EXISTENCE AND UNIQUENESS OF FIXED POINT IN PARTIALLY ORDERED CAUCHY SPACE

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Abstract
The idea of a fixed point is of great interest both in mathematics and in many areas of applied science. Quadrupled fixed point is an extension to tripled fixed point theorem. This work establishes the existence and uniqueness of a quadrupled fixed point for the $T: X^4 \rightarrow X$ map, which satisfies the quadrupled fixed point contractive condition for the partially ordered mixed monotone operator. Also, it provide a very useful additional property to ensure that the product space X^4 of the operator, equipped with a partial ordered set, has some conditions to ensure its uniqueness. Our results extend and summarize some of the results in the literature.
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1. Introduction

Metric fixed point theory is a branch of mathematical analysis that focuses on the existence and uniqueness of fixed points under metric conditions, both in the field of mapping and in the mapping itself. The fundamental result of this theory is as follows: Let (X, d) be a complete metric space and $T: X \rightarrow X$ be such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, and some $k \in [0,1)$. Then $fix(T)$ is the singleton, that is, there exists a unique $x \in X$ such that $Tx = x$ (Aniki and Rauf, 2019).
The contraction fixed point theorem is an important and practical tool for studying unknown situations in the real world. There are various extensive investigations for the fixed point theorems with contractive conditions in metric space, cone metric space, partially ordered metric space and so on. For the successful applications to the differential and integral equations, fixed point theorems for mixed monotone mappings are speedily developed recently. In (Bashkar and Lakshmikantham, 2006), the concept of coupled fixed point was introduced and proved some coupled fixed point theorems for mixed monotone mappings while (Rauf, Alata and Wahab, 2016) establish and prove some common fixed points for self-mapping in a complete cone metric space.

The notions of tripled fixed point were introduced in (Berinde and Borcut, 2011; Berinde, 2012). Meanwhile the corresponding fixed point theorem has been proved. Also, the notion of g-monotone property was introduced in (Lakshmikantham and Ciric, 2009) and the coincidence fixed point theorem was shown.

The following basic notations are useful in the statement of our results

Definition 1. (Berinde and Borcut, 2011). Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Then, the product space X^3 has the following partial order

$$(p, q, r) \leq (s, t, u) \Leftrightarrow s \geq p, t \leq q, u \geq r; \quad (p, q, r), (s, t, u) \in X^3.$$

Definition 2. (Berinde and Borcut, 2011). Let$(X, \leq)$ be a partially ordered set and $T: X^3 \rightarrow X$ be a mapping. We say that T has a mixed monotone property if $T(s, t, u)$ is monotone nondecreasing in s, monotone nonincreasing in t and monotone nondecreasing in u, that is for any $s, t, u \in X$,

$$s_1 \leq s_2 \Rightarrow T(s_1, t, u) \leq T(s_2, t, u), \quad s_1, s_2 \in X,$$
$$t_1 \leq t_2 \Rightarrow T(s, t_1, u) \geq T(s, t_2, u), \quad t_1, t_2 \in X,$$
$$u_1 \leq u_2 \Rightarrow T(s, t, u_1) \leq T(s, t, u_2), \quad u_1, u_2 \in X.$$

Definition 3. (Rauf and Aniki, 2020). An element $(s, t, u) \in X^3$ is called tripled fixed point of the mapping $T: X^3 \rightarrow X$, if

$$T(s, t, u) = s, \qquad T(t, s, u) = t, \qquad T(u, t, s) = u.$$

Definition 4. (Berinde and Borcut, 2011). A mapping $T: X^3 \rightarrow X$ is said to be (κ, μ, ψ) $-$contraction if and only if there exist three constants $\kappa \geq 0, \mu \geq 0, \psi \geq 0, \kappa + \mu + \psi < 1$, such that $\forall\ s, t, u, p, q, r \in X$,

$$d\big(T(s, t, u), T(p, q, r)\big) \leq \kappa d(s, p) + \mu d(t, q) + \psi d(u, r).$$

Matthews (1994) introduced the notion of partial metric spaces and extended the Banach contraction principle from metric spaces to partial metric spaces. Based on the notion of partial metric spaces, several authors obtained some fixed point results for mappings satisfying different contractive conditions.

Banach Principle was applied on partially ordered complete metric spaces (Rann and Reurings, 2013) and starting from their results, (Bashkar and Lakshmikantham, 2006) extended this theory to partially ordered complete metric spaces and introduce the concept of coupled fixed point on mixed-monotone operators of Picard type. Furthermore, results involving the existence and uniqueness of the coincidence points for mixed-monotone operators $T: X^2 \rightarrow X$ in the presence of a contractive type condition are given in the following results.

Theorem 1. (Aniki and Rauf, 2019). Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X^2 \rightarrow X$ be a continuous mapping having a mixed monotone property on X.

There exists $h \in [0,1)$, such that T satisfies the following contraction condition.

$$d\big(T(s,t),T(q,r)\big) \leq \frac{h}{2}[d(s,q)+d(t,r)],$$

for each $q,r,s,t \in X$, with $s \geq q$ and $t \leq r$.
If there exists $s_0, t_0 \in X$ such that

$$s_0 \leq T\big(s_{0,}t_0\big)\ and\ t_0 \geq T(t_0,s_0)$$

Then, there exists $s^*, t^* \in X$ such that
$s^* = T(s^*,t^*)$and $t^* = T(t^*,s^*)$.
However, this work aims at establishing the existence and the uniqueness of the concept of quadrupled fixed point of contractive type for mappings in partially ordered complete metric spaces

2. Main Results

In this section, we let$(X,\leq)$be a partially ordered set and d be a metric on X such that the pair (X,d) is a complete metric space. Then, X^4 is a product space with the following partial order

$$(p,q,r,s) \leq (u,v,w,x) \Leftrightarrow u \geq p, v \leq q, w \geq r, x \leq s$$
$$\forall (p,q,r,s),(u,v,w,x) \in X^4.$$

Definition 5. Let $(X,\leq)$ be a partially ordered set and $T: X^4 \to X$ be a mapping. We say that T has the mixed monotone property if $T(u,v,w,x)$ is monotone nondecreasing in u and w, and monotone nonincreasing in $\boldsymbol{v}$ and x, that is, for any $u,v,w,x \in X$,

$$u_1,u_2 \in X, u_1 \leq u_2 \Rightarrow T(u_1,v,w,x) \leq T(u_2,v,w,x),$$
$$v_1,v_2 \in X, v_1 \leq v_2 \Rightarrow T(u,v_1,w,x) \geq T(u,v_2,w,x),$$
$$w_1,w_2 \in X, w_1 \leq w_2 \Rightarrow T(u,v,w_1,x) \leq T(u,v,w_2,x),$$

and

$$x_1,x_2 \in X, x_1 \leq x_2 \Rightarrow T(u,v,w,x_1) \geq T(u,v,w,x_2).$$

Definition 6. An element $(u,v,w,x) \in X^4$ is called a quadrupled fixed point of the mapping $T: X^4 \to X$, if $T(u,v,w,x) = u, T(v,u,v,x) = v, T(w,u,v,w) = w$, and $T(x,w,v,u) = x$.
Let the pair (X,d) be a complete metric space. The mapping $T: X^4 \to X$, given by

$$d[(u,v,w,x),(p,q,r,s)] = d(u,p)+d(v,q)+d(w,r)+d(x,s),$$

defines a metric on the space X^4, which will be denoted by d for convenience purpose. Hence, the first main result is showcased in the following theorem.

Theorem 2. Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that the pair (X,d) is a complete metric space. Let $T: X^4 \to X$ be a continuous mapping having the mixed monotone property on X.Assuming there exist four constants $\vartheta,\kappa,\lambda,\mu \in [0,1)$ with $\vartheta+\kappa+\lambda+\mu < 1$ for which

$$d[T(u,v,w,x),T(p,q,r,s)] = \vartheta d(u,p) + \kappa d(v,q) + \lambda d(w,r) + \mu d(x,s), \quad (1)$$

$\forall\, u \geq p, v \leq q, w \geq r, x \leq s$. If there exist $u_0, v_0, w_0, x_0 \in X$ such that $u_0 \leq T(u_0, v_0, w_0, x_0), v_0 \geq T(v_0, u_0, v_0, x_0), w_0 \leq T(w_0, u_0, v_0, w_0)$, and $x_0 \geq T(x_0, w_0, v_0, u_0)$,then there exist $u, v, w, x \in X$ such that
$u = T(u,v,w,x), v = T(v,u,v,x), w = T(w,u,v,w)$,and $x = T(x,w,v,u)$.
It can be proved that the quadrupled fixed point in Theorem 2 is unique, provided that the product space X^4 equipped with the partial order earlier mentioned has an additional property. However, some additional conditions to ensure its uniqueness are as shown.

Theorem 3. Let $(X, \leq)$ be a partially ordered set and if d is a metric on X such that the pair (X, d) is a complete metric space. Let $T: X^4 \to X$ be a continuous mapping having mixed monotone property on X. If there exists constants $\vartheta, \kappa, \lambda, \mu \in [0,1)$ with $\gamma = \vartheta + \kappa + \lambda + \mu < 1$ for which

$$d[T(u,v,w,x),T(p,q,r,s)] = \vartheta d(u,p) + \kappa d(v,q) + \lambda d(w,r) + \mu d(x,s),$$

$\forall\, u \geq p, v \leq q, w \geq r, x \leq s$. If there exists $u_1, v_1, w_1, x_1 \in X$ such that $u_1 \leq T(u_1, v_1, w_1, x_1), v_1 \geq T(v_1, u_1, v_1, x_1), w_1 \leq T(w_1, u_1, v_1, w_1)$, and $x_1 \geq T(x_1, w_1, v_1, u_1)$,
for every $(u,v,w,x), (u_1, v_1, w_1, x_1) \in X^4$, there exists $(p,q,r,s) \in X^4$ that is comparable to (u,v,w,x) and (u_1, v_1, w_1, x_1), then we can obtain the uniqueness of the quadrupled fixed point of T.

Proof

If $(u^*, v^*, w^*, x^*) \in X^4$ is another quadrupled fixed point of T, then it can be shown that

$$d\big((u,v,w,x),(u^*,v^*,w^*,x^*)\big) = 0,$$

where

$$\lim_{n\to\infty} T^n\,(u_0, v_0, w_0, x_0) = u, \lim_{n\to\infty} T^n\,(v_0, u_0, v_0, x_0) = v, \lim_{n\to\infty} T^n\,(w_0, u_0, v_0, w_0) = w, \lim_{n\to\infty} T^n\,(x_0, w_0, v_0, u_0) = x.$$

Now, on considering two cases:

Case 1:

If (u,v,w,x) are comparable to (u^*, v^*, w^*, x^*) with respect to the ordering in X^4, then,

$$\big(T^n(u,v,w,x), T^n(v,u,v,x), T^n(w,u,v,w), T^n(x,w,v,u)\big) = (u,v,w,x)$$

that is comparable to

$$\big(T^n(u^*,v^*,w^*,x^*), T^n(v^*,u^*,v^*,x^*), T^n(w^*,u^*,v^*,w^*), T^n(x^*,w^*,v^*,u^*)\big) = (u^*,v^*,w^*,x^*)$$

Also,

$$d\big((u,v,w,x),(u^*,v^*,w^*,x^*)\big) = d(u,u^*) + d(v,v^*) + d(w,w^*) + d(x,x^*)$$

$$= d\big(T^n(u,v,w,x),T^n(u^*,v^*,w^*,x^*)\big) + d\big(T^n(v,u,v,x),T^n(v^*,u^*,v^*,x^*)\big) + d\big(T^n(w,u,v,w),T^n(w^*,u^*,v^*,w^*)\big) + d\big(T^n(x,w,v,u),T^n(x^*,w^*,v^*,u^*)\big) = \gamma^n[d(u,u^*)+d(v,v^*)+d(w,w^*)+d(x,x^*)] = \gamma^n d\big((u,v,w,x),(u^*,v^*,w^*,x^*)\big),$$

where$\gamma = \vartheta+\kappa+\lambda+\mu < 1$,this implies that $d\big((u,v,w,x),(u^*,v^*,w^*,x^*)\big) = 0$.

Case 2:

If (u,v,w,x) are not comparable to (u^*,v^*,w^*,x^*), then if there exists an upper bound or a lower bound $(p,q,r,s) \in X^4$ of (u,v,w,x), and (u^*,v^*,w^*,x^*). Then, for all $n = 1,2,\dots$,

$$\big(T^n(p,q,r,s),T^n(q,p,q,s),T^n(r,p,q,r),T^n(s,r,q,p)\big)$$

is comparable to

$$\big(T^n(u,v,w,x),T^n(v,u,v,x),T^n(w,u,v,w),T^n(x,w,v,u)\big)$$

and to

$$\big(T^n(u^*,v^*,w^*,x^*),T^n(v^*,u^*,v^*,x^*),T^n(w^*,u^*,v^*,w^*),T^n(x^*,w^*,v^*,u^*)\big).$$

We have

$$d\left(\begin{pmatrix}u\\v\\w\\x\end{pmatrix},\begin{pmatrix}u^*\\v^*\\w^*\\x^*\end{pmatrix}\right) \le d\left(\begin{pmatrix}T^n(u,v,w,x)\\T^n(v,u,v,x)\\T^n(w,u,v,w)\\T^n(x,w,v,u)\end{pmatrix},\begin{pmatrix}T^n(u^*,v^*,w^*,x^*)\\T^n(v^*,u^*,v^*,x^*)\\T^n(w^*,u^*,v^*,w^*)\\T^n(x^*,w^*,v^*,u^*)\end{pmatrix}\right)$$

$$\le d\left(\begin{pmatrix}T^n(u,v,w,x)\\T^n(v,u,v,x)\\T^n(w,u,v,w)\\T^n(x,w,v,u)\end{pmatrix},\begin{pmatrix}T^n(p,q,r,s)\\T^n(q,p,q,s)\\T^n(r,p,q,r)\\T^n(s,r,q,p)\end{pmatrix}\right) + d\left(\begin{pmatrix}T^n(p,q,r,s)\\T^n(q,p,q,s)\\T^n(r,p,q,r)\\T^n(s,r,q,p)\end{pmatrix},\begin{pmatrix}T^n(u^*,v^*,w^*,x^*)\\T^n(v^*,u^*,v^*,x^*)\\T^n(w^*,u^*,v^*,w^*)\\T^n(x^*,w^*,v^*,u^*)\end{pmatrix}\right)$$

$$\le \gamma^n\{[d(u,p)+d(v,q)+d(w,r)+d(x,s)] + [d(p,u^*)+d(q,v^*)+d(r,w^*)+d(s,x^*)]\} \to 0 \ as\ n \to \infty,$$

hence,

$$d\left(\begin{pmatrix} u \\ v \\ w \\ x \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ w^* \\ x^* \end{pmatrix}\right) = 0. \tag{2}$$

Assuming that every quadruple of elements of X have either an upper or a lower bound in X, it is established that the components of the quadrupled fixed points are equal.
The following theorem establishes the fact.

Theorem 4. In addition to the hypothesis of Theorem 2. Suppose that every quadruple of elements of X has an upper or a lower bound in X.Then, $u = v = w = x$.

Proof

If u, v, w, x are comparable, then $u = T(u, v, w, x), v = T(v, u, v, x), w = T(w, u, v, w), x = T(x, w, v, u)$ are comparable and which gives the following

$$\begin{aligned} d(u, x) &= d\big(T(u, v, w, x), T(x, w, v, u)\big) \\ &\leq \vartheta d(u, x) + \kappa d(v, w) + \lambda d(w, v) + \mu d(x, u) \\ &= (\vartheta + \mu) d(u, x) + (\kappa + \lambda) d(v, w) \\ &= (\vartheta + \kappa + \lambda + \mu) d(u, x) \\ &< d(u, x) \end{aligned} \tag{3}$$

Since u, v, w, x are comparable, then $d(u, x) = d(v, w)$. This implies that $d(u, x) = 0$ and $d(v, w) = 0$, that is, $u = x$ and $v = w$. Therefore, $u = v = w = x$.

Theorem 5. In addition to the hypothesis of Theorem 2 Suppose that $u_0, v_0, w_0, x_0 \in X$ are comparable. Then, $u = v = w = x$.

Proof

Recall that $u_0, v_0, w_0, x_0 \in X$ such that$u_0 \leq T(u_0, v_0, w_0, x_0), v_0 \geq T(v_0, u_0, v_0, x_0), w_0 \leq T(w_0, u_0, v_0, w_0)$, and $x_0 \geq T(x_0, w_0, v_0, u_0)$.
If $u_0 \leq v_0$, $v_0 \geq w_0$, and $w_0 \leq x_0$, for all $n \in \mathbb{N}, u_n \leq v_n, v_n \geq w_n$, and $w_n \leq x_n$.
Indeed, by mixed monotone property of T,

$$u_1 = T(u_0, v_0, w_0, x_0) \leq T(v_0, u_0, v_0, x_0) = v_1,$$
$$v_1 = T(v_0, u_0, v_0, x_0) \geq T(w_0, u_0, v_0, w_0) = w_1,$$

and

$$w_1 = T(w_0, u_0, v_0, w_0) \leq T(x_0, w_0, v_0, u_0) = x_1$$

Assuming that $u_n \leq v_n, v_n \geq w_n$, and $w_n \leq x_n$ for some n. Then,

$$u_{n+1} = T^{n+1}(u_0, v_0, w_0, x_0)$$
$$= T\big(T^n(u_0, v_0, w_0, x_0), T^n(v_0, u_0, v_0, x_0), T^n(w_0, u_0, v_0, w_0), T^n(x_0, w_0, v_0, u_0)\big)$$

$$= T(u_n, v_n, w_n, x_n) \leq T(v_n, u_n, v_n, x_n) = v_{n+1},$$

and similarly for w_n and x_n. Now

$$d(u,v) \leq d\big(u, T^{n+1}(u_0, v_0, w_0, x_0)\big) + d(T^{n+1}(u_0, v_0, w_0, x_0), v)$$
$$\leq d\big(u, T^{n+1}(u_0, v_0, w_0, x_0)\big) + d\big(T^{n+1}(u_0, v_0, w_0, x_0), T^{n+1}(v_0, u_0, v_0, x_0)\big) + d\big(v, T^{n+1}(v_0, u_0, v_0, x_0)\big)$$
$$= d\big(u, T^{n+1}(u_0, v_0, w_0, x_0)\big) + d\begin{bmatrix} T\big(T^n(u_0, v_0, w_0, x_0), T^n(v_0, u_0, v_0, x_0), T^n(w_0, u_0, v_0, w_0), T^n(x_0, w_0, v_0, u_0)\big), \\ T\big(T^n(v_0, u_0, v_0, x_0), T^n(u_0, v_0, w_0, x_0), T^n(v_0, u_0, v_0, x_0), T^n(x_0, w_0, v_0, u_0)\big) \end{bmatrix} + d\big(v, T^{n+1}(v_0, u_0, v_0, x_0)\big)$$
$$\leq d\big(u, T^{n+1}(u_0, v_0, w_0, x_0)\big) + \gamma^{n+1}[d(u_0, v_0) + d(w_0, v_0) + d(w, p) + d(x_0, x_0)] + d\big(v, T^{n+1}(v_0, u_0, v_0, x_0)\big) \to 0 \ as\ n \to \infty,$$

This implies that $d(u,v) = 0$, hence $u = v$. Similarly, it can be shown that $d(u,w) = 0, d(u,x) = 0, d(v,w) = 0$ and $d(v,x) = 0$.

3. Conclusion

This work demonstrates the existence and uniqueness of a quadruple fixed point in a partially ordered metric space in the presence of a contractive condition, which is a continuation of the quadruple fixed point generalization.

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