AN IMPROVED SIXTH ORDER MULTI-DERIVATIVE BLOCK METHOD USING LEGENDRE POLYNOMIAL FOR THE SOLUTION OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract
This study focuses on derivation of uniform order six multi-derivative hybrid methods at $k = 4$ by Legendre series as our bases function for the solution of General Third Order Ordinary Differential Equations. All the discrete schemes used to form our block method were gotten from a single continuous formulation of the method. The results from the two test problems used shows that the proposed method is almost as the classical solutions.
Keywords :(Multi-derivative, Legendre polynomial, Continuous formulation, third order ODEs)

Introduction

The third-order order ordinary differential equations which of the form

$$y''' = f(x, y, y', y''), \qquad y(a) = y_0, \; y'(a) = \delta_0, \; y''(a) = \alpha_0 \qquad (1)$$

where f.is a a continuous differentiable functions is conventionally solved by first reducing it to its equivalent system of first order differential equations and then applying various methods of first order methods to solve them. These approaches are extensively discussed in the literature just to mention a few such as ([1],[2],[3],[4],[5],and [7]). Although there has been tremendous success with this approach, it has some draw backs. For instance due to dimension of the problem after it has been reduced to a system of first order equations, the approach waste a lot of Computer time and human efforts, also the problems may not be posed after its reduction. Hence, there is need to develop a high order method to handle this class of problem (1) directly.

Several authors has proposed various methods of solve

$$y''' = f(x, y), \qquad y(a) = y_0, \; y'(a) = \delta_0, \; y''(a) = \alpha_0 \qquad (2)$$

directly in literature. Among them are ([2],[6], and [8]) to mention a few.

Recently, ([2]) proposed some multi-derivative hybrid block methods of order 5 for direct solution (1).

In this paper, we developed a uniform order six Block methods through the interpolation and collocation approach, the Legendre polynomial was used as the bases of our method. The single continuous formulation proposed from this method is of the form

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} = h^3\Big[\beta_0 f_n + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} + \beta_4 f_{n+4}\Big] \quad (3)$$

All the discrete schemes used in this our new method is evaluated from (3) and its derivatives at various grid and off grid points which are of uniform order Six.

The paper is organized as follows. In section 2, we discussed Legendre series which is our bases for this method. Section 3 is devoted to specification of our method. Analysis and implementation of the method are given in section 4. Section 5 is Numerical experiments to ascertain the efficiency of this method. Finally section 6 is the summary and conclusion of this paper.

Definition 1.0: Zero Stable

The method in (3) is said to be zero stable if no root of the first characteristic polynomial

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j$$

has modulus greater than one and if every root with modulus one is simple

1.0 Development of the method

We seek a polynomial of the form

$$y(x) = \sum_{n=0}^{m+t-1} a_n p_n(x) \quad , n = 0,1,2, \ldots. \quad (4)$$

The first, second and third derivatives of (4) are obtained as

$$y'(x) = \sum_{n=1}^{m+t-1} a_n p'_n(x) \quad (5)$$

$$y''(x) = \sum_{n=2}^{m+t-1} a_n p''_n(x) \quad (6)$$

$$y'''(x) = \sum_{n=3}^{m+t-1} a_n p'''_n(x)$$
$$= f(x, y, y', y'') \qquad (7)$$

Hence, our proposed linear multi-step method for (1) directly is

$$y(x) = \sum_{n=0}^{m+t-1} \alpha_n(x) y_{n+j}$$
$$= h^3 \sum_{n=0}^{m+t-1} \beta_j(x) f_{n+j} \qquad (8)$$

where m and t are points of interpolation and collocation chosen, $h = x_{n+1} - x_n$ and both $\alpha_j(x)$ and $\beta_j(x)$ are the parameters to be determined. Also $(m + t - 1)$ is the range of the series used in the method.

From $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$ which is Rodrigue formula .The Legendre series are:.

$$P_0(x) = 1, \quad P_1(x) = x \ , \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \ P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

$$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$

Proposition 1.0

The Legendre polynomials satisfy the following

$(2n - 1)\ P_n(x) = \ P'_{n+1}(x) - \ P'_{n-1}(x)$

Proof

From Rodrigues' formula we have

$$P'_k(x) = \frac{d}{dx}\left(\frac{1}{2^k k!} \frac{d^k}{dx^k}[(x^2 - 1)^k]\right) = \frac{2k}{2^k k!}\frac{d^k}{dx^k}[x(x^2 - 1)^{k-1}]$$

$$= \frac{1}{2^{k-1}(k - 1)!}\frac{d^{k-1}}{dx^{k-1}}[((2k - 1)x^2 - 1)\ (x^2 - 1)^{k-2}]$$

For $k = n + 1$ we get

$$P'_{n+1}(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{[(2(n+1)-1)\ x^2-1)-1)\ ](x^2-1)^{n-1}\}$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}\{[(2n+1)\ x^2-1)\ ](x^2-1)^{n-1}\} \qquad (i)$$

Also from Rodrigues' formula at $k = (n-1)$ we have

$$P'_{n-1}(x) = \frac{d}{dx}\left(\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)^{n-1}]\right)$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}[(x^2-1)^{n-1}]$$

$$= \frac{n}{2^n 2^{-1} n!}\frac{d^n}{dx^n}[(x^2-1)^{n-1}]$$

$$= \frac{2n}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^{n-1}] \qquad (ii)$$

Subtracting equation (ii) from (i), i.e. $P'_{n+1}(x) - P'_{n-1}(x)$, we have

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}\{[(2n+1)\ x^2-1)\ ](x^2-1)^{n-1}\} - \frac{2n}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^{n-1}]$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^{n-1}\{[(2n+1)\ x^2-1)] - 2n\ \}$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^{n-1}(2n+1)\ (x^2-1)$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n(2n+1)$$

$$= \frac{2n+1}{2^n n!}\ \frac{d^n}{dx^n}[(x^2-1)^n]$$

Hence

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{2n+1}{2^n n!}\ \frac{d^n}{dx^n}[(x^2-1)^n] = (2n+1)\ P_n(x)$$

2.0 Specification of the Method

In this proposed method the step length $k = 4, m = 6$ and $t = 3$. We interpolate (4) at $x = x_{n+j}$, $j = 0,1,2$ and also collocate (7) at $= x_{n+j}$, $j = 1, \frac{3}{2}$, $2, \frac{5}{2}$, 3,and 4 to obtain the following non linear system of equations.

$$\left.\begin{aligned} &\sum_{n=0}^{m+t-1} a_n p_n(x) = \alpha_n(x) y_{n+j}\ , j = 0,1,2 \\ &\sum_{n=3}^{m+t-1} a_n p'''_n(x) = h^3(\beta_n f_{n+j})\ , j = 0,1,\frac{3}{2},2\ ,\frac{5}{2}\ ,3,4 \end{aligned}\right\}$$

When using Maple 17 Mathematical software to determine the unknown parameters in (9) we obtain the continuous formula as

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} = h^3\Big[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_{\frac{3}{2}}(x)f_{n+\frac{3}{2}} + \beta_2(x)f_{n+2} + \beta_{\frac{5}{2}}(x)f_{n+\frac{5}{2}} + \beta_3(x)f_{n+3} + \beta_4 f_{n+4}\Big] \qquad (10)$$

where

$$\alpha_0(x) = \left[\frac{1}{6}\left(\frac{6h^2+1}{h^2}\right) - \frac{3}{2}\frac{\xi}{h} + \frac{1}{3}\,\frac{\frac{3}{2}(\xi^2) - \frac{1}{2}}{h^2}\right],$$

$$\alpha_1 = \left[-\frac{1}{3}\left(\frac{1}{h^2}\right) + \frac{2\xi}{h} - \frac{2}{3}\,\frac{\frac{3}{2}(\xi^2) - \frac{1}{2}}{h^2}\right]$$

$$\alpha_2(x) = \left[\frac{1}{6}\left(\frac{1}{h^2}\right) - \frac{1}{2}\frac{\xi}{h} + \frac{1}{3}\,\frac{\frac{3}{2}(\xi^2) - \frac{1}{2}}{h^2}\right]$$

$$\beta_1(x) = \Big[-\frac{1}{136080h^5}(156609h^6 + 48762h^4 + 2367h^2 + 10) + \frac{1}{22680h^5}(32217h^7 + 45360h^5 + 5823h^3 + 104h)(\xi) - \frac{1}{748440h^5}(1722699h^6 + 766260h^4 + 43395h^2 + 200)\left(\frac{3}{2}(\xi^2) - \frac{1}{2}\right) + \frac{1}{26730h^5}(35640h^5 + 7117h^3 + 156h)\left(\frac{5}{2}(\xi^3) - \frac{3}{2}\xi\right) - \frac{1}{270270h^5}(110682h^4$$

$+10257h^2 + 60)\left(\frac{35}{8}(\xi^4) - \frac{15}{4}(\xi^2) + \frac{3}{8}\right)$

$+ \frac{1}{110565h^5}(8411h^3 + 312h)\left(\frac{63}{8}(\xi^5) - \frac{35}{4}(\xi^3) + \right.$

$\left.\frac{15}{8}\ \xi\right) - \frac{1}{93555h^5}(789h^2 + 8)\left(\frac{231}{16}(\xi^6) - \frac{315}{16}(\xi^4) + \frac{105}{16}(\xi^2) - \frac{5}{16}\right)$

$+ \frac{16}{405405h^5}(13h)$

$\left(\frac{429}{16}(\xi^7) - \frac{693}{16}(\xi^5) + \frac{315}{16}(\xi^3) - \frac{35}{16}\ \xi\right) - \frac{16}{1216215h^5}\left(\frac{6435}{128}(\xi^8) - \frac{3003}{32}(\xi^6)\right.$

$\left.\left. + \frac{3465}{64}(\xi^4) - \frac{315}{32}(\xi^2) + \frac{35}{128}\right)\right]$

$\beta_{\frac{3}{2}}(x) = \left[\frac{1}{14175h^5}(44295h^6 + 18774h^4 + 1080h^2 + 5) - \frac{2}{4725h^5}(7935h^7 + \right.$

$15120h^5 + 2475h^3 + 50h)(\xi) + \frac{2}{31185h^5}(97449h^6 + 59004h^4 + 3960h^2 + 20)$

$\left(\frac{3}{2}(\xi^2) - \frac{1}{2}\right) - \frac{8}{22275h^5}(11880h^5 + 3025h^3 + 75h)\left(\frac{5}{2}(\xi^3) - \frac{3}{2}\xi\right)$

$+ \frac{16}{225225h^5}(21307h^4$

$+2340h^2 + 15)\left(\frac{35}{8}(\xi^4) - \frac{15}{4}(\xi^2) + \frac{3}{8}\right) - \frac{16}{36855h^5}(715h^3 + 30h)\left(\frac{63}{8}(\xi^5) - \frac{35}{4}(\xi^3)\right.$

$+$

$\left.\frac{15}{8}\ \xi\right) + \frac{64}{155925h^5}(90h^2 + 1)\left(\frac{231}{16}(\xi^6) - \frac{315}{16}(\xi^4) + \frac{105}{16}(\xi^2) - \frac{5}{16}\right) - \frac{64}{135135h^5}(5h)$

$\left(\frac{429}{16}(\xi^7) - \frac{693}{16}(\xi^5) + \frac{315}{16}(\xi^3) - \frac{35}{16}\ \xi\right) + \frac{128}{2027025h^5}\left(\frac{6435}{128}(\xi^8) - \frac{3003}{32}(\xi^6)\right.$

$\left.\left. + \frac{3465}{64}(\xi^4) - \frac{315}{32}(\xi^2) + \frac{35}{128}\right)\right]$

$\beta_2(x) = \left[- \frac{1}{115120h^5}(63474h^6 + 30051h^4 + 1971h^2 + 10)\right.$

$+ \frac{1}{2520h^5}(11094h^7 +$

$$22680h^5 + 4248h^3 + 96h)(\xi)$$
$$- \frac{1}{83160h^5}(698214h^6 + 472230h^4 + 36135h^2 + 200)$$
$$\left(\frac{3}{2}(\xi^2) - \frac{1}{2}\right) + \frac{1}{2970h^5}(17820h^5 + 5192h^3 + 144h)\left(\frac{5}{2}(\xi^3) - \frac{3}{2}\xi\right)$$
$$- \frac{1}{30030h^5}(68211h^4$$
$$+8541h^2 + 60)\left(\frac{35}{8}(\xi^4) - \frac{15}{4}(\xi^2) + \frac{3}{8}\right)$$
$$+ \frac{1}{12285h^5}(61361h^3 + 288h)\left(\frac{63}{8}(\xi^5) - \frac{35}{4}(\xi^2) +\right.$$
$$\left.\frac{15}{8}\xi\right) - \frac{1}{10395h^5}(657h^2 + 8)\left(\frac{231}{16}(\xi^6) - \frac{315}{16}(\xi^4) + \frac{105}{16}(\xi^2) - \frac{5}{16}\right) + \frac{16}{45045h^5}(12h)$$
$$\left(\frac{429}{16}(\xi^7) - \frac{693}{16}(\xi^5) + \frac{105}{16}(\xi^2) + \frac{315}{16}(\xi^3) - \frac{35}{16}\xi\right) - \frac{16}{135135h^5}\left(\frac{6435}{128}(\xi^8)\right.$$
$$- \frac{3003}{32}(\xi^6)$$
$$\left.\left. + \frac{3465}{64}(\xi^4) - \frac{315}{32}(\xi^2) + \frac{35}{128}\right)\right]$$
$$\beta_{\frac{5}{2}}(x) = \left[\frac{1}{8505h^5}(24723h^6 + 12474h^4 + 900h^2 + 5) - \frac{2}{2835h^5}(4251h^7 +\right.$$
$$9072h^5 + 1845h^3 + 46h)(\xi)$$
$$+ \frac{2}{93555h^5}(271953h^6 + 196020h^4 + 16500h^2 + 100)$$
$$\left(\frac{3}{2}(\xi^2) - \frac{1}{2}\right) - \frac{8}{13365h^5}(7128h^5 + 2255h^3 + 69h)\left(\frac{5}{2}(\xi^3) - \frac{3}{2}\xi\right)$$
$$+ \frac{16}{135135h^5}(14157h^4$$
$$+1950h^2 + 15)\left(\frac{35}{8}(\xi^4) - \frac{15}{4}(\xi^2) + \frac{3}{8}\right)$$
$$- \frac{16}{110565h^5}(2665h^3 + 138h)\left(\frac{63}{8}(\xi^5) - \frac{35}{4}(\xi^2) +\right.$$
$$\left.\frac{15}{8}\xi\right) + \frac{64}{93555h^5}(75h^2 + 1)\left(\frac{231}{16}(\xi^6) - \frac{315}{16}(\xi^4) + \frac{105}{16}(\xi^2) - \frac{5}{16}\right) - \frac{64}{405405h^5}(23h)$$

$$\left(\frac{429}{16}(\xi^7)-\frac{693}{16}(\xi^5)+\frac{105}{16}(\xi^2)+\frac{315}{16}(\xi^3)-\frac{35}{16}\ \xi\right)+\frac{128}{1216215h^5}\left(\frac{6435}{128}(\xi^8)-\frac{3003}{32}(\xi^6)+\frac{3465}{64}(\xi^4)-\frac{315}{32}(\xi^2)+\frac{35}{128}\right)\Bigg]$$

$$\beta_3(x)=\Bigg[-\frac{1}{45360h^5}(40521h^6+21294h^4+1647h^2+10)+\frac{1}{7560h^5}(6897h^7+15120h^5+3249h^3+88h)(\xi)-\frac{1}{249480h^5}(445731h^6+334620h^4+30195h^2+200)\left(\frac{3}{2}(\xi^2)-\frac{1}{2}\right)+\frac{1}{8910h^5}(11880h^5+3971h^3+132h)\left(\frac{5}{2}(\xi^3)-\frac{3}{2}\xi\right)-\frac{1}{90090h^5}(48334h^4+7137h^2+60)\left(\frac{35}{8}(\xi^4)-\frac{15}{4}(\xi^2)+\frac{3}{8}\right)+\frac{1}{36855h^5}(4693h^3+264h)\left(\frac{63}{8}(\xi^5)-\frac{35}{4}(\xi^2)+\frac{15}{8}\ \xi\right)-\frac{1}{31185h^5}(549h^2+8)\left(\frac{231}{16}(\xi^6)-\frac{315}{16}(\xi^4)+\frac{105}{16}(\xi^2)-\frac{5}{16}\right)+\frac{16}{135135h^5}(11h)\left(\frac{429}{16}(\xi^7)-\frac{693}{16}(\xi^5)+\frac{105}{16}(\xi^2)+\frac{315}{16}(\xi^3)-\frac{35}{16}\ \xi\right)-\frac{16}{405405h^5}\left(\frac{6435}{128}(\xi^8)-\frac{3003}{32}(\xi^6)+\frac{3465}{64}(\xi^4)-\frac{315}{32}(\xi^2)+\frac{35}{128}\right)\Bigg]$$

$$\beta_4(x)=\Bigg[\frac{1}{680400h^5}(29790h^6+16443h^4+1395h^2+10)-\frac{1}{113400h^5}(5010h^7+11340h^5+2610h^3+80h)(\xi)+\frac{1}{748440h^5}(65538h^6+51678h^4+5115h^2+40)$$

$$\left(\frac{3}{2}(\xi^2)-\frac{1}{2}\right)-\frac{1}{133650h^5}(8910h^5+3190h^3+120h)\left(\frac{5}{2}(\xi^3)-\frac{3}{2}\xi\right)$$
$$+\frac{1}{1351350h^5}(37323h^4$$
$$+6045h^2+60)\left(\frac{35}{8}(\xi^4)-\frac{15}{4}(\xi^2)+\frac{3}{8}\right)-\frac{1}{110565}(754h^3+48h)\left(\frac{63}{8}(\xi^5)-\frac{35}{4}(\xi^2)+\right.$$
$$\left.\frac{15}{8}\xi\right)+\frac{1}{467775h^5}(465h^2+8)\left(\frac{231}{16}(\xi^6)-\frac{315}{16}(\xi^4)+\frac{105}{16}(\xi^2)-\frac{5}{16}\right)$$
$$-\frac{16}{405405h^5}(2h)$$
$$\left(\frac{429}{16}(\xi^7)-\frac{693}{16}(\xi^5)+\frac{105}{16}(\xi^2)+\frac{315}{16}(\xi^3)-\frac{35}{16}\xi\right)+\frac{16}{6081075h^5}\left(\frac{6435}{128}(\xi^8)\right.$$
$$-\frac{3003}{32}(\xi^6)$$
$$\left.\left.+\frac{3465}{64}(\xi^4)-\frac{315}{32}(\xi^2)+\frac{35}{128}\right)\right]$$

Evaluating (10) at $= x_{n+j}$, $j = \frac{3}{2}, \frac{5}{2}$, 3, 4 , which yield the following discrete schemes to form our main Block of the method.

$$y_{n+\frac{3}{2}}+\frac{1}{8}y_n-\frac{3}{4}y_{n+1}-\frac{3}{8}y_{n+2}$$
$$=h^3\left[-\frac{7139}{92160}f_{n+1}+\frac{341}{3840}f_{n+\frac{3}{2}}-\frac{1451}{10240}f_{n+2}+\frac{1097}{11520}f_{n+\frac{5}{2}}\right.$$
$$\left.-\frac{883}{30720}f_{n+3}+\frac{127}{92160}f_{n+4}\right]$$
$$y_{n+\frac{5}{2}}-\frac{3}{8}y_n+\frac{5}{4}y_{n+1}-\frac{15}{8}y_{n+2}=h^3\left[\frac{4297}{18432}f_{n+1}-\frac{53}{256}f_{n+\frac{3}{2}}+\frac{1009}{2048}f_{n+2}-\frac{667}{2304}f_{n+\frac{5}{2}}\right.$$
$$\left.+\frac{179}{2048}f_{n+3}-\frac{77}{18432}f_{n+4}\right]$$
$$y_{n+3}-y_n+3y_{n+1}-3y_{n+2}$$
$$=h^3\left[\frac{28}{45}f_{n+1}-\frac{8}{15}f_{n+\frac{3}{2}}+\frac{7}{5}f_{n+2}-\frac{32}{45}f_{n+\frac{5}{2}}+\frac{7}{30}f_{n+3}-\frac{1}{90}f_{n+4}\right]$$
$$y_{n+4}-3y_n+8y_{n+1}-6y_{n+2}$$
$$=h^3\left[\frac{169}{90}f_{n+1}-\frac{8}{5}f_{n+\frac{3}{2}}+\frac{23}{5}f_{n+2}-\frac{88}{45}f_{n+\frac{5}{2}}+\frac{11}{10}f_{n+3}-\frac{1}{45}f_{n+4}\right]$$
(11)

Equation (11) is our main Block method, which are of uniform order 6 with Error constants of

$$\left[-\frac{9493}{6193520},\frac{28649}{6193520},\frac{149}{120960},\frac{107}{30240}\right]^T$$

The first and second derivative of (10) are evaluated at $x = x_n$ as

$$7560hy'_n + 11340y_n - 15120y_{n+1} + 3780y_{n+2}$$
$$= h^3[\,10739f_{n+1} - 25392f_{n+\frac{3}{2}} + 33282f_{n+2}$$
$$-22672f_{n+\frac{5}{2}} + 6897f_{n+3} - 334f_{n+4}]$$
$$2520h^2y''_n - 2520y_n + 5040y_{n+1} - 2520y_{n+2}$$
$$= h^3[-17401f_{n+1} + 47248f_{n+\frac{3}{2}} - 63474f_{n+2}$$
$$+43952f_{n+\frac{5}{2}} - 13507f_{n+3} + 662f_{n+4}]$$

(12)

Also the first derivative of (10) is evaluated at $x = x_{n+j}$, $j = 1, \frac{3}{2}, \frac{5}{2}, 3, 4$ as follows

$$7560hy'_{n+1} + 3780y_n - 3780y_{n+2} = h^3[-2234f_{n+1} + 3312f_{n+\frac{3}{2}} - 4473f_{n+2}$$
$$+2992f_{n+\frac{5}{2}} - 900f_{n+3} + 43f_{n+4}]$$
$$241920hy'_{n+\frac{3}{2}} + 241920y_{n+1} - 241920y_{n+2} = h^3[-449f_{n+1} - 9312f_{n+\frac{3}{2}} - 99f_{n+2}$$
$$-320f_{n+\frac{5}{2}} + 105f_{n+3} - 5f_{n+4}]$$
$$2520hy'_{n+2} - 1260y_n + 5040y_{n+1} - 3780y_{n+2} = h^3[783f_{n+1} - 752f_{n+\frac{3}{2}} + 1506f_{n+2}$$
$$-976f_{n+\frac{5}{2}} + 293f_{n+3} - 14f_{n+4}]$$
$$241920hy'_{n+\frac{5}{2}} - 241920y_n + 725760y_{n+1} - 483840y_{n+2}$$
$$= h^3[150533f_{n+1} - 129024f_{n+\frac{3}{2}}$$
$$+338139f_{n+2} - 181024f_{n+\frac{5}{2}} + 55899f_{n+3} - 2683f_{n+4}]$$
$$7560hy'_{n+3} - 11340y_n + 30240y_{n+1} - 18900y_{n+2} = h^3[7058f_{n+1} - 5808f_{n+\frac{3}{2}}$$
$$+16785f_{n+2} - 6832f_{n+\frac{5}{2}} + 2784f_{n+3} - 127f_{n+4}]$$
$$7560hy'_{n+4} - 18900y_n + 45360y_{n+1} - 26460y_{n+2} = h^3[12103f_{n+1} - 11376f_{n+\frac{3}{2}}$$
$$+34218f_{n+2} - 15440f_{n+\frac{5}{2}} + 13005f_{n+3} + 250f_{n+4}]$$

(13)

Also the second derivative of (10) is evaluated at $x = x_{n+j}$, $j = 1, \frac{3}{2}, \frac{5}{2}, 3, 4$ to obtain the following discrete schemes as follows.

$$2520hy''_{n+1} - 2520y_n + 5040y_{n+1} - 2520y_{n+2} = h^3[11701f_{n+1} - 2928f_{n+\frac{3}{2}} + 3327f_{n+2}$$
$$-2192f_{n+\frac{5}{2}} + 654f_{n+3} - 31f_{n+4}]$$

$$40320hy''_{n+\frac{3}{2}} - 40320y_n + 80640y_{n+1} - 40320y_{n+2} = h^3[25433f_{n+1} - 27248f_{n+\frac{3}{2}}$$
$$+43005f_{n+2} - 29584f_{n+\frac{5}{2}} + 8987f_{n+3} - 433f_{n+4}]$$

$$7560hy''_{n+2} - 7560y_n + 15120y_{n+1} - 7560y_{n+2} = h^3[4693f_{n+1} - 3408f_{n+\frac{3}{2}} + 10674f_{n+2}$$
$$-6128f_{n+\frac{5}{2}} + 1815f_{n+3} - 86f_{n+4}]$$

$$40320hy''_{n+\frac{5}{2}} - 40320y_n + 80640y_{n+1} - 40320y_{n+2} = h^3[25209f_{n+1} - 19632f_{n+\frac{3}{2}}$$
$$+68541f_{n+2} - 21968f_{n+\frac{5}{2}} + 8763f_{n+3} - 433f_{n+4}]$$

$$2520hy''_{n+3} - 2520y_n + 5040y_{n+1} - 2520y_{n+2} = h^3[1562f_{n+1} - 1136f_{n+\frac{3}{2}} + 3999f_{n+2}$$
$$-400f_{n+\frac{5}{2}} + 1046f_{n+3} - 31f_{n+4}]$$

$$2520hy''_{n+4} - 2520y_n + 5040y_{n+1} - 2520y_{n+2} = h^3[1863f_{n+1} - 2928f_{n+\frac{3}{2}} + 8430f_{n+2}$$
$$-6224f_{n+\frac{5}{2}} + 5757f_{n+3} + 662f_{n+4}] \quad (14)$$

All the derivatives discrete schemes gotten from equation (10) are of uniform order 6.

3.0 Block Stability Analysis of the method 11

The block method (11) will be arranged in matrix equation form as

$$\begin{bmatrix} -\frac{3}{4} & 1 & -\frac{3}{8} & 0 & 0 & 0 \\ \frac{5}{4} & 0 & -\frac{15}{8} & 1 & 0 & 0 \\ 3 & 0 & -3 & 0 & 1 & 0 \\ 8 & 0 & -6 & 0 & 0 & 1 \\ \frac{-15120}{7560} & 0 & \frac{3780}{7560} & 0 & 0 & 0 \\ \frac{5040}{2520} & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11340}{7560} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{7139}{92160} & \frac{341}{3840} & -\frac{1451}{10240} & \frac{1097}{11520} & -\frac{883}{30720} & \frac{127}{92160} \\ \frac{4297}{18432} & -\frac{53}{256} & \frac{1009}{2048} & -\frac{667}{2304} & \frac{179}{2048} & -\frac{77}{18432} \\ \frac{28}{45} & -\frac{8}{15} & \frac{7}{15} & -\frac{32}{45} & \frac{7}{30} & -\frac{1}{90} \\ \frac{169}{90} & -\frac{8}{5} & \frac{23}{5} & -\frac{88}{45} & \frac{11}{10} & -\frac{1}{45} \\ \frac{10739}{7560} & -\frac{25392}{7560} & \frac{33282}{7560} & -\frac{22672}{7560} & \frac{6897}{7560} & -\frac{334}{7560} \\ \frac{-17401}{2520} & \frac{47248}{2520} & \frac{-63474}{2520} & \frac{43952}{2520} & \frac{-13507}{2520} & \frac{662}{2520} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \quad (15)$$

Let $A^0 = \begin{bmatrix} -\frac{3}{4} & 1 & -\frac{3}{8} & 0 & 0 & 0 \\ \frac{5}{4} & 0 & -\frac{15}{8} & 1 & 0 & 0 \\ 3 & 0 & -3 & 0 & 1 & 0 \\ 8 & 0 & -6 & 0 & 0 & 1 \\ \frac{-15120}{7560} & 0 & \frac{3780}{7560} & 0 & 0 & 0 \\ \frac{5040}{2520} & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$ and

$$B^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & -\frac{11340}{760} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We normalized the block method (13) with the inverse of $A^{(0)}$ and applied the condition $\rho(\lambda) = \det\left[\lambda A^{(0)}(A^{(0)})^{-1} - B^{(0)}(A^{(0)})^{-1}\right] = 0$

$$\rho(\lambda) = \left[\lambda\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right] = 0$$

$$= det\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix} = 0$$

$= \lambda^6 - \lambda^5 = 0$

Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \ \lambda_6 = 1$.

From definition (1.0), the method (11) is zero stable and since the order of the method is $p \geq 1$, the method is consistent, thus convergent.

4.0 Numerical Experiments

The newly constructed hybrid block method is demonstrated with two problems to ascertain their efficiency

Problem 1

$$\boldsymbol{y''' - y'' + y' - y = 0}\ , \qquad \boldsymbol{y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01}$$

Exact Solution: $y(x) = Cosx$

Problem 2

$$\boldsymbol{y''' + 5y'' + 7y' + 3y = 0}\ , \qquad \boldsymbol{y(0) = 1, y'(0) = 1, y''(0) = -1, h = 0.01}$$

Exact Solution: $y(x) = \frac{1}{2}\, e^{-3x} + \frac{1}{2}\, e^{-x} + 3x\, e^{-x}$

Table 1: Approximate solution and Absolute errors of Problem 1

x	**Exact Solution**	**New Method**	**Error of ([2])**	**New Error**
0.01	0.999950000416665	0.9999500004166234	9.795241 E(-08)	1.250432 E(-09)
0.02	0.999800006666578	0.99979999990347	3.94772281 E(-07)	6.676247 E(-09)
0.03	0.999550033748988	0.999550022354949	8.87494207 E(-07)	1.1394059 E(-08)
0.04	0.999200106660978	0.999200127075472	1.57315477 E(-06)	2.0414474 E(-08)
0.05	0.998750260394966	0.998750354328703	2.434769007 E(-06)	9.3933737 E(-08)
0.06	0.998200539935204	0.998200806658417	3.451213284 E(-06)	2.66723213 E(-07)
0.07	0.997551000253280	0.99755144730079	4.5971893 E(-06)	5.44476799 E(-07)
0.08	0.996801706302619	0.996802713206786	5.843188693 E(-06)	1.006904167 E(-06)
0.09	0.995952733011994	0.995954393431063	7.155458262 E(-06)	1.660419069 E(-06)
0.1	0.995004165278026	0.99500672564932	8.495965767 E(-06)	2.607286906E(-06)

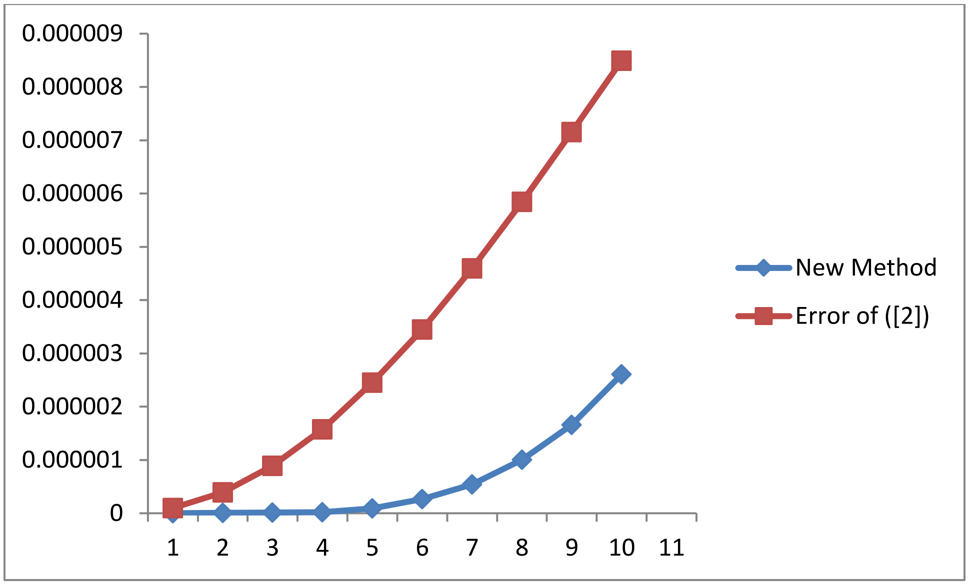

Figure 1: Error graph of Problem 1

Table 2: Approximate solution and Absolute errors of Problem 2

x	Exact Solution	New Method	Error of ([2])	New Error
0.01	1.00994917866131	1.00994921388938	3.523068 E(-08)	3.522808 E(-08)
0.02	1.01979352384391	1.019793708946230	1.8511547 E(-07)	1.8510234 E(-07)
0.03	1.02952845742923	1.029528759961010	3.02566314 E(-07)	3.0253182 E(-07)
0.04	1.03914967063302	1.039149633958600	2.5440446 E(-07)	3.667438 E(-08)
0.05	1.04865311413799	1.048652176281550	7.945598 E(-08)	9.3785644 E(-07)
0.06	1.058034988542920	1.058032230830200	8.0684677 E(-07)	2.75771272 E(-06)
0.07	1.067291735118320	1.067286169572400	2.02409638 E(-06)	5.56554592 E(-06)
0.08	1.076420026859390	1.076410368628170	3.816761174 E(-06)	9.65823122 E(-06)
0.09	1.085416759827270	1.085401685676860	6.260320350 E(-06)	1.507415041 E(-05)
0.1	1.094279044769630	1.094256987472220	9.42082743 E(-06)	2.205729741E(-05)

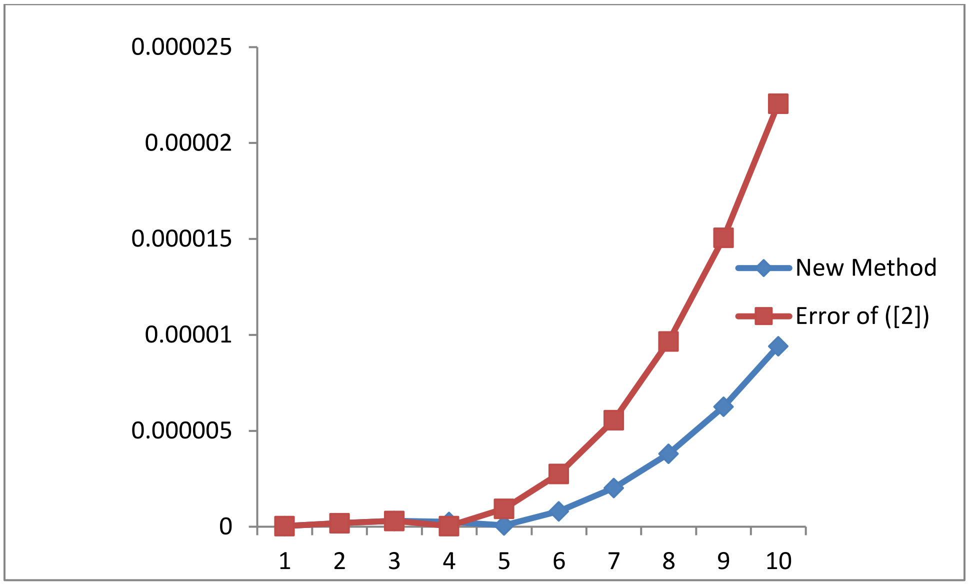

Figure 2: Error graph of Problem 2

5.0 Discussion of Results

Tables 1 and 2 displays the Approximate solutions and Absolute errors of problem 1 and 2. The figures 1 and 2 display the error graphs of the same problems.

Conclusion

We want to conclude that we are able to derived some uniform order Six multi-derivatives schemes through Legendre series as our basic functions which formed our Block method for the solution of general third order ordinary differential equations. The Block Method derived is used to obtain the solution of General Third Order ODEs directly without any reduction to the equivalent system of first Order ODEs. The results obtained from the two problems tested with this Block method displayed superiority over the existing method. (see error graphs of 1 and 2).

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