CONSTRUCTION OF VECTOR SPACE USING PERMUTATION PATTERNS

By

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Abstract

In this paper, Vector space is constructed from a set of permutations. Appropriate field, bases, dimension and span are found by testing the various Vector space axioms. Moreover, it is shown that some vectors from the span are linearly dependent and that every vector in the Vector space is linearly dependent. Our results generalize and improve the results in the literature.

Introduction

Several papers have been written on the algebraic structure G_p'. G_p' is a set of permutations.

Garba and Abubakar (2015) used modular arithmetic to construct an algebraic structure $G_p =$ $\{w_1, w_2 \cdots w_{p-1}\}$, where each $wi = \left((1)(1+i)_{mp}(1+2i)_{mp}\ldots(1+(p-1)i)_{mp}\right)$. The subscript mp indicates the numbers are taken modulo p, $1 \le n \le p$.

Each w_i is called a cycle and the elements in each w_i are distinct and called successors. A special cycle $w_p = \{(p)(p)(p)...(p)\}$ was embedded into the algebraic structure G_p, where the special cycle is of length 1 and the successors are not distinct. The resulting structure was defined as $G_p' = G_p \bigcup \{w_p\}$. A concatenation map was defined as follows:

$$\varphi_{i,j} = G_p' \times G_p' \to G_p'$$

where $1 \le i,\ j \le p$, $p \ge 5$ is prime. Then (G_p', φ) is an abelian group because it satisfies closure, commutativity, associativity, inverse and identity laws. Some algebraic and number theoretic properties of the structure were enumerated. The length of each cycle was defined as $\pi(w) := |\Delta^1 f(w)|$ where each $\Delta^1 f(w)$ is the difference between the first and the last successors in a cycle w_i. Also, an operator $\pi : G_p \to X$ was defined. This operator defines an isomorphism on G_p. In their work, Garba and Ojonugwa (2017) defined a Γ_1-non deranged permutation group $G_p^{\Gamma_1}$ as the permutation group with fixed element on the first column from the left. $G_p^{\Gamma_1} = \{w_i\}$, $1 \le i \le (p-1)$, $p \ge 5$.

Vector spaces stem from affine geometry, via the introduction of coordinates in the plane or three-dimensional space. Around 1636, French mathematicians Renscartes and Pierre de Fermat founded analytic geometry by identifying solutions to an equation of two variables with points on a plane curve, Bourbaki(1969). To achieve geometric solutions without using coordinates, Bolzano introduced, in 1804, certain operations on points, lines and planes, which are predecessors of vectors. Mbius in 1827 introduced the notion of barycentric coordinates. Bellavitis introduced the notion of a bipoint, i.e., an oriented segment one of whose ends is the origin and the other one a target, Dorier(1995). Vectors were reconsidered with the presentation of complex numbers by Argand and Hamilton and the inception of quaternions by the latter, Hamilton(1853). They are elements in R^2 and R^4; treating them using linear combinations goes back to Laguerre in 1867, who also defined systems of linear equations.

In 1857, Cayley introduced the matrix notation which allows for a harmonization and simplification of linear maps. Around the same time, Grassmann studied the barycentric calculus initiated by Mbius. He envisaged sets of abstract objects endowed with operations, Grassman(2000). In his work, the concepts of linear independence and dimension, as well as scalar products are present. Actually Grassmann's 1844 work exceeds the framework of vector spaces, since his considering multiplication, too, led him to what are today called algebras. Italian mathematician Peano was the first to give the modern definition of vector spaces and linear maps in 1888, Peano(1888).

An important development of vector spaces is due to the construction of function spaces by Henri Lebesgue. This was later formalized by Banach and Hilbert, around 1920, Banach(1922). At that time, algebra and the new field of functional analysis began to interact, notably with key concepts such as spaces of p-integrable functions and Hilbert spaces, Dorier(1995) and Moore(1995). Also at this time, the first studies concerning infinite-dimensional vector spaces were done.

Molodtsov(1999) defined soft sets over the universe U as a parameterized family of subsets of U. Sezgin and Atagun(2016) building on the work of Molodtsov(1999) and Molodtsov et al(2006), defined a soft Vector space as the soft set (F,A) over a Vector space with some conditions. Omid et al(2021) defined and studied the composition Vector spaces as a type of tri-operational algebra. Marat(2021) provided a characterization of finite dimensionality of Vector spaces in terms of the right sided invertibility of linear operators on them.

In this work, we wish to test that the algebraic structure $G_p^{'}$ is a Vector space. We select an appropriate field which is the Galois field F_p, where p is prime and find the Bases which we proved as consisting of the additive and multiplicative identities of $G_p^{'}$. We show that all elements of $G_p^{'}$ can be written as a finite linear combinations of the elements of the Bases. The Dimension of $G_p^{'}$ was taken as the cardinality of the Bases. We determined the Subspace by showing that it satisfies all the requirements of a Vector space. Finally, we established that every vector is linearly dependent by showing that the sum of all vectors gives us the zero vector.

Main Results

Proposition 2.1

Let F_p be a Galois group of order p, where p is prime. Then $G_p^{'}$ is a vector space over $F_p = Z/pZ = \{0,1,2,...,p-1\}$.

Proof.

We need to show that $\alpha w_i \to w_{(\alpha i) \bmod p} \in G_p^{'}$ is closed with respect to vector addition and scalar multiplication.

Closure of vector addition:

$$+ : G_p^{'} \times G_p^{'} \to G_p^{'}$$

such that

$$+(w_i, w_j) \to w_{(i+j) \bmod p} \in G_p^{'}$$

Closure of scalar multiplication:

$$\times : F_p \times G_p' \to G_p'$$

such that $\forall w_i \in G_p'$ and $\forall\ \alpha \in F_p$

$$\alpha w_i \to w_{(\alpha i) \bmod p} \in G_p'$$

Proposition 2.2

There exists a subspace $\{\ w_p\ \}$ of G_p'.

Proof.

$\{\ w_p\ \}$ is closed with respect to vector addition and scalar multiplication by elements of F_p.

Closure of vector addition: $w_p + w_p = w_{(2p) \bmod p} = w_p$

Closure of scalar multiplication: $\forall\ \alpha \in F_p$, $\alpha\ w_p = w_{(\alpha p) mod p} = w_p$.

Proposition 2.3

$\{w_1, w_p\}$ is the basis
for G_p'.

Proof.

For all $w_i \in G_p'$, $w_i = \alpha\ w_i + \beta\ w_p = w_\alpha + w_{\beta p} = w_{(\alpha+\beta p) mod p} = w_\alpha$ where $i=\alpha$ and $\beta \in F_p$

For example, consider the cycle $w_3 \in G_7'$ $w_3 = 3w_1 + \beta w_7\ \forall\ \beta \in F_p$

The Dimension of G_p' is 2 because the basis $\{w_1,w_p\}$ has cardinality 2.

Proposition 2.4

Span $\{G_p'\} = G_p'$.

Proof.

The span of G_p' is the set of all linear combinations of all the vectors in G_p' of the form $\alpha\, w_1 + \beta\, w_p$ where α and $\beta \in F_p$.

Every element in G_p' can be written in the form $\alpha\, w_1 + \beta\, w_p$ where $\alpha, \beta \in F_p$.

Therefore G_p' is a Span over itself.

Proposition 2.5

With m >p any m vectors from Span($w_1,\ldots,w_p$) are linearly dependent.

Proof.

One can assume that m = p + 1 and then prove the lemma by induction on p. Let p = 1. Take two vectors w_j and w_k from Span(w_i). Then for some $\alpha, \beta \in F_p$ we have

$$w_j = \alpha w_i, \;\; w_k = \beta w_i$$

If $\alpha = 0$, then $w_j = 0$. Hence w_j and w_k are linearly dependent. If $\alpha = 0$, then

$$w_k = \beta w_i = (\beta w_j \alpha^{-1}) = w_j\,(\beta \alpha^{-1})$$

Hence, w_j and w_k are linearly dependent again. Now suppose that the lemma is true for some p = m - 1 (m >1) and prove it for p = m. Let w_1, w_2,. . . , w_{m+1} be some m + 1 vectors from Span($w_1,\ldots,w_m$) say,

$$w_r = \sum_{s=1}^{m} w_s \alpha_{sr}\,, \qquad \text{k=1,...,m+1}$$

If $\alpha_{1,r} = 0$ for all r, then $w_2,\ldots,w_m$ belong to the Span($w_2,\ldots,w_m$). By induction, they are linearly dependent. A fortiori, $w_1,\ldots,w_m,w_{m+1}$ are linearly dependent.

Proposition 2.6

Every vector in G_p' is linearly dependent.

Proof.

We sum all the elements of G_p' as follows:

$$(w_1 + w_{p-1}) + (w_2 + w_{p-2}) + \ldots + (w_{(p-1)/2} + w_{(p+1)/2}) + w_p = w_p$$

.

Hence all the elements of G_p' are linearly dependent.

Conclusion

We have seen that the abelian group G_p' is a Vector space because it satisfies all the axioms of a Vector space. We have shown the bases, dimension, span, and subspace. We have shown that the vectors are all linearly dependent.

Conflict of Interest

The authors declare no conflict of interest.

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