EXAMINIING THE CONVERSION PROCESS OF LINEAR PROGRAMMING PROBLEM INTO A NON- SYMMETRIC GAME, AND ITS SPECIAL STRUCTURE

[1]Folakemi M. Okafor, [2]Emmanuel Iyasere Adiri, [3]Onyeka Patience
National Mathematical Centre, Abuja, Nigeria and Mathematics Department, College of Education, Warri, Delta State Nigeria. folayanfe@gmail.com
emmanueladiri2@gmail.com

Abstract
This paper is aimed at examining the relationship that exists between the non-symmetric games and linear programming where a Linear Programming Problem was converted to non-symmetric game. The results of our numerical computations show that the non-symmetric game of the dual LPP is the transpose of the primal LPP.
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Introduction

In literature, there abound many research works on the conversion of game problem into linear programming problem. Notably among these are Hillier and Lieberman (2001), Taha (2002), Ekoko (2011), Okafor, etal,. (2018) and Okafor& Adiri (2020). The reason for the formulation of game problem, dynamic programming problem, transportation problem etc. to Linear Programming problem is not unconnected with the fact that linear programming is one of the most applicable areas of Operations Research. More so, there are various computer programs that are available to solve LP problems using the simplex method or variations of it.

This research work centers on the formulation of a linear programming problem as a game problem. Our conversion from Linear Programming Problem to a non- symmetric game problem is important due to its special structures and its application to certain real life situations such as the comparative-advantage problem of international trade where we assume positive amounts of all limited resources and all international prices positive (Takayama,1967).

Conversion of LPP to Non- Symmetric Games

Often in linear-programming we encounter problems with special properties such as;

i. All the right-hand coefficients (b_i) in the constraints are of the same algebraic sign;

ii. All the (c_j) coefficients in the linear expression to be maximized (or minimized) are of the same algebraic sign;

iii. All the (a_{ij}) coefficients are of the same algebraic sign.

The simple comparative – advantage problem of international trade is of this form when we assume positive amounts of all limited resources and all international prices positive, also, the simple minimum-diet problem is also of this special type, since all the minimum requirements of nutrients are positive and all food prices are positive.

The only problem of interest with the above special properties can be put into the maximization form as follows;

$$\left.\begin{array}{l} \text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{Subject to} \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ ----------------- \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ \qquad x_j \ge 0,\ j = 1(1)n \\ \text{where } c_j \ge 0, b_i \ge 0, a_{ij} \ge 0 \end{array}\right\} \tag{2.1}$$

It follows that we can imagine our problem as having only b's and c's that are definitely positive, with all the rows and columns corresponding to zero (c_i) or (b_j) having been removed from the problem matrix.

We consider problem of the form (2.1) but with $b_i > 0$, $c_i > 0$, and no restriction on the a_{ij} except that we must have a feasible problem with finite z = z*.

We can rewrite LPP (2.1) above by dividing each constraint *i* by b_i. Hence each term on the L.H.S becomes:

$$\frac{a_{ij}x_j}{b_i} \tag{2.2}$$

The system (2.1) could be rewritten in terms of the variable u_j as follows:

Let $u_j = c_jx_j$ i.e. $x_j = \dfrac{u_j}{c_j}$ (2.3)

Hence every term

$$\frac{a_{ij}x_j}{b_i} = \frac{a_{ij}u_j}{b_ic_j} = A_{ij}u_j$$

Where

$$A_{ij} = \frac{a_{ij}}{b_i C_j}$$

Therefore, LPP (2.1) above can be written as

$$\left.\begin{array}{l} \text{Maximize } z = u_1 + u_2 + \cdots + u_n \\ \text{Subject to} \\ A_{11}u_1 + A_{12}u_2 + \cdots + A_{1n}u_n \le 1 \\ A_{21}u_1 + A_{22}u_2 + \cdots + A_{2n}u_n \le 1 \\ ------------------ \\ A_{m1}u_1 + A_{m2}u_2 + \cdots + A_{mn}u_n \le 1 \\ \quad u_j \ge 0,\ j = 1(1)n \end{array}\right\} \qquad (2.4)$$

The non-symmetric game in general form is thus

$$\begin{bmatrix} A_{11} & A_{12} \cdots A_{1n} \\ A_{21} & A_{22} \cdots A_{2n} \\ ----- & ----- \\ A_{m1} & A_{m2} \cdots A_{mn} \end{bmatrix} \qquad (2.5)$$

Hence, converting the dual LPP (2.1) which is

$$\left.\begin{array}{l} \text{Minimize } z^* = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m \\ \text{Subject to} \\ a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\ ------------------ \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \\ \quad y_1 \ge 0, \cdots, y_m \ge 0 \\ \text{where } c_j \ge 0, b_j \ge 0, a_{ij} \ge 0 \end{array}\right\} \qquad (2.6)$$

We can rewrite the dual LPP (2.6) above by dividing each constraint i by c_j.
i.e.

$$\frac{a_{ji} y_i}{c_j} \qquad (2.7)$$

and replace each y_i by w_i.
Let $w_i = b_i y_i$

This implies,

$$y_i = \frac{w_i}{b_i}$$

Then substituting for y_i in equation (2.7) above, we have

$$\frac{a_{ji} y_i}{c_j} = \frac{a_{ji} w_i}{c_j b_i} = A_{ji} w_i$$

Where

$$A_{ji} = \frac{a_{ji}}{c_j b_i}$$

Therefore, the dual LPP (2.6) above can be written as

$$\left.\begin{array}{l} \text{Minimize } z^* = w_1 + w_2 + \cdots + w_m \\ \text{Subject to} \\ A_{11} w_1 + A_{21} w_2 + \cdots + A_{m1} w_m \geq 1 \\ A_{12} w_1 + A_{22} w_2 + \cdots + A_{m2} w_m \geq 1 \\ ----------------- \\ A_{1n} w_1 + A_{2n} w_2 + \cdots + A_{mn} w_m \geq 1 \\ \qquad w_i \geq 0, i = 1(1)n \end{array}\right\} \quad (2.8)$$

The non-symmetric game in general form thus becomes

$$\begin{bmatrix} A_{11} & A_{21} \cdots A_{m1} \\ A_{12} & A_{22} \cdots A_{m2} \\ --------- & \\ A_{1n} & A_{2n} \cdots A_{mn} \end{bmatrix} \quad (2.9)$$

Numerical Illustration of Conversion of LPP to Non-Symmetric Game

We discussed in detail the conversion of LPP to non-symmetric game earlier. Worthy of note that not every type of linear programming problem can be converted to the special non-symmetric games. It is observed that only LPPs whose R.H.S. coefficients b_i have the same algebraic sign can be converted to non-symmetric game.

Example (Ekoko, 2011)

As an illustration, let us consider the following LPP:

$$\left.\begin{aligned} &Max\ z = 4x_1 + 3x_2 + 6x_3 \\ &s.t. \\ &3x_1 + x_2 + 3x_3 \le 30 \\ &2x_1 + 2x_2 + 3x_3 \le 40 \\ &x_1,\ \ x_2,\ \ x_3 \ge 0 \end{aligned}\right\} \qquad (3.1)$$

We can rewrite LPP in system (3.1) above by dividing each constraint *i* by b_i. Hence each term on the L.H.S. becomes:

$$\frac{a_{ij} x_j}{b_i}$$

Thereafter, rewrite the LPP in system (3.1) in term of the variable u_j as follows:

Let

$$u_j = c_j x_j \text{ i.e. } x_j = \frac{u_j}{c_j}$$

Hence every term

$$\frac{a_{ij} x_j}{b_i} = \frac{a_{ij} u_j}{b_i c_j} = A_{ij} u_j$$

Where

$$Aij = \frac{a_{ij}}{b_i c_i}$$

Therefore, LPP (3.1) above can be rewritten as

$$\left.\begin{aligned} &\text{Max } z = u_1 + u_2 + u_3 \\ &\text{subject to} \\ &A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \le 1 \\ &A_{21}u_1 + A_{22}u_2 + A_{23}u_3 \le 1 \\ &A_{31}u_1 + A_{32}u_2 + A_{33}u_3 \le 1 \\ &u_1, u_2, u_3 \ge 0 \end{aligned}\right\} \qquad (3.2)$$

Where $u_1 = 4x_1$, $u_2 = 3x_2$, $u_3 = 6x_3$

Since $A_{ij} = \dfrac{a_{ij}}{b_i c_j}$,

$$A_{11} = \frac{a_{11}}{b_1 c_1} = \frac{3}{(30)(4)} = \frac{3}{120} = \frac{1}{40}$$

$$A_{12} = \frac{a_{12}}{b_1 c_2} = \frac{1}{(30)(3)} = \frac{1}{90}$$

$$A_{13} = \frac{a_{13}}{b_1 c_3} = \frac{3}{(30)(6)} = \frac{3}{180} = \frac{1}{60}$$

$$A_{21} = \frac{a_{21}}{b_2 c_1} = \frac{2}{(40)(4)} = \frac{2}{160} = \frac{1}{80}$$

$$A_{22} = \frac{a_{22}}{b_2 c_2} = \frac{2}{(40)(3)} = \frac{2}{120} = \frac{1}{60}$$

$$A_{23} = \frac{a_{23}}{b_2 c_3} = \frac{3}{(40)(6)} = \frac{3}{240} = \frac{1}{80}$$

As stated in section (2.0) the non-symmetric game in general form is given as:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Therefore, the non-symmetric game payoff matrix corresponding to the LPP (3.1) can be stated as:

$$\begin{bmatrix} \frac{1}{40} & \frac{1}{90} & \frac{1}{60} \\ \frac{1}{80} & \frac{1}{60} & \frac{1}{80} \end{bmatrix}$$

Let us consider the conversion of the dual LPP (3.1). The dual LPP of the LPP (3.1) can be written as

$$\left.\begin{aligned} &\text{Minimize } z^* = 30y_1 + 40y_2 \\ &s.t. \\ &3y_1 + 2y_2 \geq 4 \\ &y_1 + 2y_2 \geq 3 \\ &3y_1 + 3y_2 \geq 6 \\ &y_1, y_2, y_3 \geq 0 \end{aligned}\right\} \quad (3.3)$$

As discussed earlier, we can rewrite the dual LPP (3.3) above by dividing each constraint by c_j.

i.e.

$$\frac{a_{ji} y_i}{c_j} = \frac{a_{ji} w_i}{c_j b_i}$$

Hence the LPP (3.3) can be written as

Minimize $z^* = w_1 + w_2$

$s.t.$

$$A_{11}w_1 + A_{21}w_2 \geq 1$$
$$A_{12}w_1 + A_{22}w_2 \geq 1$$
$$A_{13}w_1 + A_{23}w_2 \geq 1$$

The non-symmetric game of the dual LPP in general forms is:

$$\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{bmatrix}$$

Where

$$A_{ji} = \frac{a_{ji}}{c_j b_i}$$

Hence

$$A_{11} = \frac{a_{11}}{c_1 b_1} = \frac{3}{(30)(4)} = \frac{3}{120} = \frac{1}{40}$$

$$A_{12} = \frac{a_{12}}{c_1 b_2} = \frac{1}{(30)(3)} = \frac{1}{90}$$

$$A_{13} = \frac{a_{13}}{c_1 b_3} = \frac{3}{(30)(6)} = \frac{3}{180} = \frac{1}{60}$$

$$A_{21} = \frac{a_{21}}{c_2 b_1} = \frac{2}{(40)(4)} = \frac{2}{160} = \frac{1}{80}$$

$$A_{22} = \frac{a_{22}}{c_2 b_2} = \frac{2}{(40)(3)} = \frac{2}{120} = \frac{1}{60}$$

$$A_{23} = \frac{a_{23}}{c_2 b_3} = \frac{3}{(40)(6)} = \frac{3}{240} = \frac{1}{80}$$

Therefore, the conversion of the dual LPP (3.3) to the corresponding non-symmetric game yields

$$\begin{bmatrix} \frac{1}{40} & \frac{1}{80} \\ \frac{1}{90} & \frac{1}{60} \\ \frac{1}{60} & \frac{1}{80} \end{bmatrix}$$

Discussion of Results

As expected, non-symmetric game payoff matrix is the transpose of that of the primal LP problem.

It is interesting to note that the non-symmetric game of the dual LPP is the transpose of the non-symmetric game of the primal LPP.

Conclusion

The researchers found that LP problem can be converted to non-symmetric game. From available literature, it is observed that much attention had not been given to this area.

The results of our numerical computations show that the non-symmetric game of the dual LPP is the transpose of the primal LPP. This type of conversion of LPP to non-symmetric game problems has been found to be associated with specific economic problem. These include the comparative-advantage problem of international trade in which we assume positive amounts of all limited resources and all international prices positive. A second example is the minimum-diet problem in which all the minimum requirements of nutrients are positive and all food prices are positive

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