A Robust Block Unification of Multi-Step Methods for the Solution of Boundary Layer Flow

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Abstract
A numerical method is proposed in this paper to directly solve third order boundary value problems that could be coupled with second order boundary value problem(s). The Modified Boundary Value Methods (MBVMs) with continuous coefficients are applied. These are referred to as Block unification multi-step methods (BUMMs) which are derived and used to obtain methods applied through the block unification approach. The computed results are compared with some numerical results to show efficiency and accuracy advantages.
Keywords: *Block Unification Method, Boundary Value Problems, Modified Boundary Value Method*

1. Introduction

Mathematical models developed in science, engineering and technology help understand physical phenomena in these fields. These models are expressed in equations in which a function and its derivatives play significant roles. These equations arise not only in fields like physical science but also in fields like operation research, psychology, medicine, economics, engineering, etc., ranging from models that describe neural works to the deflection of a curved beam that has a constant or varying cross section and as a result faster and more accurate numerical methods are required since most of them defy analytical solutions (Jikantoro et al., 2018).

Steady flow of viscous incompressible fluids has attracted considerable attention in recent years due to its crucial role in numerous engineering applications. Numerical analysts encounter actually a wide variety of challenges in obtaining suitable algorithms for computing flow and heat transfer of viscous fluids (Bataller, 2010). The most common approach for problems in unbounded domains is to apply polynomials that are orthogonal over unbounded domains, other direct method is based on rational approximation (Parand et al., 2011).

Boundary layer flow problems in ordinary differential equations have been discussed in many papers in recent years. The paper of (Akdi and Sedra, 2014) combined the standard adomian decomposition method and a finite difference scheme, while taking note of their respective advantages and disadvantages, to solve the Blasius problem. This way the coupled method offset the limitations of the individual methods.

Collocation approximation was applied in deriving schemes that were applied as a block method to solve special third order initial value problems in (Olabode, 2009). Other researches include the works of Abdullah et al. (2013:2013) who had developed a fifth order block method using constant step size with shooting technique to solve third order non-linear boundary value problems and developed a fourth order two-point block method for solving non-linear third order boundary value problems respectively. A continuous linear multistep method was used in Jator (2008) to generate multiple finite difference methods that were assembled into a single block matrix that was used to solve third order BVPs. Multiple Finite Difference Methods from a linear multistep method of step 4 obtained in Jator (2009) were used to solve third order boundary value problems directly. A family of three step hybrid methods independent of first and second derivative components using Taylor approach were proposed to solve special third order ODEs in Jikantoro et al. (2018) directly. In the work of Ahmed (2017), the variational iteration method was used to get numerical solutions to third order ordinary boundary value problems after reducing them to a system of first order ODEs. The paper of Bhatti et al. (2018) solved the resulting coupled differential equations from stagnation point flow over a permeable shrinking sheet by applying successive linearisation method and spectral collocation method.

Boundary Value Methods in Block Unification Approach have been considered recently in the solution of ordinary differential equations. These methods are developed from the linear multistep method in which a main method is developed and additional methods are derived from the main method. Boundary value methods applied as block unification method to solve second order boundary value problems were implemented in the work of Biala and Jator (2017).

2. Derivation of Method

In this section, the construction of the block unification multistep method through the interpolation and collocation approach is discussed. This method will be used to produce several discrete schemes for solving third order ordinary differential equations.

The starting point is to construct the modified boundary value method (MBVM), for third order ordinary differential equations, which has the form

$$U(x)=\alpha_v(x)y_{n+v}+\alpha_{v-1}(x)y_{n+v-1}+\alpha_0(x)y_n+h^3\sum_{j=0}^{k}\beta_j(x)f_{n+j}+h^3\beta_w(x)f_{n+w}, \tag{1}$$

Where $v=\begin{cases}\dfrac{k}{2} & for\ even\, k\\ \dfrac{k+1}{2} & for\ odd\ k\end{cases}$

$\alpha_0(x),\alpha_{v-1}(x),\alpha_v,\beta_j,\beta_w$ are continuous coefficients and v is chosen to be half the step number so that the formula derived from (1) satisfies the root condition.

The main and additional methods are then obtained by evaluating (1) at x_{n+j} where

$j=1(1)2v,\ j\neq v-1,v$ to obtain the formula of the following form:

$$y_{n+j}+\alpha_v y_{n+v}+\alpha_{v-1}y_{n+v-1}+\alpha_0 y_n=h^3\sum_{i=o}^{k}\beta_i f_{n+i}+h^3\beta_w f_{n+w} \tag{2}$$

The approach above is applied to construct another block unification multi-step method (BUMM), for second order ordinary differential equations, which has the form

$$U(x)=\alpha_v(x)s_{n+v}+\alpha_{v-1}(x)s_{n+v-1}+\alpha_0(x)s_n+h^2\sum_{j=0}^{k}\beta_j(x)m_{n+j}+h^2\beta_w(x)m_{n+w}, \tag{3}$$

The first and second derivative formulas for (1) are used to generate additional methods by evaluating $U'(x)$ and $U''(x)$ at $x_{n+j}, j=0(1)k$ so also is the first derivative formula for (3) used to generate its additional methods. The construction of (1) is discussed in the following theorem.

Theorem 2.1 *Let* $T_j(x),\ j=0(1)(k+4)$ *be the Chebyshev Polynomial used as basis function and* W *a vector given by* $W=(y_n,y_{n+v-1},y_{n+v},f_n,f_{n+1},...,f_k)^T$ *where* T *is the transpose. Consider the matrix V defined as*

$$V=\begin{pmatrix} T_0(x_n) & T_1(x_n) & \dots & T_{k+4}(x_n) \\ T_0(x_{n+v-1}) & T_1(x_{n+v-1}) & \dots & T_{k+4}(x_{n+v-1}) \\ T_0(x_{n+v}) & T_1(x_{n+v}) & \dots & T_{k+4}(x_{n+v}) \\ T_0'''(x_n) & T_1'''(x_n) & \dots & T_{k+4}'''(x_n) \\ T_0'''(x_{n+1}) & T_1'''(x_{n+1}) & \dots & T_{k+4}'''(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots \\ T_0'''(x_{n+k}) & T_1'''(x_{n+k}) & \dots & T_{k+4}'''(x_{n+k}) \end{pmatrix}$$

and obtained by replacing the jth column of V by the vector W and let (2) satisfy

$$U(x_{n+j})=y_{n+j} \qquad j=0,v-1,v \ \text{ and } \ j=0,v-2,v-1,v$$

$$U'''(x_{n+j})=f_{n+j} \qquad j=0(1)k \tag{4}$$

then the continuous representation (1) is equivalent to

$$U(x)=\sum_{j=0}^{k+4}\frac{\det(V_j)}{\det(V)}T_j(x) \tag{5}$$

3. **Numerical Method: Block Unification Multistep Method**

To derive an implicit three step method for the third order ordinary differential equations with one off-grid point, the following specifications were considered, r = 3, s= 5, k = 3, v $=\frac{7}{3}$, to give the continuous form as:

$$y(x)=\alpha_0 y_n+\alpha_1 y_{n+1}+\alpha_2 y_{n+2}+h^3[\beta_0 f_n+\beta_1 f_{n+1}+\beta_2 f_{n+2}+\beta_{\frac{7}{3}} f_{n+\frac{7}{3}}+\beta_3 f_{n+3}] \tag{6}$$

Evaluating equation (6) at points $x=x_{n+3}, x=x_{n+\frac{7}{3}}$ gives

$$y_{n+3}=y_n-3y_{n+1}+3y_{n+2}+\frac{1}{140}h^3 f_n+\frac{37}{80}h^3 f_{n+1}+\frac{13}{20}h^3 f_{n+2}-\frac{81}{560}h^3 f_{n+\frac{7}{3}}+\frac{1}{40}h^3 f_{n+3}$$

$$y_{n+\frac{7}{3}} = \frac{2}{9}y_n - \frac{7}{9}y_{n+1} + \frac{14}{9}y_{n+2} + \frac{137}{76545}h^3 f_n + \frac{2911}{29160}h^3 f_{n+1} + \frac{139}{1215}h^3 f_{n+2} - \frac{1067}{22680}h^3 f_{n+\frac{7}{3}} + \frac{169}{43740}h^3 f_{n+3} \qquad (7)$$

For $n = 0(3)(N-3)$

The first derivative formulae are

$$hy'_n = -\frac{3}{2}y_n + 2y_{n+1} - \frac{1}{2}y_{n+2} + \frac{167}{2940}h^3 f_n + \frac{577}{1680}h^3 f_{n+1} - \frac{101}{420}h^3 f_{n+2} + \frac{793}{3920}h^3 f_{n+\frac{7}{3}} - \frac{11}{420}h^3 f_{n+3}$$

$$hy'_{n+1} = -\frac{1}{2}y_n + \frac{1}{2}y_{n+2} - \frac{11}{1470}h^3 f_n - \frac{173}{1120}h^3 f_{n+1} + \frac{1}{105}h^3 f_{n+2} - \frac{27}{1568}h^3 f_{n+\frac{7}{3}} + \frac{1}{336}h^3 f_{n+3}$$

$$hy'_{n+2} = \frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} + \frac{13}{2940}h^3 f_n + \frac{367}{1680}h^3 f_{n+1} + \frac{27}{140}h^3 f_{n+2} - \frac{351}{3920}h^3 f_{n+\frac{7}{3}} + \frac{1}{140}h^3 f_{n+3}$$

$$hy'_{n+\frac{7}{3}} = \frac{5}{6}y_n - \frac{8}{3}y_{n+1} + \frac{11}{6}y_{n+2} + \frac{680}{107163}h^3 f_n + \frac{310459}{816480}h^3 f_{n+1} + \frac{25679}{51030}h^3 f_{n+2} - \frac{38813}{211680}h^3 f_{n+\frac{7}{3}} + \frac{3865}{244944}h^3 f_{n+3}$$

$$hy'_{n+3} = \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} + \frac{9}{980}h^3 f_n + \frac{2393}{3360}h^3 f_{n+1} + \frac{89}{84}h^3 f_{n+2} - \frac{27}{1568}h^3 f_{n+\frac{7}{3}} + \frac{39}{560}h^3 f_{n+3} \qquad (8)$$

for $n = 0(3)(N-3)$

And the second derivative formulae are

$$h^2 y''_n = y_n - 2y_{n+1} + y_{n+2} - \frac{389}{1260}h^3 f_n - \frac{227}{240}h^3 f_{n=1} + \frac{53}{60}h^3 f_{n+2} - \frac{81}{112}h^3 f_{n+\frac{7}{3}} + \frac{17}{180}h^3 f_{n+3}$$

$$h^2 y''_{n+1} = y_n - 2y_{n+1} + y_{n+2} + \frac{53}{2520}h^3 f_n + \frac{1}{12}h^3 f_{n=1} - \frac{11}{40}h^3 f_{n+2} + \frac{27}{140}h^3 f_{n+\frac{7}{3}} - \frac{1}{45}h^3 f_{n+3}$$

$$h^2 y''_{n+2} = y_n - 2y_{n+1} + y_{n+2} + \frac{1}{180}h^3 f_n + \frac{39}{80}h^3 f_{n=1} + \frac{49}{60}h^3 f_{n+2} - \frac{27}{80}h^3 f_{n+\frac{7}{3}} + \frac{1}{36}h^3 f_{n+3}$$

$$h^2 y''_{n+\frac{7}{3}} = y_n - 2y_{n+1} + y_{n+2} + \frac{407}{68040}h^3 f_n + \frac{29}{60}h^3 f_{n=1} + \frac{641}{648}h^3 f_{n+2} - \frac{71}{420}h^3 f_{n+\frac{7}{3}} + \frac{29}{1215}h^3 f_{n+3}$$

$$h^2 y''_{n+3} = y_n - 2y_{n+1} + y_{n+2} + \frac{1}{540}h^3 f_n + \frac{31}{60}h^3 f_{n=1} + \frac{79}{120}h^3 f_{n+2} + \frac{81}{140}h^3 f_{n+\frac{7}{3}} + \frac{11}{45}h^3 f_{n+3}$$

(9)

for $n = 0(3)(N-3)$

To derive an implicit three step method for the second order ordinary differential equations with one off-grid point, the following specifications were considered, r = 3, s= 5, k = 3, v $= \frac{7}{3}$, to give the continuous form as:

$$d(x) = \alpha_0 d_n + \alpha_1 d_{n+1} + \alpha_2 d_{n+2} + h^2[\beta_0 g_n + \beta_1 g_{n+1} + \beta_2 g_{n+2} + \beta_{\frac{7}{3}} g_{n+\frac{7}{3}} + \beta_3 g_{n+3}] \quad (10)$$

Evaluating equation (10) at points $x = x_{n+3}, x = x_{n+\frac{7}{3}}, x = x_n$ gives

$$d_{n+3} = -d_{n+1} + 2d_{n+2} - \frac{1}{140}h^2 g_n + \frac{29}{240}h^2 g_{n+1} + \frac{41}{60}h^2 g_{n+2} - \frac{81}{560}h^2 g_{n+\frac{7}{3}} + \frac{7}{120}h^2 g_{n+3}$$

$$d_{n+\frac{7}{3}} = -\frac{1}{3}d_{n+1} + \frac{4}{3}d_{n+2} - \frac{313}{153090}h^2 g_n + \frac{1093}{29160}h^2 g_{n+1} + \frac{1817}{7290}h^2 g_{n+2} - \frac{527}{7560}h^2 g_{n+\frac{7}{3}} + \frac{317}{43740}h^2 g_{n+3}$$

$$d_n = 2d_{n+1} - d_{n+2} + \frac{8}{105}h^2 g_n + \frac{209}{240}h^2 g_{n+1} - \frac{1}{15}h^2 g_{n+2} + \frac{81}{560}h^2 g_{n+\frac{7}{3}} - \frac{1}{40}h^2 g_{n+3}$$

(11)

For $n = 0(3)(N-3)$

The first derivative formulae are

$$hd_n = -d_{n+1} + d_{n+2} - \frac{101}{315}h^2 g_n - \frac{13293}{10080}h^2 g_{n+1} + \frac{441}{630}h^2 g_{n+2} - \frac{729}{1120}h^2 g_{n+\frac{7}{3}} + \frac{91}{1008}h^2 g_{n+3}$$

$$hd_{n+1} = -d_{n+1} + d_{n+2} + \frac{23}{2520}h^2 g_n - \frac{973}{3360}h^2 g_{n+1} - \frac{11}{24}h^2 g_{n+2} + \frac{297}{1120}h^2 g_{n+\frac{7}{3}} - \frac{19}{720}h^2 g_{n+3}$$

$$hd_{n+2} = -d_{n+1} + d_{n+2} - \frac{2}{315}h^2 g_n + \frac{11}{96}h^2 g_{n+1} + \frac{399}{630}h^2 g_{n+2} - \frac{297}{1120}h^2 g_{n+\frac{7}{3}} + \frac{19}{720}h^2 g_{n+3}$$

$$hd_{n+\frac{7}{3}} = -d_{n+1} + d_{n+2} - \frac{403}{68040}h^2 g_n + \frac{371}{3360}h^2 g_{n+1} + \frac{2611}{3240}h^2 g_{n+2} - \frac{1755}{18144}h^2 g_{n+\frac{7}{3}} + \frac{383}{19440}h^2 g_{n+3}$$

$$hd_{n+3} = -d_{n+1} + d_{n+2} - \frac{10}{1008}h^2 g_n + \frac{161}{1120}h^2 g_{n+1} + \frac{133}{280}h^2 g_{n+2} + \frac{729}{1120}h^2 g_{n+\frac{7}{3}} + \frac{1211}{5040}h^2 g_{n+3}$$

(12)

for $n = 0(3)(N-3)$

4. Analysis of Basic Properties

Theorem 4.1: Let (y_i, y_i', y_i'') be an approximation to the solution vector $(y(x_i), y'(x_i), y''(x_i))$ for the third order ordinary equations from boundary layer flow. If $e_i = |y(x_i) - y_i|, e_i' = |y'(x_i) - y_i'|, e_i'' = |y''(x_i) - y_i''|$, where the exact solution given by the vector $(y(x), y'(x), y''(x))$ is several times differentiable and if $\|E\| = \|Y - \bar{Y}\|$, then the BVMs are said to be convergent of order $k+3$ which implies that $\|E\| = O(h^{k+3})$, where k is the step number.

Proof: Consider the exact form of the system formed from equations (7) to (9) given by

$$PY - h^3 QF(Y) + C + L(h) = 0 \tag{13}$$

where $L(h)$ is the truncation error vector obtained from the formulae (7) to (9). The approximate

form of the system is given by

$$P\bar{Y} - h^3 QF(\bar{Y}) + C = 0 \tag{14}$$

where $\bar{Y}$ is the approximate solution of vector Y.

Using the mean value theorem after subtracting (13) from (14) and letting $E = |\bar{Y} - Y| = (e_1, \ldots e_N, e'_1, \ldots e'_N, e''_1, \ldots e''_N)^T$, we get the error system

$$(P - h^3 QB)E = L(h) \tag{15}$$

where B is the Jacobian matrix and its entries $B_{rs}, r, s = 1,2,3,$ are defined as

$$B_{rs} = \begin{pmatrix} \frac{\partial f_1^{(r-1)}}{\partial y_1^{(s-1)}} & \cdots & \frac{\partial f_1^{(r-1)}}{\partial f_N^{(s-1)}} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_N^{(r-1)}}{\partial f_1^{(s-1)}} & \cdots & \frac{\partial f_N^{(r-1)}}{\partial f_N^{(s-1)}} \end{pmatrix}$$

From (15) and $L(h)$

$$E = (P - h^3 QB)^{-1} L(h)$$

$$E = SL(h)$$

$$\|E\| = \|SL(h)\|$$

$$= O(h^{-3})O(h^{k+6})$$

$$= O(h^{k+3})$$

Which show that the methods are convergent and the global errors are of order $O(h^{k+3})$

5. Numerical Examples and Results Discussion

Here, numerical examples such as Blasius equation and Falkner-Skan equation are considered. Solutions in tables 1 and 2 were compared with solutions using Runge-Kutta method. Tables 3 and 4 have solutions being validated with solutions in other papers.

Problem 1: Blasius Equation

$$2y''' + yy'' = 0$$

$$y(0) = 0, y'(0) = 0, y'(\infty) = 1$$

Table 1: Comparison of the Solutions from Proposed Methods and Runge-Kutta Method

		Proposed Method			Runge-Kutta		
x	N	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$	$y''(0)$	$y(x_\infty)$	$y''(x_\infty)$
1.0	9	1.021157329	0.5063049940	0.9381906626	1.021157016	0.506305291	0.93810
2.0	17	0.5442717691	1.051664551	0.3810337080	0.5442717609	1.051664633	0.38103
3.0	25	0.4045496973	1.679698960	0.1689551177	0.4045497078	1.6796990467	0.16895
4.0	33	0.3527462516	2.432249676	0.06202511200	0.3527462779	2.432249926	0.06202
5.0	41	0.33256595103	3.3170985421	0.0155692563	0.3325659529	3.3170985488	0.01556

From Table 1 above, a comparison of the proposed method and Runge-Kutta shows a good performance of the method.

Problem 2: Falkner-Skan Equation

$$y'''(\eta) + \beta_0 y(\eta) y''(\eta) + \beta\left(1 - y'(\eta)^2\right) = 0$$

$$f(0) = 0, y'(0) = 0, \lim_{\eta \to \infty} y'(\eta) = 1$$

Table 2: Comparison of the Errors from Proposed Methods and Runge-Kutta Method

		Proposed Method		Runge-Kutta Method		
x	N	$y''(x_\infty)$	$y(x_\infty)$	N	$y''(x_\infty)$	$y(x_\infty)$
0.1	9	0.5223955323	0.6065298823	27	0.522394253	0.606530550
0.2	17	0.03825982349	1.510386946	51	0.0382595394	1.510388234
0.3	25	0.0014085063	2.502848721	75	0.0014082032	2.502849911
0.4	33	0.0000245898	3.502571462	99	0.0000245779	3.502571249

The proposed method has a good performance compared with the existing Runge-Kutta method. This is shown in the table 2 above.

Problem 3

$$y''' - y'^2 + 1 + yy' - M(y'-1) = 0$$

$$d'' + yd' - y'd - Md = 0$$

$$y(0) = k, y'(0) = \alpha, y'(\infty) = 1$$

$$\mathrm{d}(0) = 1, \mathrm{d}(\infty) = 0$$

Table 3: Numerical comparison for the stretching case $(\alpha \succ 0)$ *with the existing results for M = k = 0* .

$\alpha \succ 0$	$f''(0)$	$f''(0)$	$-d'(0)$	$-d'(0)$
	[14]	Proposed Method	[14]	Proposed Method
0	1.23258765	1.232583905	0.81130132	0.8113170417
0.1	1.14656100	1.146557577	0.863451660	0.8634652926
0.2	1.051129994	1.051127244	0.91330283	0.9133157941
0.3	0.94681611	0.9468142651	0.96111587	0.30112933847
0.5	0.71329495	0.7132950814	1.05145843	1.051476251
1	0	0	1.25331413	1.253359472
2	-1.88730667	-1.887402684	1.58956678	1.589740624
3	-10.26474931	-10.26844767	2.33809899	2.3399380450

Table 4: Numerical comparison for the shrinking case $(\alpha \prec 0)$ *with the existing results for M = k = 0.*

$\alpha \prec 0$	$f''(0)$	$f''(0)$	$-d'(0)$	$-d'(0)$
	Bhatti (2018)	Proposed Method	Bhatti(2018)	Proposed Method
-0.25	1.40224081	1.402238699	0.66857275	0.6686022783
-0.5	1.49566976	1.495675888	0.50144758	0.5015139670
-0.75	1.48929824	1.489330566	0.29376251	0.2939313809
-1.0	1.32881688	1.328961913	0	0
-1.15	1.08223117	1.082786939	-0.29799548	-0.2961961037
-1.2465	0.58428167	0.6173065669	-0.94776590	-0.8869752488
-1.2474		0.5741833003		-0.9561670930

In tables 3 and 4,which show the numerical comparison for Hartmann number, M, and suction/injection parameter, k, for different values of stretching and shrinking parameter, it can be observed that when $M = k = 0$ for both cases of α , the results from the proposed method are in good agreement with existing literature.

CONCLUSION

In this paper, BUMMs have been proposed using the boundary value technique to solve boundary layer flow problems in ordinary differential equations. This has been done by applying the method directly to the differential equations. The efficiency of the methods was given in the Tables 1 and 2. In the two tables, the accuracy of the results can be comparable as the proposed methods have a good performance in comparison to the Runge-Kutta method. The methods were also applied to problem 3 and results were validated by comparing results with those from other literature. The results have a good agreement with the ones found in literature that comparison was made with.

REFERENCES

Abdullah, A. S., Majid, Z. A. and Senu, N. (2013). Solving Third Order Boundary Value Problem with Fifth Order Block Method. *Mathematical Methods in Engineering and Economics,* ISBN: 978-1-61804-230-9

Abdullah, A. S., Majid, Z. A. and Senu, N. (2013). Solving Third Order Boundary Value Problem Using Fourth Order Block Method. *Applied Mathematical Sciences,* Vol. 7(53), 2629-2645

Ahmed, J. (2017). Numerical Solutions of Third-Order Boundary Value Problems Associated with Draining and Coating Flows. *Kyungpook Mathematical Journal*, Vol. 57,651-665

Akdi, M. and Sedra, M. B. (2014). Numerical Solution of the Blasius Problem. *The African Review of Physics*, 9(2014):0022

Bataller, R. C. (2010). Numerical Comparisons of Blasius and Sakiadis Flows. *MATEMATIKA*, 26(2), 187-196

Bhatti, M. M., Ali Abbas, M. & Rashidi, M. M. (2018). A robust numerical method for solving stagnation point flow over a permeable shrinking sheet under the influence of MHD. *Applied Computation and Mathematics,* 316(2018): 381-389

Biala, T. A. & Jator, S. N. (2017). A Family of Boundary Value Methods for Systems of Second Order Boundary Value Problems. *International Journal of Differential Equations*, Hindawi Publishing Corporation

Jator, S. N. (2008). On the Numerical Integration of Third Order Boundary Value Problems by a Linear Multistep Method. *International Journal of Pure and Applied Mathematics,* Vol. 46(3), 375-388

Jator, S. N. (2009). Novel Finite Difference Schemes for Third Order Boundary Value Problems. *International Journal of Pure and Applied Mathematics*, 53(1), 37-54

Jator, S., Okunlola, T., Biala, T. & Adeniyi, R. (2018). Direct Integrators for the General Third-Order Ordinary Differential Equations with an Application to the Korteweg–de Vries Equation. *Int. J. Appl. Comput. Math*, 4(110)

Jikantoro, Y. D., Ismail, F., Senu, N. & Ibrahim, Z. B. (2018). A New Integrator for Special Third Order Differential Equations with Application to Thin Film Flow Problem. *Indian J. Pure Appl. Math.,* Vol. 49(1), 151-167

Olabode, B. T. (2009). An Accurate Scheme by Block Method for Third Order Ordinary Differential Equations. *The Pacific Journal of Science and Technology,* Vol. 10(1), 136-142

Parand, K., Pakniat, N. & Delafkar, Z. (2011). Numerical Solution of the Falkner-Skan Equation with Stretching Boundary by Collocation Method. *International Journal of Nonlinear Science,* International Journal of Nonlinear Science, Vol. 11(3), 275-283