Numerical Solution of Optimal Control Problems Using Improved Euler Method

By

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Abstract

This paper discussed the extension of the improved Euler method to the solution of optimal control problems with the state constrained by differential equations. The method combined the classical method with the numerical algorithm of Euler by embedding each of the boundary conditions from the Hamiltonian into the algorithm of Euler and allow the system to undergo the iterative process of the Euler until the gradient norm of the objective function approaches certain tolerance(according to [1], [2], [12] and [13]. A suitable formula for updating the control variable $u(t)$ *as akin to [2] at each one-dimensional search was developed. The numerical results generated by this process demonstrated the stability and robustness of the method as it triumphs over reasonable number of problems.*

Keywords*: Euler method, Optimal control problems, Hamiltonian, Algorithm process, Control variable.*

Introduction

In this paper, we considered the minimization of general form of optimal control problems without delay in the state equations. In the past, the favored approach for solving optimal control problems was that of indirect methods. In the indirect method, the calculus of variation is employed to obtain the first-order optimality conditions. These conditions result in a two-point or multi-point (for complex problems) boundary value problems which have special structure because it arises from taking the derivative of a Hamiltonian which is of the form:

$$\frac{\partial H(x(t),U(t),\lambda(t),t)}{\partial \lambda(t)} = \dot{x}(t) \qquad (1)$$

and $$-\frac{\partial H(x(t),U(t),\lambda(t),t)}{\partial \lambda(t)} = \dot{\lambda}(t) \qquad (2)$$

Where $x(t)$ is the state variable or trajectory of the system, $u(t)$ the control input or variable, $H = L + \lambda^T a - N^T b$ above is called the Augmented Hamiltonian function which we shall derived later. The beauty of the indirect method is that the state and the co-state or ad-joints are solved for, and the resulting solution is readily verified to be an external trajectory. According to [17], the major disadvantage of the method is that the boundary value problem is often extremely difficult to solve particularly for problems with interior point constraints. In order to overcome this great disadvantage, the first-order optimality condition which arises from taking the derivative of the Hamiltonian *H* above was embedded in the algorithm of the improved Euler method with the objective of developing a robust method of solution that is capable of solving different classes of optimal control problems.

2. Developmental Approach to Optimal Control Problems

In order to establish necessary conditions that guaranteed optimality in optimal control problems, we consider a plant whose system is described by the first order differential equation below:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{3}$$

With a Bolza form of performance measure

$$J(.) = s(x(t))\big|_{t=t_f} + \int_{t_0}^{t_f} v(x(t), u(t), t)dt \tag{4}$$

Where equation (3) is state equation, (4) is the objective function, the first and the second term of (4) are terminal cost and integral cost function respectively [14].

Our objective is to find the control input $u(t)$ and the trajectory $x(t)$ of a system from initial time t_0 to the final time t_f such that the performance index (4) subject to the constraint (3) is minimized.

Equation (4) can be written in the form

$$J(.) = \int_{t_0}^{t_f} v(x(t), u(t), t)dt + \int_{t_0}^{t_f} \frac{\partial s(x(t), t)}{\partial t} dt \tag{5}$$

For (5) to be simplified, we apply chain rule to get

$$J(.)=\int_{t_0}^{t_f} v(x(t),u(t),t)dt+\int_{t_0}^{t_f}\left[\left(\frac{\partial s(.)}{\partial x(t)}\right)^T \dot{x}(t)+\frac{\partial s(.)}{\partial t}.1\right]dt \qquad (6)$$

Transformation of constraints equation (3) and (4) to unconstraint one via Lagrange Multiplier method we have.

$$\text{J(.)}=\int_{t_0}^{t_f} v(x(t),u(t),t)dt+\int_{t_0}^{t_f}\left[\left(\frac{\partial s(.)}{\partial x(t)}\right)^T \dot{x}(t)+\frac{\partial s(.)}{\partial t}.1\right]dt$$

$$+\int_{t_0}^{t_f}\lambda^T(t)[f(x(t),u(t),t)-\dot{x}(t)]dt \qquad (7)$$

According to (1), the Lagrange function form of (7) can be written as

$$\int_{t_0}^{t_f} L(x(t),u(t),\lambda(t),t)dt \qquad (8)$$

Where (8) i.e.

$$\int_{t_0}^{t_f} L(x(t),u(t),\lambda(t),t)dt = v(x(t),u(t),\lambda(t),t)+$$

$$+\lambda^T(t)[f(x(t),u(t),t)]+\left[\left(\frac{\partial s(x(t),t)}{\partial x(t)}\right)\right]^T \dot{x}(t)+\frac{\partial s(x(t),t)}{\partial t}-\lambda^T(t)\dot{x}(t)$$

Which can be written in Hamiltonian form as

$$H(x(t),u(t),\lambda^T(t),t)]+\left[\left(\frac{\partial s(x(t),t)}{\partial x(t)}\right)\right]^T \dot{x}(t)+\frac{\partial s(x(t),t)}{\partial t}-\lambda^T(t)\dot{x}(t)$$

Where H is the Hamiltonian function

If the objective function is perturbed according to [19], we have

$$J_a(.) = \int_{t_0}^{t_f+\delta t_f} [v(x^*(t)+\delta x(t), u^*(t)+\delta u(t), t) + \left(\frac{\partial s(.)}{\partial x(t)}\right)_*^T (\dot{x}^*(t)+\delta \dot{x}(t)) +$$
$$\left(\frac{\partial s(.)}{\partial t}\right)_* + \lambda^T(t) f(x^*(t)+\delta x(t), u^*(t)+\delta u(t), t) - (\dot{x}^*(t)+\delta \dot{x}(t))]dt \qquad (9)$$

$$= \int_{t_0}^{t_f+\delta t_f} L_p(.)dt = \int_{t_0}^{t_f} L_p(.)dt + \int_{t_f}^{t_f+\delta t_f} L_p(.)dt \cong \int_{t_0}^{t_f} L_p(.)dt + L(.)/_{t=t_f} \delta t_f$$

Where $L_p(.)$ is the perturbed model of Lagrange multiplier

The Variation of the functional value can be expressed as

$$\Delta J_a = J_a(.)dt - J(.) = \int_{t_0}^{t_f+\delta t_f} L_p(.)dt - \int_{t_0}^{t_f} L(.)dt \cong \int_{t_0}^{t_f} L_p(.)dt + L(.)/_{t=t_f} \delta t_f - \int_{t_0}^{t_f} L(.)dt$$

$$\int_{t_0}^{t_f} [L_p(.) - L(.)]dt = \int_{t_0}^{t_f} [L_p(x^*(t)+\delta x(t), u^*(t)+\delta u(t), t) - L(x^*(t), u^*(t), \lambda^*(t), t)]dt$$
$$+\dot{x}^*(t), u^*(t), \lambda^*(t), t)/_{t=t_f} \delta t_f \qquad (10)$$

Using the Taylor series expansion and integration by parts rule to (10) as in [18], [19] and applied by [17], Also considering the first variation of the functional we obtain

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial L(.)}{\partial x(t)} - \frac{d}{dt}\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_*^T \delta x(t)\right] dt +$$
$$\int_{t_0}^{t_f} \left(\frac{\partial L(.)}{\partial u(t)}\right)^T \delta u(t)dt + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_*^T \delta x(t)/_{t=t_f} + L(.)_*/_{t=t_f} \delta t_f \qquad (11)$$

Lemma 1

Let g(t) and x(t) be continuous and integrable over a close interval t_0 and t_f then,

$\int_{t_0}^{t_f} g(t)\delta x(t)dt = 0$ iff $g(t) = 0$ at every point over the integral $\left[t_0, t_f\right]$

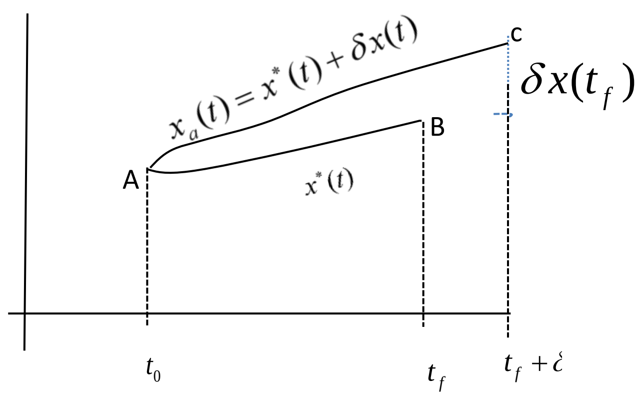
From (11) using lemma 1, we have

$$\frac{\partial L(.)}{\partial x(t)} - \frac{d}{dt}\left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_* = 0 \tag{12}$$

$$\left(\frac{\partial L(.)}{\partial u(t)}\right)_* = 0 \tag{13}$$

Finally

$$\delta J \cong L(.)_* /_{t=t_f}\ \delta t_f + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_* \delta x(t) /_{t=t_f} \tag{14}$$



$$\delta x_f = \delta x(t_f) + \dot{x}_a /_{t=t_f}\ \delta t_f = \delta x(t_f) + (\dot{x}^*(t) + \delta \dot{x}(t)) /_{t=t_f}\ \delta t_f$$

$$\cong \delta x(t_f) + (\dot{x}^*(t_f)\delta t_f \tag{15}$$

Substitute (15) into (14) we have

$$\delta J \cong L(.)_* /_{t=t_f}\ \delta t_f + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_* /_{t=t_f}\ (\delta x_f - \dot{x}(t_f))\delta t_f \tag{16}$$

Simplification of (16) give

$$\delta J \cong L(.) - \left[\frac{\partial L(.)}{\partial \dot{x}(t)}\dot{x}(t)\right]_* /_{t=t_f}\ \delta t_f + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_*^T /_{t=t_f}\ \delta x_f$$

If $\delta J = 0$ which is regarded as necessary condition, then

$$L(.) - \left[\frac{\partial L(.)}{\partial \dot{x}(t)}\dot{x}(t)\right]_* /_{t=t_f}\ \delta t_f + \left(\frac{\partial L(.)}{\partial \dot{x}(t)}\right)_*^T /_{t=t_f}\ \delta x_f = 0 \qquad (17)$$

Where

$$L(.) = H(x(t), u(t), \lambda(t), t) + \frac{\partial s(.)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t) \qquad (18)$$

Using (18) in (12), (13) and (17). From (12), we get

$$\frac{\partial}{\partial x(t)}\left[H(x(t), u(t), \lambda(t), t) + \frac{\partial s(.)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t)\right] - \frac{d}{dt}\left[\frac{\partial[H(x(t), u(t), \lambda(t), t) + \frac{\partial s(.)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(.)}{\partial t} - \lambda^T(t)\dot{x}(t)]}{\partial \dot{x}(t)}\right] \qquad (19)$$

From chain rule, if $f(x(t), y(t), z(t))$, then

$$\frac{d}{dt}f(.) = \frac{\partial f(.)}{\partial x(t)}\dot{x}(t) + \frac{\partial f(.)}{\partial y(t)}\dot{y}(t) + \frac{\partial f(.)}{\partial z(t)}\dot{z}(t) \qquad (20)$$

Application of (20) in (19) give

$$\frac{\partial}{\partial x(t)}[H(.) + \frac{ds(.)}{dt} - \lambda^T(t)\dot{x}(t)] - \frac{d}{dt}\left[\frac{\partial s(.)}{\partial x(t)} - \lambda^T(t)\right]$$

From (19), $\frac{\partial s(.)}{\partial \dot{x}(t)}$ and $\frac{\partial s(.)}{\partial t}$ can be written combinely as $\frac{d}{dt}s(.)$

$$\left(\frac{\partial H(.)}{\partial x(t)}\right)_* = -\dot{\lambda}(t) \qquad (21)$$

Equation (21) is called the co-state and free from $\dot{x}(t)$

Also from (13),

$$\left(\frac{\partial L(.)}{\partial u(t)}\right)_* = 0, \Rightarrow \left(\frac{\partial L(.)}{\partial u(t)}\right)_* = \left(\frac{\partial H(.)}{\partial u(t)}\right)_* = 0 \qquad (22)$$

Where $L(.)$ remain as defined in (18)

The similar version of (21) which was gotten from the Lagrange equation is

$$\left(\frac{\partial H(.)}{\partial \lambda(t)}\right)_* = \dot{x}(t) \qquad (23)$$

Equation (23) is called the state equation if the system equation is express in state pace form.

Finally, the boundary condition (17) can be written in Hamiltonian form as

$$\left\{ H(.) + \left(\frac{\partial s}{\partial x(t)}\right)^T - \lambda^T(t)\dot{x}(t) - [\frac{\partial s}{x(t)} - \lambda(t)]\dot{x}(t) \right\}_* /_{t=t_f}\ \delta t_f + \left[\frac{\partial s}{x(t)} - \lambda(t)\right]_* /_{t=t_f}\ \delta x_f = 0$$

Therefore

$$\left[H(.) + \frac{\partial s}{\partial x(t)} \right]_* /_{t=t_f}\ \delta t_f + \left[\frac{\partial s}{x(t)} - \lambda(t)\right]_*^T /_{t=t_f}\ \delta x_f = 0 \qquad (24)$$

If equation (21)-(24) are solved, we get our trajectory $x(t)$ and the control input $u(t)$ which minimizes the P.I.

Algorithmic steps

Given the constraint

$\dot{x}(t) = f(x(t), u(t), t)$ and the performance index $J(.) = s(x(t))|_{t=t_f} + \int_{t_0}^{t_f} v(x(t), u(t), t)dt$

Step 1: Form $H(x(t), u(t), \lambda(t), t)$

Step 2: Compute (21), (22), (23) using the boundary condition (24)

3.0 Embedding Improved Euler

The application of improved Euler's to optimal control problems need the in-depth knowledge of optimization and numerical analysis. The algorithm of this embedment of the improved Euler as called in this research can be summarized as follows:

Step 1: derive the Hamiltonian using the multiplier method

Step2: Solve equation (21)- (23) and boundary condition (25)

Step3: Determine the numerical value of the co state and the state using the Euler's algorithm

Step4: if $x_i = x_{i-1}$ or H(x(t), u(t),λ(t)) approaches zero, stop, else

step 5: Update t_{i+1} with $t_{i+1} = t_i + t_{i-1}$ and repeat step 1- step 3.

4.0 Problems, Results and Discussion

We shall present some optimal control problems and the result generated by the algorithm when applied to such life problems

4.1 Problems

Problem 1: Lagrange form of Optimal Control problem without delay

$$\underset{(x,u,\lambda)}{Minimize}\ J = \int_0^1 [0.5u^2(t)]dt$$

$$\text{Subject to}: \dot{x}_1 = x_2(t)$$

$$\dot{x} = -x_1(t) + u(t)$$

$$x_1(0) = 1,\ x_2(0) = 2\ \ \lambda_1(0) = 1,\ \ \lambda_2(0) = 0$$

Problem 2: Lagrange form of Optimal Control problem with weighted matrix as coefficient

$$\underset{(x,u,\lambda)}{Minimize}\ J = 0.5\int_0^1 [x^T(t)Px(t) + u^T(t)Ru(t)]dt$$

$$\text{Subject to}: \dot{x}_1 = 2x_2(t)$$

$$\dot{x}_2 = -x_1(t) - 3x_2(t) + u(t)$$

$$x_1(0)=10\,,\, x_2(0)=-5\,,\;\lambda_1(0)=1,\;\lambda_2(0)=0\,,\;R=1\,,\;\;P=\begin{pmatrix}1 & 0\\ 0 & 10\end{pmatrix}$$

$t_0 \le t \le t_f$

4.2 Numerical Results

The numerical results are presented in the tables below:

Table 4.1: Numerical solution of problem 1

values of T	$x_1^*(t)$	$x_2^*(t)$	$u^*(t)$	NO. of iter	$J(x_j,u_j,\lambda_j)$
$t=0.2$	1.34779	1.47399	-1.32769	**4**	0.551911
$t=0.4$	1.58910	0.940452	-1.04505	4	0.0489287
$t=0.6$	1.72553	0.430117	-0.720742	4	-0.380566
$t=0.8$	1.76453	-0.0296308	-0.36770	5	-0.668764
$t=0.9$	1.75122	-0.233235	-0.184773	5	-0.744561
$t=1.0$	1.71857	-0.416123	$1.11022*10^{-12}$	4	1.768370

Table 4.2: Numerical solution of problem 2

values of T	$x_1^*(t)$	$x_2^*(t)$	$u^*(t)$	NO. of iter	$J(x_j,u_j,\lambda_j)$
$t=0.2$	1.47401	0.587855	-0.333073	**4**	-0.259827
$t=0.4$	1.58386	0.0498879	0.061901	4	0.238474
$t=0.6$	1.55925	-0.139084	0.207944	4	0.636421
$t=0.8$	1.48821	-0.20775	0.200358	5	0.635307
$t=0.9$	1.44389	-0.236722	0.132121	5	0.440433
$t=1.0$	1.39258	-0.279923	$2.84397*10^{-14}$	4	0.0927813

where

$x_1^*(t)$: is the value of $x(t)$ that satisfied the optimality condition

$x_2^*(t)$: is the value of $x(t)$ $u(t)$ that satisfied the optimality condition

$u^*(t)$: is the value of that satisfied the optimality condition

4.3 Comments

It can be seen from the above results in table 4.1 and 4.2 that two the three tested problems have similar characteristics:

*value of the state and the control variables $x^*(t)$ and $u^*(t)$ change for different values of t as J was approaching the optimal solution .

*The values of the control u(t)was decreasing as t was approaching terminal point i.e t_f

4.4 Conclusions

From the results and comment above, it can be concluded that the method of embedding the Hamiltonian into the algorithm of improved Euler is less strenuous when compared with the stress involve in development and execution of control operator as seen in [1] , [12] and [13], robust and reliable as it showed $x^*(t)$ as a trajectory whose curve with respect to t can be sketched.

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