ON THE STUDY OF THE BASIC PROPERTIES OF MULTIVARIATE AUTOREGRESSIVE MODELS

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Abstract

Models containing numerous response and predictive time variables, known as multivariate autoregressive models, establish a link between each response and the lag terms of both the response and predictive variables. Multivariate time series models, like univariate time series models, contain some basic properties that distinguish each model. The basic features of Multivariate Autoregressive Models (MARM), also known as Vector Autoregressive Models (VARM), are investigated in this study. The work focuses on using model parameter estimations to derive the variance, autocorrelations, and cross-autocorrelations of multi-dimensional VAR models. The features of broad VAR models, such as variances, autocovariances, cross-autocovariances, autocorrelations, and cross-autocorrelations, are calculated and validated using empirical examples.

***Keywords**: Vector Autoregressive Models, Autocovariance/Cross-Autocovariance and Autocorrelation/Cross-autocorrelations*

Introduction

Multivariate time series models are time series models that include numerous response and predictive time variables and establish a relationship between the response and lag terms of both the response and predictive time processes. The models are multi-parameter extensions of univariate time series that account for the distributions of predictive time variables in each response variable. Every reaction time variable is a function of its previous values, much like in univariate time series. What distinguishes multivariate time series models from the univariate is that each response time variable is a function of its past values and other time variables. With multivariate time series models, there is a feed-forward and feedback mechanism between each response and predictive time variables, which implies that each response time variable in a given multivariate time series model is a predictor term to another in a set of multivariate time series models. The models are also known as VAR models. The VAR models are n-dimensional response and predictor time variables characterized by autoregressive processes. If the Vector time variable is characterized by a moving average process, the model is known as Vector Moving Average

(VMA) or Moving Average Vector (MAV) model. VAR model is described by [2] as one of the most successful, flexible, and easy-to-use models to capture the dynamic behaviour of multivariate time series. VAR models are popular models in studying the behaviour and feed-forward and feedback mechanism amongst macroeconomic variables. The flexibility of multivariate time series models, which gives room for the study of the effect of other time variables on another is the advantage it has over the univariate time series model. These are evident when modelling multiple set of economic and financial time series. Identification of the basic properties that characterize the behaviour of a model has always been the tradition in time series and other areas of statistics. In univariate time series, there are underlying properties that explain the nature of a given model. Fundamentally, some basic properties of a univariate autoregressive model include the variance, autocovariance, autocorrelation structures, etc. As an extension of the univariate autoregressive model, Vector Autoregressive Models are a set of models that have gained prominence in multivariate time series modelling due to their wide areas of applications in economic and financial studies.

The interest in this paper is on the basic properties of multivariate autoregressive models. These include variances, autocovariances, cross-covariances, autocorrelations, and cross-autocorrelations. Properties of Bayesian Multivariate Autoregressive Time Series Models are presented, which compared the small sample performance of Bayesian Multivariate Vector Autoregressive time series models relative to frequent power and parameter estimation bias, [6]. Some basic properties of n-dimensional vector time series which include cross-autocovariances and cross-autocorrelations have been obtained by [1]. The work did not consider the estimates of the model coefficients, rather used lagged variables for computations of cross-autocovariances and cross-autocorrelations for stationary multivariate time series. Still on multivariate time series are frequentist and Bayesian methods considered for the estimation of structural vector autoregressive models by [4]. Network vector autoregressive (NAR) model have been introduced by [10]. The work focused on a large-scale social network with a continuous response observed for each node at equally spaced time points. The response from different nodes constitutes an ultra-high-dimensional vector whose time series dynamics is to be investigated. The NAR assumes each node’s response at a given time is a function of its previous time value, the average of its connected neighbours, a set of node-specific covariates and independent noise. The NAR model is estimated using an ordinary least squares type of estimators, and its asymptotic properties are investigated. Investigation of the stationarity of multivariate time series using autocorrelations and cross-autocorrelations with three response time variables representing urban, rural and average consumer price indices and Nigeria's crude oil production quantity and price volatilities have been carried out by [8] and [9]. The two papers adopted lag terms in obtaining the autocorrelations and cross-autocorrelations to investigate the stationarity of multivariate time series through the positive definiteness property. On the probabilistic properties of autoregressive models, [7]. The theoretical properties were presented and demonstrated using simulated and real-world examples.

A number of published works have little or no contributions in building up models for computations of variances, autocovariances, cross-autocovariances, autocorrelations, cross-autocorrelations from multivariate time series models. To bridge the gap, this paper proposes the use of parametric model to develop models for estimations of the fundamental properties of Vector Autoregressive Models (VARM).

Statistical Method

This section considers the general Multivariate Autoregressive Time Series and proposed method of derivation of its properties.

Definition

According to [3], Multivariate Autoregressive Models are presented as,

$$Y_{it} = \varphi_i + \sum_{k=1}^{p}\sum_{j=1}^{n} \beta_{k.ij} Y_{jt-k} + \epsilon_{it}, i = 1, \dots, m \qquad (1)$$

where, $Y_{1t(i=1,\dots,m)}$ response time variables, Y_{jt-k} are the lag terms with parameter matrix $\beta_{k.ij}$, φ_i are constants and ϵ_{it} errors associated with the response time variables. On the assumption that $Y_{1t(i=1,\dots,m)}$ are stationary time series processes distributed about the constant origin,

$$E(Y_{1t}) = E(Y_{2t}) = \cdots = E(Y_{mt}) = 0 => \varphi_1 = \varphi_2 = \cdots = \varphi_m = 0.$$

For $i = 1,2, \dots, m; j = 1, \dots, n \text{ } and \text{ } k = 1,2,$…,P. Equation (1) is expanded as follows,

$$\begin{aligned} Y_{1t} &= \varphi_{1.11}Y_{1t-1} + \varphi_{1.12}Y_{2t-1} + \cdots + \varphi_{1.1n}Y_{nt-1} + \varphi_{2.11}Y_{1t-2} + \varphi_{2.12}Y_{2t-2} + \cdots \\ &+ \varphi_{2.1n}Y_{nt-2} + \cdots + \varphi_{p.11}Y_{1t-p} + \varphi_{p.12}Y_{2t-p} + \cdots + \varphi_{p.1n}Y_{nt-p} \\ &+ \epsilon_{1t} \end{aligned} \qquad (2)$$

$$Y_{2t} = \varphi_{1.21}Y_{1t-1} + \varphi_{1.22}Y_{2t-1} + \cdots + \varphi_{1.2n}Y_{nt-1} + \varphi_{2.21}Y_{1t-2} + \varphi_{2.22}Y_{2t-2} + \cdots + \varphi_{2.2n}Y_{nt-2} + \cdots + \varphi_{p.21}Y_{1t-p} + \varphi_{p.22}Y_{2t-p} + \cdots + \varphi_{p.2n}Y_{nt-p} + \epsilon_{2t} \quad (3)$$

$$\vdots \qquad\qquad \vdots$$

$$Y_{mt} = \varphi_{1.m1}Y_{1t-1} + \varphi_{1.m2}Y_{2t-1} + \cdots + \varphi_{1.mn}Y_{nt-1} + \varphi_{2.m1}Y_{1t-2} + \varphi_{2.m2}Y_{2t-2} + \cdots + \varphi_{2.mn}Y_{nt-2} + \cdots + \varphi_{p.m1}Y_{1t-p} + \varphi_{p.m2}Y_{2t-p} + \cdots + \varphi_{p.mn}Y_{nt-p} + \epsilon_{mt} \quad (4)$$

Equations (2), (3) and (4) are the p^{th} order Multivariate Autoregressive Models.
The coefficients $\beta_{1.ij}$ are first time lag parameters, indicating the contributions of $Y_{jt-1(j=1,\dots,n)}$ to the response time variable, $Y_{it(i=1,\dots,m)}$. $\epsilon_{it(i=1,\dots,m)} \sim N(0, \sigma^2_{\epsilon_{it}})$.

Variance, Auto-covariance and Cross-Autocovariance

Recall that,

$$COV(Y_t, Y_{t+k}) = E(Y_t Y_{t+k}) - E(Y_t)E(Y_{t+k}), \quad (5)$$

$$E(Y_t) = E(Y_{t+k}) = 0$$

Variance, Autocovariance and Autocorrelation of Y_{1t}

Variance of Y_{1t}

Multiplying Equation (2) by Y_{1t} and taking the expectations,

$$E(Y_{1t}Y_{1t}) = E\left[Y_{1t}\left(\varphi_{1.11}Y_{1t-1} + \varphi_{1.12}Y_{2t-1} + \cdots + \varphi_{1.1n}Y_{nt-1} + \varphi_{2.11}Y_{1t-2} + \varphi_{2.12}Y_{2t-2} + \cdots + \varphi_{2.1n}Y_{nt-2} + \cdots + \varphi_{p.11}Y_{1t-p} + \varphi_{p.12}Y_{2t-p} + \cdots + \varphi_{p.1n}Y_{nt-p} + \epsilon_{1t}\right)\right] \quad (6)$$

$$E(Y_{1t}^2) = \varphi_{1.11}E(Y_{1t}Y_{1t-1}) + \varphi_{1.12}E(Y_{1t}Y_{2t-1}) + \cdots + \varphi_{1.1n}E(Y_{1t}Y_{nt-1}) + \varphi_{2.11}E(Y_{1t}Y_{1t-2}) + \varphi_{2.12}E(Y_{1t}Y_{2t-2}) + \cdots + \varphi_{2.1n}E(Y_{1t}Y_{nt-2}) + \cdots + \varphi_{p.11}E(Y_{1t}Y_{1t-p}) + \varphi_{p.12}E(Y_{1t}Y_{2t-p}) + \cdots + \varphi_{p.1n}E(Y_{1t}Y_{nt-p}) + E(Y_{1t}\epsilon_{1t}) \quad (7)$$

$E(Y_{1t}^2) = \gamma_{1t,1t}$ (Variance of Y_{1t})

$$\gamma_{1t,1t} = \varphi_{1.11}\gamma_{1t,1t(1)} + \varphi_{1.12}\gamma_{1t,2t(1)} + \cdots + \varphi_{1.1n}\gamma_{1t,nt(1)} + \varphi_{2.11}\gamma_{1t,1t(2)} + \varphi_{2.12}\gamma_{1t,2t(2)} + \cdots + \varphi_{2.1n}\gamma_{1t,nt(2)} + \cdots + \varphi_{p.11}\gamma_{1t,1t(p)} + \varphi_{p.12}\gamma_{1t,2t(p)} + \cdots + \varphi_{p.1n}\gamma_{1t,nt(p)} + \sigma_{\epsilon_{1t}}^2 \quad (8)$$

Equation (8) further reduces to

$$\gamma_{1t,1t} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{1t,jt(k)} \quad (9)$$

From Equation (9), $\gamma_{1t,1t(k=1,\ldots,p)}$ are autocovariances of Y_{1t}; $\gamma_{1t,2t(k=1,\ldots,p)},\cdots,$ $\gamma_{1t,nt(k=1,\ldots,p)}$ are cross-autocovariances of Y_{1t} and $Y_{2t},\ldots,Y_{nt}$; $\varphi_{k.1j}$ is the parameter of j^{th} predictive variable to Y_{1t} at k^{th} lag and $E(Y_{1t}\epsilon_{1t}) = \sigma_{\epsilon_{1t}}^2$(correlated stationary processes at zero time lag), which represent variance of the error term of Y_{1t}.

Autocovariances and Autocorrelations of Y_{1t}

i. Multiplying Equation (2) by Y_{1t-1} and taking the expectations,

$$E(Y_{1t}Y_{1t-1}) = \varphi_{1.11}E(Y_{1t-1}Y_{1t-1}) + \varphi_{1.12}E(Y_{1t-1}Y_{2t-1}) + \cdots + \varphi_{1.1n}E(Y_{1t-1}Y_{nt-1} + \varphi_{2.11}E(Y_{1t-1}Y_{1t-2}) + \varphi_{2.12}E(Y_{1t-1}Y_{2t-2}) + \cdots + \varphi_{2.1n}E(Y_{1t-1}Y_{nt-2}) + \cdots + \varphi_{p.11}E(Y_{1t-1}Y_{1t-p}) + \varphi_{p.12}E(Y_{1t-1}Y_{2t-p}) + \cdots + \varphi_{p.1n}E(Y_{1t-1}Y_{nt-p}) + E(Y_{1t-1}\epsilon_{1t}) \quad (10)$$

$E(Y_{1t}Y_{1t-1}) = \gamma_{1t,1t(1)}$ (autocovariance of $Y_{1t}\ at\ k = 1$)

$$\gamma_{1t,1t(1)} = \varphi_{1.11}\gamma_{1t,1t} + \varphi_{1.12}\gamma_{1t,2t} + \cdots + \varphi_{1.1n}\gamma_{1t,nt} + \varphi_{2.11}\gamma_{1t,1t(1)} + \varphi_{2.12}\gamma_{1t,2t(1)} + \cdots + \varphi_{2.1n}\gamma_{1t,nt(1)} + \cdots + \varphi_{p.11}\gamma_{1t,1t(p-1)} + \varphi_{p.12}\gamma_{1t,2t(p-1)} + \cdots + \varphi_{p.1n}\gamma_{1t,nt(p-1)} \quad (11)$$

$E(Y_{1t-1}\epsilon_{1t}) = 0$ (uncorrelated processes)

$$\gamma_{1t,1t(1)} = \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\gamma_{1t(1),jt(k)} \tag{12}$$

Dividing Equation (12) by $\gamma_{1t,1t}$ produces the autocorrelation

$$\rho_{1t,1t(1)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\gamma_{1t(1),jt(k)}}{\gamma_{1t,1t}} \tag{13}$$

$\rho_{1t,1t(1)}$ is the autocorrelation of Y_{1t} and $Y_{1t(1)}$

ii. Multiplying Equation (2) by Y_{1t-2} and taking the expectations,

$$\begin{aligned} &E(Y_{1t}Y_{1t-2}) \\ &= \varphi_{1.11}E(Y_{1t-2}Y_{1t-1}) + \varphi_{1.12}E(Y_{1t-2}Y_{2t-1}) + \cdots + \varphi_{1.1n}E(Y_{1t-2}Y_{nt-1} \\ &+ \varphi_{2.11}E(Y_{1t-2}Y_{1t-2}) + \varphi_{2.12}E(Y_{1t-2}Y_{2t-2}) + \cdots + \varphi_{2.1n}E(Y_{1t-2}Y_{nt-2}) + \cdots \\ &+ \varphi_{p.11}E(Y_{1t-2}Y_{1t-p}) + \varphi_{p.12}E(Y_{1t-2}Y_{2t-p}) + \cdots + \varphi_{p.1n}E(Y_{1t-2}Y_{nt-p}) \\ &+ E(Y_{1t-2}\epsilon_{1t}) \end{aligned} \tag{14}$$

$E(Y_{1t}Y_{1t-2}) = \gamma_{1t,1t(2)}$ (autocovariance of Y_{1t} at $k = 2$)

$$\begin{aligned} &\gamma_{1t,1t(2)} \\ &= \varphi_{1.11}\gamma_{1t,1t(1)} + \varphi_{1.12}\gamma_{2t,1t(1)} + \cdots + \varphi_{1.1n}\gamma_{nt,1t(1)} + \varphi_{2.11}\gamma_{1t,1t} + \varphi_{2.12}\gamma_{1t,2t} + \cdots \\ &+ \varphi_{2.1n}\gamma_{1t,nt} + \cdots + \varphi_{p.11}\gamma_{1t,1t(p-2)} + \varphi_{p.12}\gamma_{1t,2t(p-2)} + \cdots \\ &+ \varphi_{p.1n}\gamma_{1t,nt(p-2)} \end{aligned} \tag{15}$$

$E(Y_{1t-2}\epsilon_{1t}) = 0$ (uncorrelated processes)

$$\gamma_{1t,1t(2)} = \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\gamma_{1t(2),jt(k)} \tag{16}$$

Dividing Equation (16) by $\gamma_{1t,1t}$ produces the autocorrelation

$$\rho_{1t,1t(2)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{1t(2),jt(k)}}{\gamma_{1t,1t}} \tag{17}$$

$\rho_{1t,1t(2)}$ is the autocorrelation of Y_{1t} and $Y_{1t(2)}$

iii. Multiplying Equation (2) by Y_{1t-p} and taking the expectations,

$$\begin{aligned} &E\left(Y_{1t}Y_{1t-p}\right) \\ &= \varphi_{1.11}E(Y_{1t-p}Y_{1t-1}) + \varphi_{1.12}E(Y_{1t-p}Y_{2t-1}) + \cdots + \varphi_{1.1n}E(Y_{1t-p}Y_{nt-1} \\ &+ \varphi_{2.11}E(Y_{1t-p}Y_{1t-2}) + \varphi_{2.12}E(Y_{1t-p}Y_{2t-2}) + \cdots + \varphi_{2.1n}E(Y_{1t-p}Y_{nt-2}) + \cdots \\ &+ \varphi_{p.11}E(Y_{1t-p}Y_{1t-p}) + \varphi_{p.12}E(Y_{1t-p}Y_{2t-p}) + \cdots + \varphi_{p.1n}E(Y_{1t-p}Y_{nt-p}) \\ &+ E\left(Y_{1t-p}\epsilon_{1t}\right) \end{aligned} \tag{18}$$

$E\left(Y_{1t}Y_{1t-p}\right) = \gamma_{1t,1t(p)}$ (autocovariance of Y_{1t} at $k = p$)

$$\begin{aligned} \gamma_{1t,1t(p)} = {} & \varphi_{1.11}\gamma_{1t,1t(p-1)} + \varphi_{1.12}\gamma_{2t,1t(p-1)} + \cdots + \varphi_{1.1n}\gamma_{nt,1t(p-1)} + \varphi_{2.11}\gamma_{1t,1t(p-2)} \\ & + \varphi_{2.12}\gamma_{2t,1t(p-2)} + \cdots + \varphi_{2.1n}\gamma_{nt,1t(p-2)} + \cdots + \varphi_{p.11}\gamma_{1t,1t} + \varphi_{p.12}\gamma_{1t,2t} \\ & + \cdots + \varphi_{p.1n}\gamma_{1t,nt} \end{aligned} \tag{19}$$

$E\left(Y_{1t-p}\epsilon_{1t}\right) = 0$ (uncorrelated processes)

$$\begin{aligned} &\gamma_{1t,1t(p)} \\ &= \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{1t(p),jt(k)} \end{aligned} \tag{20}$$

Dividing Equation (20) by $\gamma_{1t,1t}$ produces the autocorrelation

$$\rho_{1t,1t(p)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{1t(p),jt(k)}}{\gamma_{1t,1t}} \tag{21}$$

$\rho_{1t,1t(p)}$ is the autocorrelation of Y_{1t} and $Y_{1t(p)}$

Generally,

$$\rho_{1t,1t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{1t(l),jt(k)}}{\gamma_{1t,1t}} & , l = \pm 1, \pm 2, \pm \cdots \end{cases} \tag{22}$$

Variance, Autocovariances and Autocorrelations of Y_{2t}

Variance of Y_{2t}

Multiplying Equation (3) by Y_{2t} and taking the expectations,

$$\begin{aligned} E(Y_{2t}^2) = {} & \varphi_{1.21}E(Y_{2t}Y_{1t-1}) + \varphi_{1.22}E(Y_{2t}Y_{2t-1}) + \cdots + \varphi_{1.2n}E(Y_{2t}Y_{nt-1}) \\ & + \varphi_{2.21}E(Y_{2t}Y_{1t-2}) + \varphi_{2.22}E(Y_{2t}Y_{2t-2}) + \cdots + \varphi_{2.2n}E(Y_{2t}Y_{nt-2}) + \cdots \\ & + \varphi_{p.21}E(Y_{2t}Y_{1t-p}) + \varphi_{p.22}E(Y_{2t}Y_{2t-p}) + \cdots + \varphi_{p.2n}E(Y_{2t}Y_{nt-p}) \\ & + E(Y_{2t}\epsilon_{2t}) \qquad (23) \end{aligned}$$

$E(Y_{1t}^2) = \gamma_{2t,2t}$ (Variance of Y_{2t})

$$\begin{aligned} \gamma_{2t,2t} = {} & \varphi_{1.21}\gamma_{2t,1t(1)} + \varphi_{1.22}\gamma_{2t,2t(1)} + \cdots + \varphi_{1.2n}\gamma_{2t,nt(1)} + \varphi_{2.21}\gamma_{2t,1t(2)} \\ & + \varphi_{2.22}\gamma_{2t,2t(2)} + \cdots + \varphi_{2.2n}\gamma_{2t,nt(2)} + \cdots + \varphi_{p.21}\gamma_{2t,1t(p)} + \varphi_{p.22}\gamma_{2t,2t(p)} \\ & + \cdots + \varphi_{p.2n}\gamma_{2t,nt(p)} \\ & + \sigma_{\epsilon_{2t}}^2 \qquad (24) \end{aligned}$$

Equation (24) further reduces to

$$\gamma_{2t,2t} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{2t,jt(k)} \tag{25}$$

From Equation (25), $\gamma_{2t,2t(k=1,\ldots,p)}$ are autocovariances of Y_{2t}, $\gamma_{2t,1tk=1,\ldots,p1)},\cdots,$ $\gamma_{2t,nt(k=1,\ldots,p)}$ are cross-autocovariances of Y_{2t} and $Y_{1t},\ldots,Y_{2t}$ and Y_{nt}, $\varphi_{k.2j}$ is the parameter of j^{th} predictive variable to Y_{2t} at k^{th} lag and $E(Y_{2t}\epsilon_{2t}) = \sigma_{\epsilon_{2t}}^2$(correlated stationary processes at zero time lag), which represent variance of the error term of Y_{2t}.

Autocovariance and Autocorrelation of Y_{2t}

i. Multiplying Equation (3) by Y_{2t-1} and taking the expectations,

$$\begin{aligned} &E(Y_{2t}Y_{2t-1}) \\ &= \varphi_{1.21}E(Y_{2t-1}Y_{1t-1}) + \varphi_{1.22}E(Y_{2t-1}Y_{2t-1}) + \cdots + \varphi_{1.2n}E(Y_{2t-1}Y_{nt-1} \\ &+ \varphi_{2.21}E(Y_{2t-1}Y_{1t-2}) + \varphi_{2.22}E(Y_{2t-1}Y_{2t-2}) + \cdots + \varphi_{2.2n}E(Y_{2t-1}Y_{nt-2}) + \cdots \\ &+ \varphi_{p.21}E(Y_{2t-1}Y_{1t-p}) + \varphi_{p.22}E(Y_{2t-1}Y_{2t-p}) + \cdots + \varphi_{p.2n}E(Y_{2t-1}Y_{nt-p}) \\ &+ E(Y_{2t-1}\epsilon_{2t}) \end{aligned} \tag{26}$$

$E(Y_{2t}Y_{2t-1}) = \gamma_{2t,2t(1)}$ (autocovariance of $Y_{2t}\ at\ k = 1$)

$$\begin{aligned} &\gamma_{2t,2t(1)} \\ &= \varphi_{1.21}\gamma_{2t,1t} + \varphi_{1.22}\gamma_{2t,2t} + \cdots + \varphi_{1.2n}\gamma_{2t,nt} + \varphi_{2.21}\gamma_{2t,1t(1)} + \varphi_{2.22}\gamma_{2t,2t(1)} + \cdots \\ &+ \varphi_{2.2n}\gamma_{2t,nt(1)} + \cdots + \varphi_{p.21}\gamma_{2t,1t(p-1)} + \varphi_{p.22}\gamma_{2t,2t(p-1)} + \cdots \\ &+ \varphi_{p.2n}\gamma_{2t,nt(p-1)} \end{aligned} \tag{27}$$

$E(Y_{2t-1}\epsilon_{2t}) = 0$ (uncorrelated processes)

$$\begin{aligned} &\gamma_{2t,2t(1)} \\ &= \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\gamma_{2t(1),jt(k)} \end{aligned} \tag{28}$$

Dividing Equation (28) by $\gamma_{2t,2t}$ produces the autocorrelation

$$\rho_{2t,2t(1)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.2j}\gamma_{2t(1),jt(k)}}{\gamma_{2t,2t}} \tag{29}$$

$\rho_{2t,2t(1)}$ is the autocorrelation of $Y_{2t}\ and\ Y_{2t(1)}$

ii. Multiplying Equation (3) by Y_{2t-2} and taking the expectations,

$$\begin{aligned} &E(Y_{2t}Y_{2t-2}) \\ &= \varphi_{1.21}E(Y_{2t-2}Y_{1t-1}) + \varphi_{1.22}E(Y_{2t-2}Y_{2t-1}) + \cdots + \varphi_{1.2n}E(Y_{2t-2}Y_{nt-1} \\ &+ \varphi_{2.21}E(Y_{2t-2}Y_{1t-2}) + \varphi_{2.22}E(Y_{2t-2}Y_{2t-2}) + \cdots + \varphi_{2.2n}E(Y_{2t-2}Y_{nt-2}) + \cdots \\ &+ \varphi_{p.21}E(Y_{2t-2}Y_{1t-p}) + \varphi_{p.22}E(Y_{2t-2}Y_{2t-p}) + \cdots + \varphi_{p.2n}E(Y_{2t-2}Y_{nt-p}) \\ &+ E(Y_{2t-2}\epsilon_{2t}) \end{aligned} \tag{30}$$

$E(Y_{2t}Y_{2t-2}) = \gamma_{2t,2t(2)}$ (autocovariance of Y_{2t} $at\ k = 2$)

$$\gamma_{2t,2t(2)} = \varphi_{1.21}\gamma_{2t,2t(1)} + \varphi_{1.22}\gamma_{2t,2t(1)} + \cdots + \varphi_{1.2n}\gamma_{nt,2t(1)} + \varphi_{2.21}\gamma_{1t,2t} + \varphi_{2.22}\gamma_{2t,2t} + \cdots + \varphi_{2.2n}\gamma_{2t,nt} + \cdots + \varphi_{p.21}\gamma_{2t,1t(p-2)} + \varphi_{p.22}\gamma_{2t,2t(p-2)} + \cdots + \varphi_{p.2n}\gamma_{2t,nt(p-2)} \quad (31)$$

$E(Y_{2t-2}\epsilon_{2t}) = 0$ (uncorrelated processes)

$$\gamma_{2t,2t(2)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{2t(2),jt(k)} \quad (32)$$

Dividing Equation (32) by $\gamma_{2t,2t}$ produces the autocorrelation

$$\rho_{2t,2t(2)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{2t(2),jt(k)}}{\gamma_{2t,2t}} \quad (33)$$

$\rho_{2t,2t(2)}$ is the autocorrelation of Y_{2t} and $Y_{2t(2)}$

iii. Multiplying Equation (3) by Y_{2t-p} and taking the expectations,

$$E(Y_{2t}Y_{2t-p}) = \varphi_{1.21}E(Y_{2t-p}Y_{1t-1}) + \varphi_{1.22}E(Y_{2t-p}Y_{2t-1}) + \cdots + \varphi_{1.2n}E(Y_{2t-p}Y_{nt-1} + \varphi_{2.21}E(Y_{2t-p}Y_{1t-2}) + \varphi_{2.22}E(Y_{2t-p}Y_{2t-2}) + \cdots + \varphi_{2.2n}E(Y_{2t-p}Y_{nt-2}) + \cdots + \varphi_{p.21}E(Y_{2t-p}Y_{1t-p}) + \varphi_{p.22}E(Y_{2t-p}Y_{2t-p}) + \cdots + \varphi_{p.2n}E(Y_{2t-p}Y_{nt-p}) + E(Y_{2t-p}\epsilon_{2t}) \quad (34)$$

$E(Y_{2t}Y_{2t-p}) = \gamma_{2t,2t(p)}$ (autocovariance of Y_{2t} $at\ k = p$)

$$\gamma_{2t,2t(p)} = \varphi_{1.21}\gamma_{1t,2t(p-1)} + \varphi_{1.22}\gamma_{2t,2t(p-1)} + \cdots + \varphi_{1.2n}\gamma_{nt,2t(p-1)} + \varphi_{2.21}\gamma_{1t,2t(p-2)} + \varphi_{2.22}\gamma_{2t,2t(p-2)} + \cdots + \varphi_{2.2n}\gamma_{nt,2t(p-2)} + \cdots + \varphi_{p.21}\gamma_{1t,2t} + \varphi_{p.22}\gamma_{2t,2t} + \cdots + \varphi_{p.2n}\gamma_{2t,nt} \quad (35)$$

$E(Y_{2t-p}\epsilon_{2t}) = 0$ (uncorrelated processes)

$$\gamma_{2t,2t(p)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{2t(p),jt(k)} \quad (36)$$

Dividing Equation (36) by $\gamma_{1t,1t}$ produces the autocorrelation

$$\rho_{2t,2t(p)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{2t(p),jt(k)}}{\gamma_{2t,2t}} \quad (37)$$

$\rho_{2t,2t(p)}$ is the autocorrelation of Y_{2t} *and* $Y_{2t(p)}$

Therefore,

$$\rho_{2t,2t(l)} = \begin{cases} 1 & ,l = 0 \\ \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{2t(l),jt(k)}}{\gamma_{2t,2t}} & ,l = \pm1,2,\pm,\ldots \end{cases} \quad (38)$$

Generally,

$$\rho_{it,it(l)} = \begin{cases} 1 & ,l = 0 \\ \frac{\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\varphi_{k.ij}\gamma_{it(l),jt(k)}}{\gamma_{it,it}} & ,l = \pm1,\pm2,\pm\cdots \end{cases} \quad (39)$$

Cross-Autocovariance and Cross-Autocorrelations

The Cross-autocovariance is the covariance between Y_{it} *and* $Y_{jt(k)}$.

The approach of obtaining the cross-autocorrelations is not different from the earlier section, except the cross-autocovariances and cross-autocorrelations involve two distinct time variables Y_{it} *and* Y_{jt},

Multiplying Equation (2) by Y_{2t-1} and taking the expectations, we have

$$\begin{aligned} E(Y_{1t}Y_{2t-1}) = {} & \varphi_{1.11}E(Y_{2t-1}Y_{1t-1}) + \varphi_{1.12}E(Y_{2t-1}Y_{2t-1}) + \cdots + \varphi_{1.1n}E(Y_{2t-1}Y_{nt-1}) \\ & + \varphi_{2.11}E(Y_{2t-1}Y_{1t-2}) + \varphi_{2.12}E(Y_{2t-1}Y_{2t-2}) + \cdots + \varphi_{2.1n}E(Y_{2t-1}Y_{nt-2}) \\ & + \cdots + \varphi_{p.11}E(Y_{2t-1}Y_{1t-p}) + \varphi_{p.12}E(Y_{2t-1}Y_{2t-p}) + \cdots \\ & + \varphi_{p.1n}E(Y_{2t-1}Y_{nt-p}) \\ & + E(Y_{2t-1}\epsilon_{1t}) \end{aligned} \tag{40}$$

$E(Y_{1t}Y_{2t-1}) = \gamma_{1t,2t(1)}$ (Cross-covariance of Y_{1t} and $Y_{2t(1)}$)

$$\begin{aligned} & \gamma_{1t,2t(1)} \\ & = \varphi_{1.11}\gamma_{2t,1t} + \varphi_{1.12}\gamma_{2t,2t} + \cdots + \varphi_{1.1n}\gamma_{2t,nt} + \varphi_{2.11}\gamma_{2t,1t(1)} + \varphi_{2.12}\gamma_{2t,2t(1)} + \cdots \\ & + \varphi_{2.1n}\gamma_{2t,nt(1)} + \cdots + \varphi_{p.11}\gamma_{2t,1t(p-1)} + \varphi_{p.12}\gamma_{2t,2t(p-1)} + \cdots \\ & + \varphi_{p.1n}\gamma_{2t,nt(p-1)} \end{aligned} \tag{41}$$

$E(Y_{2t-1}\epsilon_{1t}) = 0$(uncorrelated stationary processes at zero time lag)

$$\gamma_{1t,2t(1)} = \sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\gamma_{2t(1),jt(k)} \tag{42}$$

Dividing Equation (42) by $\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}$ produces the autocorrelation

$$\rho_{1t,2t(1)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n} \varphi_{k.1j}\gamma_{2t(1),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \tag{43}$$

$\rho_{1t,2t(1)}$ is the cross-autocorrelation of Y_{1t} and $Y_{2t(1)}$

Also, Multiplying Equation (2) by Y_{2t-2} and taking the expectations, we have

$$E(Y_{1t}Y_{2t-2}) = \varphi_{1.11}E(Y_{2t-2}Y_{1t-1}) + \varphi_{1.12}E(Y_{2t-2}Y_{2t-1}) + \cdots + \varphi_{1.1n}E(Y_{2t-2}Y_{nt-1}) + \varphi_{2.11}E(Y_{2t-2}Y_{1t-2}) + \varphi_{2.12}E(Y_{2t-2}Y_{2t-2}) + \cdots + \varphi_{2.1n}E(Y_{2t-2}Y_{nt-2}) + \cdots + \varphi_{p.11}E(Y_{2t-2}Y_{1t-p}) + \varphi_{p.12}E(Y_{2t-2}Y_{2t-p}) + \cdots + \varphi_{p.1n}E(Y_{2t-2}Y_{nt-p}) + E(Y_{2t-2}\epsilon_{1t}) \quad (44)$$

$E(Y_{1t}Y_{2t-2}) = \gamma_{1t,2t(2)}$ (Cross-covariance of Y_{1t} *and* $Y_{2t(2)}$)

$$\gamma_{1t,2t(2)} = \varphi_{1.11}\gamma_{1t,2t(1)} + \varphi_{1.12}\gamma_{2t,2t(1)} + \cdots + \varphi_{1.1n}\gamma_{2t(1),nt} + \varphi_{2.11}\gamma_{2t,1t} + \varphi_{2.12}\gamma_{2t,2t} + \cdots + \varphi_{2.1n}\gamma_{2t,nt} + \cdots + \varphi_{p.11}\gamma_{2t,1t(p-2)} + \varphi_{p.12}\gamma_{2t,2t(p-2)} + \cdots + \varphi_{p.1n}\gamma_{2t,nt(p-2)} \quad (45)$$

$E(Y_{2t-2}\epsilon_{1t}) = 0$(uncorrelated stationary processes at zero time lag)

$$\gamma_{1t,2t(2)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{2t(2),jt(k)} \quad (46)$$

Dividing Equation (46) by $\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}$ produces the autocorrelation

$$\rho_{1t,2t(2)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{2t(2),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \quad (47)$$

$\rho_{1t,2t(2)}$ is the cross-autocorrelation of Y_{1t} *and* $Y_{2t(2)}$

Multiplying Equation (2) by Y_{2t-p} and taking the expectations,

$$\rho_{1t,2t(p)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{2t(p),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \quad (48)$$

Therefore,

$$\rho_{1t,2t(l)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.1j}\gamma_{2t(l),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}}, l = 0, \pm 1, \pm 2, \pm, \ldots \qquad (49)$$

Multiplying Equation (3) by Y_{1t-1} and taking the expectations,

$$\begin{aligned} &E(Y_{2t}Y_{1t-1}) \\ &= \varphi_{1.21}E(Y_{1t-1}Y_{1t-1}) + \varphi_{1.22}E(Y_{1t-1}Y_{2t-1}) + \cdots + \varphi_{1.2n}E(Y_{1t-1}Y_{nt-1}) \\ &+ \varphi_{2.21}E(Y_{1t-1}Y_{1t-2}) + \varphi_{2.22}E(Y_{1t-1}Y_{2t-2}) + \cdots + \varphi_{2.2n}E(Y_{1t-1}Y_{nt-2}) + \cdots \\ &+ \varphi_{p.21}E(Y_{1t-1}Y_{1t-p}) + \varphi_{p.22}E(Y_{1t-1}Y_{2t-p}) + \cdots + \varphi_{p.2n}E(Y_{1t-1}Y_{nt-p}) \\ &+ E(Y_{1t-1}\epsilon_{2t}) \end{aligned} \qquad (50)$$

$E(Y_{2t}Y_{1t-1}) = \gamma_{2t,1t(1)}$ (Cross-covariance of Y_{2t} *and* $Y_{1t(1)}$)

$$\begin{aligned} &\gamma_{2t,1t(1)} \\ &= \varphi_{1.21}\gamma_{1t,1t} + \varphi_{1.22}\gamma_{1t,2t} + \cdots + \varphi_{1.2n}\gamma_{1t,nt} + \varphi_{2.21}\gamma_{1t,2t(1)} + \varphi_{2.22}\gamma_{1t,2t(1)} + \cdots \\ &+ \varphi_{2.2n}\gamma_{1t,nt(1)} + \cdots + \varphi_{p.21}\gamma_{1t,1t(p-1)} + \varphi_{p.22}\gamma_{1t,2t(p-1)} + \cdots \\ &+ \varphi_{p.2n}\gamma_{1t,nt(p-1)} \end{aligned} \qquad (51)$$

$E(Y_{1t-1}\epsilon_{2t}) = 0$(uncorrelated stationary processes at zero time lag)

$$\gamma_{2t,1t(1)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{1t(1),jt(k)} \qquad (52)$$

Dividing Equation (52) by $\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}$ produces the autocorrelation

$$\rho_{2t,1t(1)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{1t(1),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \qquad (53)$$

$\rho_{2t,1t(1)}$ is the cross-autocorrelation of Y_{2t} *and* $Y_{1t(1)}$

Also, multiplying Equation (3) by Y_{1t-2} and taking the expectations,

$$\begin{aligned} &E(Y_{2t}Y_{1t-2}) \\ &= \varphi_{1.21}E(Y_{1t-2}Y_{1t-1}) + \varphi_{1.22}E(Y_{1t-2}Y_{2t-1}) + \cdots + \varphi_{1.2n}E(Y_{1t-2}Y_{nt-1}) \\ &+ \varphi_{2.21}E(Y_{1t-2}Y_{1t-2}) + \varphi_{2.22}E(Y_{1t-2}Y_{2t-2}) + \cdots + \varphi_{2.2n}E(Y_{1t-2}Y_{nt-2}) + \cdots \\ &+ \varphi_{p.21}E(Y_{1t-2}Y_{1t-p}) + \varphi_{p.22}E(Y_{1t-2}Y_{2t-p}) + \cdots + \varphi_{p.2n}E(Y_{1t-2}Y_{nt-p}) \\ &+ E(Y_{1t-2}\epsilon_{2t}) \end{aligned} \tag{54}$$

$E(Y_{2t}Y_{1t-2}) = \gamma_{2t,1t(2)}$ (Cross-covariance of Y_{2t} *and* $Y_{1t(2)}$)

$$\begin{aligned} &\gamma_{2t,1t(2)} \\ &= \varphi_{1.21}\gamma_{1t,1t(1)} + \varphi_{1.22}\gamma_{2t,1t(1)} + \cdots + \varphi_{1.2n}\gamma_{nt,1t(1)} + \varphi_{2.21}\gamma_{1t,1t} + \varphi_{2.22}\gamma_{1t,2t} + \cdots \\ &+ \varphi_{2.2n}\gamma_{1t,nt} + \cdots + \varphi_{p.21}\gamma_{1t,1t(p-2)} + \varphi_{p.22}\gamma_{1t,2t(p-2)} + \cdots \\ &+ \varphi_{p.2n}\gamma_{1t,nt(p-2)} \end{aligned} \tag{55}$$

$E(Y_{1t-2}\epsilon_{2t}) = 0$(uncorrelated stationary processes at zero time lag)

$$\gamma_{2t,1t(2)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{1t(2),jt(k)} \tag{56}$$

Dividing Equation (56) by $\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}$ produces the autocorrelation

$$\rho_{2t,1t(2)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{1t(2),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \tag{57}$$

$\rho_{2t,1t(2)}$ is the cross-autocorrelation of Y_{2t} *and* $Y_{1t(2)}$

Multiplying Equation (3) by Y_{1t-p} and taking the expectations,

$$\rho_{2t,1t(p)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{1t(p),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \quad (58)$$

Therefore,

$$\rho_{2t,1t(l)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.2j}\gamma_{1t(l),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}}, l = 0, \pm 1, \pm 2, \pm, \dots \quad (59)$$

Multiplying Equation (4) by Y_{1t-1} and taking the expectations, we have

$$\begin{aligned} E(Y_{mt}Y_{1t-1}) &= \varphi_{1.m1}E(Y_{1t-1}Y_{1t-1}) + \varphi_{1.m2}E(Y_{1t-1}Y_{2t-1}) + \cdots + \varphi_{1.mn}E(Y_{1t-1}Y_{nt-1}) \\ &+ \varphi_{2.m1}E(Y_{1t-1}Y_{1t-2}) + \varphi_{2.m2}E(Y_{1t-1}Y_{2t-2}) + \cdots + \varphi_{2.mn}E(Y_{1t-1}Y_{nt-2}) \\ &+ \cdots + \varphi_{p.m1}E(Y_{1t-1}Y_{1t-p}) + \varphi_{p.m2}E(Y_{1t-1}Y_{2t-p}) + \cdots \\ &+ \varphi_{p.mn}E(Y_{1t-1}Y_{nt-p}) \\ &+ E(Y_{1t-1}\epsilon_{mt}) \end{aligned} \quad (60)$$

$E(Y_{mt}Y_{1t-1}) = \gamma_{mt,1t(1)}$ (Cross-covariance of Y_{mt} *and* $Y_{1t(1)}$)

$$\begin{aligned} \gamma_{mt,1t(1)} &= \varphi_{1.m1}\gamma_{1t,1t} + \varphi_{1.m2}\gamma_{1t,2t} + \cdots + \varphi_{1.mn}\gamma_{1t,nt} + \varphi_{2.m1}\gamma_{1t,1t(1)} + \varphi_{2.m2}\gamma_{1t,2t(1)} \\ &+ \cdots + \varphi_{2.mn}\gamma_{1t,nt(1)} + \cdots + \varphi_{p.m1}\gamma_{1t,1t(p-1)} + \varphi_{p.m2}\gamma_{1t,2t(p-1)} + \cdots \\ &+ \varphi_{p.mn}\gamma_{1t,nt(p-1)} \end{aligned} \quad (61)$$

$E(Y_{1t-1}\epsilon_{mt}) = 0$(uncorrelated stationary processes at zero time lag)

Equation (61) further reduces to

$$\gamma_{mt,1t(1)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{1t(1),jt(k)} \quad (62)$$

Dividing Equation (64) by $\sqrt{\gamma_{1t,1t}\gamma_{mt,mt}}$, it produces cross-autocorrelation presented as

$$\rho_{mt,1t(1)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{1t(1),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{mt,mt}}} \tag{63}$$

Multiplying Equation (4) by Y_{1t-2} and taking the expectations, we have

$$\begin{aligned} E(Y_{mt}Y_{1t-2}) = {} & \varphi_{1.m1}E(Y_{1t-2}Y_{1t-1}) + \varphi_{1.m2}E(Y_{1t-2}Y_{2t-1}) + \cdots + \varphi_{1.mn}E(Y_{1t-2}Y_{nt-1}) \\ & + \varphi_{2.m1}E(Y_{1t-2}Y_{1t-2}) + \varphi_{2.m2}E(Y_{1t-2}Y_{2t-2}) + \cdots + \varphi_{2.mn}E(Y_{1t-2}Y_{nt-2}) \\ & + \cdots + \varphi_{p.m1}E(Y_{1t-2}Y_{1t-p}) + \varphi_{p.m2}E(Y_{1t-2}Y_{2t-p}) + \cdots \\ & + \varphi_{p.mn}E(Y_{1t-2}Y_{nt-p}) \\ & + E(Y_{1t-2}\epsilon_{mt}) \end{aligned} \tag{64}$$

$E(Y_{mt}Y_{1t-2}) = \gamma_{mt,1t(2)}$ (Cross-covariance of Y_{mt} *and* $Y_{1t(2)}$)

$$\begin{aligned} \gamma_{mt,1t(2)} & \\ = {} & \varphi_{1.m1}\gamma_{1t,1t(1)} + \varphi_{1.m2}\gamma_{2t,1t(1)} + \cdots + \varphi_{1.mn}\gamma_{nt,1t(1)} + \varphi_{2.m1}\gamma_{1t,1t} + \varphi_{2.m2}\gamma_{1t,2t} + \cdots \\ & + \varphi_{2.mn}\gamma_{1t,nt} + \cdots + \varphi_{p.m1}\gamma_{1t,1t(p-2)} + \varphi_{p.m2}\gamma_{1t,2t(p-2)} + \cdots \\ & + \varphi_{p.mn}\gamma_{1t,nt(p-2)} \end{aligned} \tag{65}$$

$E(Y_{1t-2}\epsilon_{mt}) = 0$(uncorrelated stationary processes at zero time lag)

Equation (65) further reduces to

$$\gamma_{mt,1t(2)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{1t(2),jt(k)} \tag{66}$$

Dividing Equation (66) by $\sqrt{\gamma_{1t,1t}\gamma_{mt,mt}}$, it produces cross-autocorrelation presented as

$$\rho_{mt,1t(2)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{1t(2),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{mt,mt}}} \tag{67}$$

Multiplying Equation (4) by Y_{1t-p} and taking the expectations,

$$\rho_{mt,1t(p)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(p),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} \tag{68}$$

Therefore,

$$\rho_{mt,1t(l)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{1t(l),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}}, l = 0, \pm 1, \pm 2, \pm, \ldots \tag{79}$$

Multiplying Equation (4) by Y_{2t-1} and taking the expectations,

$$\begin{aligned} E(Y_{mt}Y_{2t-1}) = {} & \varphi_{1.m1}E(Y_{2t-1}Y_{1t-1}) + \varphi_{1.m2}E(Y_{2t-1}Y_{2t-1}) + \cdots + \varphi_{1.mn}E(Y_{2t-1}Y_{nt-1}) \\ & + \varphi_{2.m1}E(Y_{2t-1}Y_{1t-2}) + \varphi_{2.m2}E(Y_{2t-1}Y_{2t-2}) + \cdots + \varphi_{2.mn}E(Y_{2t-1}Y_{nt-2}) \\ & + \cdots + \varphi_{p.m1}E(Y_{2t-1}Y_{1t-p}) + \varphi_{p.m2}E(Y_{2t-1}Y_{2t-p}) + \cdots \\ & + \varphi_{p.mn}E(Y_{2t-1}Y_{nt-p}) + E(Y_{2t-1}\epsilon_{mt}) \end{aligned} \tag{70}$$

$E(Y_{mt}Y_{2t-1}) = \gamma_{mt,2t(1)}$ (Cross-covariance of Y_{mt} and $Y_{2t(1)}$)

$$\begin{aligned} \gamma_{mt,2t(1)} = {} & \varphi_{1.m1}\gamma_{1t,2t} + \varphi_{1.m2}\gamma_{2t,2t} + \cdots + \varphi_{1.mn}\gamma_{2t,nt} + \varphi_{2.m1}\gamma_{2t,1t(1)} + \varphi_{2.m2}\gamma_{2t,2t(1)} \\ & + \cdots + \varphi_{2.mn}\gamma_{2t,nt(1)} + \cdots + \varphi_{p.m1}\gamma_{2t,1t(p-1)} + \varphi_{p.m2}\gamma_{2t,2t(p-1)} + \cdots \\ & + \varphi_{p.mn}\gamma_{2t,nt(p-1)} \end{aligned} \tag{71}$$

$E(Y_{2t-1}\epsilon_{mt}) = 0$(uncorrelated stationary processes at zero time lag)

Equation (71) further reduces to

$$\gamma_{mt,2t(1)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(1),jt(k)} \tag{72}$$

Dividing Equation (72) by $\sqrt{\gamma_{2t,2t}\gamma_{mt,mt}}$, it produces cross-autocorrelation presented as

$$\rho_{mt,2t(1)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(1),jt(k)}}{\sqrt{\gamma_{2t,2t}\gamma_{mt,mt}}} \qquad (73)$$

Multiplying Equation (4) by Y_{2t-2} and taking the expectations,

$$\begin{aligned} E(Y_{mt}Y_{2t-2}) = {} & \varphi_{1.m1}E(Y_{2t-2}Y_{1t-1}) + \varphi_{1.m2}E(Y_{2t-2}Y_{2t-1}) + \cdots + \varphi_{1.mn}E(Y_{2t-2}Y_{nt-1}) \\ & + \varphi_{2.m1}E(Y_{2t-2}Y_{1t-2}) + \varphi_{2.m2}E(Y_{2t-2}Y_{2t-2}) + \cdots + \varphi_{2.mn}E(Y_{2t-2}Y_{nt-2}) \\ & + \cdots + \varphi_{p.m1}E(Y_{2t-2}Y_{1t-p}) + \varphi_{p.m2}E(Y_{2t-2}Y_{2t-p}) + \cdots \\ & + \varphi_{p.mn}E(Y_{2t-2}Y_{nt-p}) + E(Y_{2t-2}\epsilon_{mt}) \end{aligned} \qquad (74)$$

$E(Y_{mt}Y_{2t-2}) = \gamma_{mt,2t(2)}$ (Cross-covariance of Y_{mt} and $Y_{2t(2)}$)

$$\begin{aligned} \gamma_{mt,2t(2)} & \\ = {} & \varphi_{1.m1}\gamma_{1t,2t(1)} + \varphi_{1.m2}\gamma_{2t,2t(1)} + \cdots + \varphi_{1.mn}\gamma_{nt,2t(1)} + \varphi_{2.m1}\gamma_{1t,2t} + \varphi_{2.m2}\gamma_{2t,2t} + \cdots \\ & + \varphi_{2.mn}\gamma_{2t,nt} + \cdots + \varphi_{p.m1}\gamma_{2t,1t(p-2)} + \varphi_{p.m2}\gamma_{2t,2t(p-2)} + \cdots \\ & + \varphi_{p.mn}\gamma_{2t,nt(p-2)} \end{aligned} \qquad (75)$$

$E(Y_{2t-2}\epsilon_{mt}) = 0$(uncorrelated stationary processes at zero time lag)

Equation (75) further reduces to

$$\gamma_{mt,2t(2)} = \sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(2),jt(k)} \qquad (76)$$

Dividing Equation (76) by $\sqrt{\gamma_{2t,2t}\gamma_{mt,mt}}$, it produces cross-autocorrelation presented as

$$\rho_{mt,2t(2)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(2),jt(k)}}{\sqrt{\gamma_{2t,2t}\gamma_{mt,mt}}} \qquad (77)$$

Multiplying Equation (4) by Y_{2t-p} and taking the expectations,

$$\rho_{mt,2t(p)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(p),jt(k)}}{\sqrt{\gamma_{2t,2t}\gamma_{mt,mt}}} \quad (78)$$

Therefore,

$$\rho_{mt,2t(l)} = \frac{\sum_{k=1}^{p}\sum_{j=1}^{n}\varphi_{k.mj}\gamma_{2t(l),jt(k)}}{\sqrt{\gamma_{2t,2t}\gamma_{mt,mt}}}, l = 0, \pm1, \pm2, \pm, \dots \quad (79)$$

Generally,

$$\rho_{it,jt(k)} = \frac{\sum_{k=1}^{p}\varphi_{k.ij}\gamma_{it(l),jt(k)}}{\sqrt{\gamma_{it,it}\gamma_{jt,jt}}}, k, l = 0, \pm1, \pm2, \pm, \dots (k \neq l), (i \neq j) \quad (80)$$

Equation (80) is the general model for cross-autocorrelation at lag k.

RESULTS:

In this section, numerical estimates of the variances, auto-covariances, cross-autocovariances, autocorrelations and cross-autocorrelations are presented.

Preliminary Estimates

Here, regression equations for Y_{1t} and Y_{2t} with estimated parameters for the first two lags of each response and predictive time variable are presented. The lag length is arbitrary chosen for the estimate of more than one parameter in each model. Therefore,

$$Y_{it} = \sum_{k=1}^{2}\sum_{j=1}^{2}\varphi_{k.ij}Y_{jt-k} + \epsilon_{it} \quad (81)$$

$for\ i = 1; j = 1,2; k = 1,2,$

$$\hat{Y}_{1t} = -0.5327Y_{1t-1} + 0.1310Y_{2t-1} - 0.1233Y_{1t-2} - 0.0694Y_{2t-2} \quad (82)$$

$$\hat{\sigma}^2_{\epsilon_{1t}} = 0.0006930$$

$for\ i = 2; j = 1,2; k = 1,2,$

$$\hat{Y}_{2t} = -0.179Y_{1t-1} + 0.1518Y_{2t-1} - 0.053Y_{1t-2} + 0.1001Y_{2t-2} \quad (83)$$

$$\hat{\sigma}^2_{\epsilon_{2t}} = 0.001616$$

The parameter estimates in Equations (82) and (83) are the input values for the computations of the autocorrelations. Further inputs for the autocorrelations and cross-autocorrelations are the estimates of autocovariances and cross-autocovariances whose model is presented as,

$$\gamma_{it,jt(l)} = \sum_{k=1}^{2} \varphi_{k.ij}\gamma_{it(l),jt(k)} \quad (84)$$

$for\ i = 1,2; j = 1,2; l = 1,2,3$, Table 1 presents the estimates.

Table 1: Autocovariance and Cross-autocovariance Estimates

$Lag_{\ l}$	$\gamma_{1t,1t(l)}$	$\gamma_{1t,2t(l)}$	$\gamma_{2t,1t(l)}$	$\gamma_{2t,2t(l)}$
1	-0.00048692	0.00026391	-0.00014413	0.00030496
2	0.000125890	-0.00020959	0.00000518	0.00016932
3	0.000053980	0.00011425	-0.00009412	0.00016829

The lag length in Table 1 is arbitrary chosen for increasing numbers of autocorrelations and cross-autocorrelations.

Estimation of Variance of Y_{1t}

From Equation (11), k =1, 2; j = 1,2 result to the following model

$$\gamma_{1t,1t} = \emptyset_{1.11}\gamma_{1t,1t(1)} + \emptyset_{1.12}\gamma_{1t,2t(1)} + \emptyset_{2.11}\gamma_{1t,1t(2)} + \emptyset_{2.12}\gamma_{1t,2t(2)} + \sigma^2_{\epsilon_{1t}} \quad (85)$$

From the estimates of Equation (84) and Table 1, therefore,

$$\hat{\gamma}_{1t,1t} = 0.0009860$$

Autocorrelations of Y_{1t} and $Y_{1t(l)}$

From Equation (22),

$$\rho_{1t,1t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{2}\sum_{j=1}^{2}\varphi_{k.1j}\gamma_{1t(l),jt(k)}}{\gamma_{1t,1t}} & , l = \pm 1, \pm 2, \pm \cdots \end{cases} \tag{86}$$

Cross-Autocorrelations of Y_{1t} and $Y_{2t(l)}$

From Equation (49),

$$\rho_{1t,2t(l)} = \frac{\sum_{k=1}^{2}\sum_{j=1}^{2}\varphi_{k.1j}\gamma_{2t(l),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} , l = 0,1,2, \dots \tag{87}$$

Estimation of Variance of Y_{2t}

From Equation (25), k =1, 2; j = 1,2 result to the following model

$$\gamma_{2t,2t} = \emptyset_{1.21}\gamma_{2t,1t(1)} + \emptyset_{1.22}\gamma_{2t,2t(1)} + \emptyset_{2.21}\gamma_{2t,1t(2)} + \emptyset_{2.22}\gamma_{2t,2t(2)} + \sigma_{\epsilon_{2t}}^{2} \tag{88}$$

From the estimates of Equation (83) and Table 1, therefore,

$$\hat{\gamma}_{2t,2t} = 0.0017048$$

Autocorrelations of Y_{2t} *and* $Y_{2t(l)}$

From Equation (38),

$$\rho_{2t,2t(l)} = \begin{cases} 1 & , l = 0 \\ \dfrac{\sum_{k=1}^{2}\sum_{j=1}^{2}\varphi_{k.2j}\gamma_{2t(l),jt(k)}}{\gamma_{1t,1t}} & , l = \pm 1, \pm 2, \pm \cdots \end{cases} \tag{89}$$

Cross-Autocorrelations of Y_{2t} and $Y_{1t(l)}$

From Equation (59),

$$\rho_{2t,1t(l)} = \frac{\sum_{k=1}^{2}\sum_{j=1}^{2}\varphi_{k.2j}\gamma_{1t(l),jt(k)}}{\sqrt{\gamma_{1t,1t}\gamma_{2t,2t}}} , l = 0,1,2, \dots \tag{90}$$

SUMMARY OF RESULTS

Table1: Estimates from MAR Model Parameters and Direct Computations

Auto/Cross-Autocorrelation Measure	Variable	Estimates from MAR Models	Direct Computations Using Lagged Variables
Autocorrelations of X_{1t} and $X_{1t(k)}$	$\rho_{1t,1t(1)}$	-0.564	-0.497
	$\rho_{1t,1t(2)}$	0.127	0.128
	$\rho_{1t,1t(3)}$	0.040	0.055
Cross-Autocorrelations of X_{1t} and $X_{2t(k)}$	$\rho_{1t,2t(1)}$	0.205	0.208
	$\rho_{1t,2t(2)}$	-0.161	-0.165
	$\rho_{1t,2t(3)}$	0.081	0.089
Autocorrelations of X_{2t} and $X_{2t(k)}$	$\rho_{2t,2t(1)}$	0.183	0.183
	$\rho_{2t,2t(2)}$	0.102	0.101
	$\rho_{2t,2t(3)}$	0.090	0.101
Cross-Autocorrelations of X_{2t} and $X_{1t(k)}$	$\rho_{2t,1t(1)}$	-0.110	-0.112
	$\rho_{2t,1t(2)}$	0.003	0.004
	$\rho_{2t,1t(3)}$	-0.073	-0.073

SUMMARY

Multivariate time series models, such as Vector Autoregressive Models (VAMs), construct various associations between response time variables and associated predictive lag terms. The paper's main focus was on using a model parameter technique to define and evaluate the fundamental features of two-dimensional Vector Autoregressive Models. This method differs from the typical way of using lagging variables across time as input variables for variance, autocovariance, cross-autocovariance, autocorrelations, and cross-autocorrelations computations. Crude oil production quantity and price return series representing Y_{1t} and Y_{2t} were utilized as response time variables in bivariate VAR models to illustrate and validate the model developed. The variances were calculated using model parameter estimates and validated with numerical examples. To verify the autocorrelations and cross-autocorrelation models created using the model parameters, estimates obtained from ordinary least squares regression and other statistical measures were utilized as entry variables. This method differs from [1], [8] and [9], a typical methodology of using lag terms of response time variables for direct computations of the above measures. Table 1 summarizes the results obtained using the two approaches, and numerical illustrations show that the estimates from the two methods for computations of variances, autocovariances, autocorrelations, cross-autocovariances and cross-autocorrelations for Vector Autoregressive Models are accurate and comparable.

CONCLUSION

Every model in statistics has some basic properties that describe the model's distribution or process, depending on the situation. The goal of this research was to develop a model-based approach for computatoion of some basic VAR model properties. The results of this study show that the model-based method of computing autocovariances, cross-autocovariances, autocorrelations, and cross-autocorrelations in VAR models provides the same level of accuracy as the traditional method. This approach is new and complimentary to the already established method of examining the aforementioned statistical properties vector Autoregressive Models.

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