimal time with positive inventory (t_1^*), cycle length(T^*), and the economic order quantity (EOQ^*) decrease while the total variable cost $(Z(T^*,\ t_1^*))$ increases, and vice versa. In real market situation the higher the cost of an item, the higher the total variable cost. This result implies that the retailer orders a smaller quantity to enjoy the benefits of trade credit more frequently in the presence of an increased unit purchasing price and consequently shortening optimal time with positive inventory and cycle length.
3. When the unit selling price (S) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), the economic order quantity (EOQ^*) and the total variable cost $(Z(T^*,\ t_1^*))$ decrease and vice versa. In real market situation the higher the selling price of an item, the lower the demand of that item and vice versa. This means that it the unit selling price per unit increases, the retailer orders less quantity of items in order to take the benefits of the trade credit more frequently.
4. When the interest charge (I_c) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and the economic order quantity (EOQ^*) decrease while the total variable cost $(Z(T^*,\ t_1^*))$ increases when interest charge is high for both case 1 and 2 and vice versa. This means that when interest charge increases, the retailer might order fewer amounts of items. As for $M > t_1$,the increase/decrease in interest charge (I_c) does not affect the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $(Z(T^*,\ t_1^*))$, because the interest charge is zero.
5. When the interest earned (I_e) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $(Z(T^*,\ t_1^*))$ decrease

and vice versa. This implies that when the interest earned is high, the optimal time with positive inventory (t_1^*), cycle length (T^*), the economic order quantity and the total variable cost are low. Hence the retailer should order fewer items so as to effectively take the benefit of trade credit more frequently.

6. When the backlogging parameter (δ) increases, the optimal time with positive inventory (t_1^*) and cycle length (T^*) decrease while the economic order quantity (EOQ^*) increases which in turn leads to the increase in total variable cost $(Z(T^*,\ t_1^*))$ and vice versa.
7. When the shortage cost (C_b) increases, the optimal time with positive inventory (t_1^*) and total variable cost $(Z(T^*,\ t_1^*))$ will increases, cycle length (T^*), and economic order quantity (EOQ^*) decreases and vice versa. This means that when the shortages cost increase, the number of backordered items reduce drastically which in turn decreases order quantity. Hence the retailer should avoid shortages when the shortage cost is very high.

6 Conclusion

In this article, an EOQ model for non-instantaneous deteriorating items with time dependent quadratic demand rate, time dependent linear holding cost and shortages under trade credit policy. The demand rate before deterioration sets in is assumed to be time dependent quadratic and that is considered as a constant after deterioration begins. Shortages are allowed and partially backlogged. The optimal time with positive inventory and cycle length that minimise total variable cost are determined. Also, the corresponding economic order quantity (EOQ) is determined. Moreover, some useful theorems that prove the existent and uniqueness of the optimal solutions were provided and an easy-to-use method to determine the optimal time with positive inventory, cycle length and the corresponding EOQ such that total variable cost has a minimum value under various conditions were also presented. Some numerical examples are given to illustrate the theoretical result of the model. Some numerical examples are presented to demonstrate the model. Sensitivity analysis were also carried out to show the effect of changes in system parameters in decision variables. The results show that the retailer reduces total variable cost by ordering less to shorten the optimal time with positive inventory and cycle length when deterioration sets in, unit purchasing price increases, unit selling price increases, interest charge increases, shortage cost increases and interest earned decreases. The proposed model could be used in inventory control of non-instantaneous deteriorating items such as, aircrafts, computers, seasonal products, fashionable goods, android mobiles, automobiles, garments, television, computer chips, and photographic films and so on.

The proposed model can be extended by taking more realistic assumptions, such as two storage facilities, variable deterioration rate, inflation rates, reliability of items, quantity discounts, quadratic holding cost, ramp type or trapezoidal type or probabilistic demand rates, finite time horizon, multi-item inventory models and so on.

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