On Cardinality in Finite Semigroup of Full Order -Preserving Contractions

By

Shamsuddeen Habibu, Umar Mallam Abdulkarim (PhD), H.K Oduwole (PhD), Ibrahim Yusuf Kakangi and Jaiyeola Olorunshola Braimoh

Department of Mathematics, Federal College of Education (Technical), Gusau, Zamfara State, Nigeria, Department of Mathematics, Nasarawa State University, Nasarawa State, Nigeria, Department of Mathematical Science, Kaduna State University, Kaduna and Department of General Studies Education, Federal College of Education(Technical), Gusau, Zamfara State, Nigeria shamsuddeenhabeeb@gmail.com

Abstract
In this work, we considered the semigroup OCT_n consisting of all mappings of a finite set $X_n = \{1, 2, 3, - - - , n\}$ which are both order – preserving and contraction, that is mapping $\alpha : Xn \rightarrow Xn$ such that, for all $x , y \in X_n$, $x \leq y \Rightarrow x\alpha \leq y\alpha$, and $|x\alpha - y\alpha| \leq |x - y|$. In particular, we established a closed form formular for the number of elements in OCT_n

Keyword: Full Transformation, Contraction, Semigroup, order preserving

INTRODUCTION

A Semigroup is a non-empty set which is closed under an associative binary operation. There are many examples of different classes of semigroups, but the classical ones are obtained by mapping of a set into itself. This is because self of a set play similar role in semigroup theory as permutations in the theory of groups. That is, every semigroup can be represented by a semigroup of mapping of a set (Howie, 1995).

Let $X_n = \{1, 2, - - - - , n\}$. A partial transformation of X_n is any mapping $\alpha: dom(\alpha) \rightarrow x_n$, , where $dom(\alpha) \subseteq x_n$. The partial mapping is said to be a full transformation if $dom(\alpha) = x_n$. The set of all partial, full and partial one – to – one mapping of X_n are semigroups under composition of mappings. These are respectively called the full transformation semigroup , the partial transformation semigroup and systematic inverse semigroup, and are denoted by T_n, P_n and I_n respectively. These semigroups along with many of their interesting subsemigroups have been studies both algebraically and combintorially by many authors. These studies were pioneered by Howie (1966) in which he showed that a singular elements (non – invertible elements) in Tn are generated by singular idempotents in T_n (That is singular elements $e \in T_n$ satisfying $e^2 = e$). Howie

(1966) work drew the attention of many researchers for example Garba (1990, 1994a, b, c, d, e) (Ayik et al 2005, 2008), Umar (1992, 1993, 1994, 1996) and the reference there in. Combinatorial result pertaining to order of semigroups have been studied in the semigroups T_n and many of its notable subsemigroups. Adeshola (2012) studied some combinatorial identities in the semigroup OCT_n of all order- preserving full contractions.

It was proved by Howie(1966) that in every finite full transformation semigroup T_n, the subsemigroup $sing_n$, of all singular self-map of T_n, is generated by its set E=E($sing_n$), of all idempotents (that is, each element of $sing_n$ is expressable as a product of idempotents in E). the analogue of this result for semigroup of singular matrices was obtained by Erdos (1967). Different kind of combinatorial problems arises from the work of Howie. Many researchers became interested in addressing these problems with respect to different kind of generating sets.

Let X_n = {1, 2, - - - - , n}. then it is not difficult to see that for the semigroup T_n , P_n , I_n we have the following orders, which may be found in (Ganyushkin and Mazorchuk (2009):

$$/T_n/= n^n$$
$$/P_n/= (n+1)^n$$
$$/I_n/=\sum_{r=0}^{n}\binom{n}{r}^2 r!$$

The number of idempotent element in the semigroup T_n is computed by Harris and Schoenfield (1967) as $/E(T_n)/= \sum_{K=1}^{n}\binom{n}{K}\ k^{n-k}$, and for P_n , I_n were obtained by Ganyushkin and Mazorchuk (2009) as $/E(P_n)/= \sum_{K=0}^{n}\binom{n}{K}\ (k+1)^{n-k}$

$$/E(I_n)/= 2^n$$

2. PRELIMINARIES

2.1 Semigroups

A groupoid is a pair $(S, *)$ consisting of a non-empty set S and a binary operation * defined on S. we say that groupoid $(S,*)$ is a semigroup if the operation * is associative in S, that is to say, if, for all x, y and z in S, the equality $(x * y) * z = x * (y * z)$ holds if in a semigroup S the binary operation has the property that, for all x , y, in S, $xy = yx$, we say that S is a commutative semigroup. If a semigroup S contains an element 1 with the property that, for all $x \in S$, $x1 = 1x = x$ then S is called a semigroup with identity, and the element 1 is called the identity element of S.

Theorem 2.1 (Howie (1995)) A semigroup S has at most one identity.

Proof. If 1 and 1^1 are elements of S with property that $x1 = 1x = x \, and \, x \, 1^1 = 1^1 x = x$ for all x in S, then

$1^1 = 11^1$ (since 1 is an identity)

= 1 (since 1^1 is identity)

If S is a semigroup, which has no identity element, then it is very easy to adjoin an extra element 1 to S (to form a monoid out of S) given that 1s = s1 = s for all $S \in S$, and 11 = 1, it is then easy to see that $S \cup \{1\}$ becomes a monoid. Given monid, denoted by S^1, is defined by

$$S^1 = \begin{cases} S & if \ S \ has \ identity \\ S \ U \ \{1\} & otherwise \end{cases}$$

and called a semigroup with identity adjoined if necessary.

If a semigroup S with at least two elements contains an element O given that, for all $x \in S$, $0 \, x = x \, 0 = x = 0$, then s is called semigroup with zero and the element 0 as the zero element of S.

By analogy with case of S^1, for any semigroup S, we defined

$$S^0 = \begin{cases} S & if \ S \ has \ zero \\ S \ U \ \{0\} & otherwise \end{cases}$$

and refers to S^o as the semigroup obtained from S by adjoining a zero if necessary.

2.2 Subsemigroup and Ideals

A non – empty subset T of a semigroup S is called a subsemigroup of S if it is closed with respect to multiplication that is, if for all $x, y \ in \ T, xy \in T$.

If A and B are subset of a semigroup S, then we write AB to mean the set $\{ab: a \in A \ and \ b \in B\}$. and that $A^2 = a_1 a_2 : a_a, a_2 \in A$. The condition of closure in the definition of subsemigroup can be stated as $T^2 \subseteq T$.

A subsemigroup of S which is a group with respect to the multiplication inherited from S is called a subgroup of S.

2.3 Regular semigroups

An element a of a semigroup S is called regular if there exist x in S given that$x \, a \, x = a$. The semigroup S is called regular if all its elements are regular. That is if $(\forall \, a \in S)(\exists \, x \in S) \, ax \, a = a$

2.4 Ideal and Green's relations

The notion of ideals lead naturally to the consideration of certain equivalence relation on a semigroup. These equivalence relations, first introduced by Green (1951) played a fundamental role in the development of semigroup theory. Since their introduction, they have become standard tools for investigating the structure of semigroups.

If a is an element in a semigroup S, the sets

$S^1a = Sa \cup \{a\}$, $aS^1 = aS \cup \{a\}$ and $S^1aS^1 = SaS \cup Sa \cup a\,S \cup \{a\}$, are left, right and two – sided ideals of S respectively. These are respectively the smallest left, right and two sided. Ideals of S containing a. We shall call them principal left, right and two-sided ideals of S generated by a respectively.

For any two elements $a, b \in S$, we define the equivalences $\mathcal{L}, R, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ on $\mathcal{S}$ by

$a\,\mathcal{L}\,b$ if and only if $S^1a = S^1b$

$a\,\mathcal{R}\,b$ if and only if $aS^1 = bS^1$

$a\,\mathcal{J}\,b$ if and only if $S^1aS^1 = S^1bS^1$

$H = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$

These five equivalences are known as Green's relation (Howie, 1995).

Propositions 2.5 (Howie (1995)) let

1. $a\,\mathcal{L}\,\beta$ if and only if $\mathrm{I}m(\alpha) = \mathrm{I}m(\beta)$
2. $a\,\mathcal{R}\,\beta$ if and only if $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$
3. $a\,\mathcal{J}\,\beta$ if and only if $|\,im(\alpha)| = |\,im(\beta)|$
4. $\mathcal{D} = \mathcal{J}$

As a consequence of this, we see that, the J-classes in T_n are J_r and the number of L – classes is the number of distinct subset of X_n of cardinality r, that is, the binomial coefficient $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

The number of R-classes is the number of equivalences on X_n having r classes, that is, the stirling number of the second kind $S(n,r)$ defined recursively as $S(n,r) = S\,(n-1, r-1) + rS(n-1,r)$with boundary conditions $S(n,1) = S(n,n) = 1$, Also, $S(n, n-1) = \frac{n\,(n-1)}{2}$ and $S\,(n,2) = 2^{n-1}$

Therefore, a J-class J_r of T_n is visualized as an egg box in which the α - classes are the columns, the R – classes are the rows and the H - classes are the cells. The number of cells is $\frac{n}{r}$ x S (n, r), and each cell contains r! elements.

A subset Y = $\{a_1 - - -, a_r\}$, of X_n is said to be a traversal of (or orthogonal to) an equivalence ll, which classes $\{A_1, A_2, - - - A_r\}$, if each a_i in Y belongs to a unique P - class A_j. if Y is a traversal of P given that $a_i \in A_i$ for each i, then, the map

$$\in \; = \begin{pmatrix} A_1 & A_2 - - - - - A_r \\ a_r & a_2 - - - - - - a_r \end{pmatrix}$$

Is an idempotent. It is the unique idempotent in the H – class H_Y, P, in J_r corresponding to Y and P. Associated with a mapping α in T_n is a diagraph $\underset{\Gamma}{\rightarrow}(\alpha)$ whose vertices are labelled 1, 2, - - -, n and there is an edge $\iota \longrightarrow j$ if and only if $\iota\, \alpha = j$. Let $\alpha \in T_n$, we define an equivalence relation w on X_n by $\{(\iota, j) \in X_n \text{ x } X_n : (\exists\, r,\ s \geq 0)\ i\, \alpha^r = J\, \alpha^s\ \}$.
The w – classes are the connected components of $\underset{\Gamma}{\rightarrow}(\alpha)$ are called the orbitals of α. Each orbit Ω has a Kernel K(Ω), defined by $K(\Omega) = \{i \in \Omega : (\exists\, r > 0)\ i\, \alpha^r = i\}$. To see that K(Ω) is not empty for each orbit(Ω), consider an element in $i\ in\, \Omega$. The elements $i, i\,\alpha, i, \alpha^2, \ldots.$ cannot be all distinct, and so there exist $m \geq 0$ and $r \geq 1$ such that $i\, \alpha^{m+r} = i\, \alpha^m$. Thus $i\alpha^m \in K(\Omega)$
An orbit OL is said to be standard if and only if $| < |K\ (\Omega)\ | < |\ \Omega\ |$, acyclic is and only if $1 = |\ K(\Omega)| < |\ \Omega\ |$, cyclic if and only if $1 = |\ K\ (\Omega)\ | = \ |\ \Omega\ |$

Example 1. The map

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 11 & 14 \end{pmatrix}$$

In T_{14} has orbits $\Omega_1 = \{1, 2, 3, 4, 5, 6, 7\}$, $\Omega_2\ \{8, 9, 10\}$, $\Omega_3\ \{11, 12, 13,\}$ and $\Omega_4\ \{14\}$.

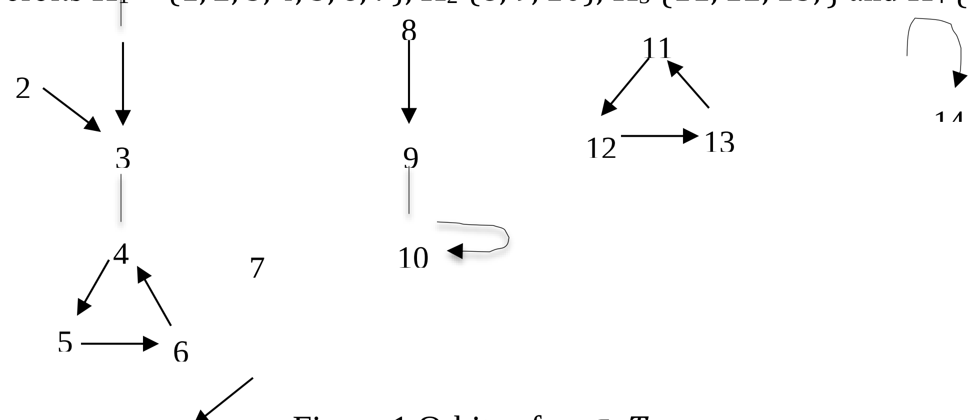

Figure 1 Orbits of $\alpha \in T_{14}$

It is clear from these diagram in fig 1. That,
$K(\Omega_1) = \{4, 5, 6,\}$, $K(\Omega_2) = \{10\}$, $K(\Omega_3) = \{11,\ 12, 13\}$
and $K(\Omega_4) = \{14\}$, therefore
Ω_1 is standard since $1 < |\ K\ (\Omega_1) < |\ \Omega_1\ |$
Ω_2 is acyclic since $1 = |\ K\ (\Omega_2) < |\ \Omega_2\ |$
Ω_3 is cyclic since $1 < |\ K\ (\Omega_3) < |\ \Omega_3\ |$
Ω_4 is trivial since $1 = |\ K\ (\Omega_4) < |\ \Omega_4\ |$
For each $\alpha \in T_n$ we define the gravity of α (Howie, 1980) by $g\ (\alpha\ ) = n + c\ (\alpha) - f(\alpha)$, where $C(\alpha)$ is the number of cyclic orbits of α and f(α) is the number of acyclic orbits plus the number of trivial orbits of α

3. MATERIAL AND METHODS

3.1 Number of order – preserving full contractions

This section is dedicated to finding an alternative method of obtaining the closed form formular for the order of the semigroup of order – preserving full contractions. The method used involves enumerating the elements of order – preserving full contraction OCT_n from the elements of first order preserving semigroups denoted by OT_n. We enumerate the elements of OCT_n for small integers n = 1, 2, 3, 4 according to the partitioning of OCT_n into J – classes. Standard tools in combinatorics such as binomial coefficient, Pascal triangles and other known identities were used. We approached the counting of elements by analysis special cases, making observation and then proceeding in establishing our observation in the general cases.

3.2 Enumeration of element in OCT_n

Since the semigroup OCT_n is a subsemigroup of OT_n. We obtain the elements of OCT_n for small values of n = 1, 2, 3, 4 by only considering order – preserving contraction mappings.

For n = 1 **Table 1: Elements of height 1 in OCT_1**

$J_1(OCT_1)$		{1}
1		$\begin{pmatrix}1\\1\end{pmatrix}$

$$| OCT_1 | = | J_1 (OCT_1) | = 1$$

For n = 2 **Table 2: Elements of height 2 in OCT_2**

$J_1(OCT_2)$	{1}	{2}
1 2	$\begin{pmatrix}1\ 2\\1\end{pmatrix}$	$\begin{pmatrix}1\ 2\\2\end{pmatrix}$

Table 3: Elements of height 2 in OCT_2

$J_2(OCT_2)$	{1, 2}
1 / 2	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$

$$| OCT_2 | = | J_1(OCT_2 | + | J_2(OCT_2) | = 2 + 1 = 3$$

For n = 3 **Table 4: Elements of height 1 in OCT_3**

$J_1(OCT_3)$	{1}	{2}	{3}

1 2 3	$\begin{pmatrix} 1\,2\,3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1\,2\,3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1\,2\,3 \\ 3 \end{pmatrix}$

Table 5: Elements of height 2 in OCT_3

$J_2(OCT_3)$	{1,2}	{1, 3}	{2, 3}
1 / 2 3	$\begin{pmatrix} 1 & 23 \\ 1 & 2 \end{pmatrix}$		$\begin{pmatrix} 1 & 23 \\ 2 & 3 \end{pmatrix}$
12 / 3	$\begin{pmatrix} 12 & 3 \\ 1 & 2 \end{pmatrix}$		$\begin{pmatrix} 12 & 3 \\ 2 & 3 \end{pmatrix}$

The empty cells in the table are those H – classes of OTn that contain no contraction mappings. This is also the case for all subsequent tables of the elements of OCT_n.

Table 6: Elements of height 3 in OCT_3

J_3 (OCT_3)	{1, 2, 3}
1/2/3	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$| OCT_3 | = |J_1(OCT_3)| + | J_2 (OCT_3) | + | J_3 (OCT_3) |$
$= 3 + 4 + 1 = 8$

For n = 4

Table 7: Elements of height 1 in OCT_4

$J_1(OCT_4)$	{1}	{2}	{3}	{4}
1 2 3 4	$\begin{pmatrix} 1\,2\,3\,4 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1\,2\,3\,4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1\,2\,3\,4 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 1\,2\,3\,4 \\ 4 \end{pmatrix}$

Table 8: Elements of height 2 in OCT_4

$J_2(OCT_4)$	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3,4}
1/ 2 3 4	$\begin{pmatrix} 1 & 234 \\ 1 & 2 \end{pmatrix}$			$\begin{pmatrix} 1 & 234 \\ 2 & 3 \end{pmatrix}$		$\begin{pmatrix} 1 & 234 \\ 3 & 4 \end{pmatrix}$
12 / 34	$\begin{pmatrix} 1\,2 & 34 \\ 1 & 2 \end{pmatrix}$			$\begin{pmatrix} 12 & 3\,4 \\ 2 & 3 \end{pmatrix}$		$\begin{pmatrix} 12 & 34 \\ 3 & 4 \end{pmatrix}$
123 / 4	$\begin{pmatrix} 123 & 4 \\ 1 & 2 \end{pmatrix}$			$\begin{pmatrix} 123 & 4 \\ 2 & 3 \end{pmatrix}$		$\begin{pmatrix} 123 & 4 \\ 3 & 4 \end{pmatrix}$

Table 9: Elements of height 3 in OCT_4

$J_3(OCT_4)$	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}
1/ 2 /3 4	$\begin{pmatrix} 1 & 2 & 34 \\ 1 & 2 & 3 \end{pmatrix}$			$\begin{pmatrix} 1 & 2 & 34 \\ 2 & 3 & 4 \end{pmatrix}$
1/23 /4	$\begin{pmatrix} 1 & 23 & 4 \\ 1 & 2 & 3 \end{pmatrix}$			$\begin{pmatrix} 1 & 23 & 4 \\ 2 & 3 & 4 \end{pmatrix}$
12/ 3 / 4	$\begin{pmatrix} 12 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$			$\begin{pmatrix} 12 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$

Table 10: Elements of height 4 in OCT_4

J_4 (OCT_4)	{1, 2, 3, 4}
1 / 2 / 3 / 4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$|OCT_4| = |J_1(OCT_4)| + |J_2(OCT_4| + |J_3(OCT_4)| + |J_4(OCT_4)| = 4 + 9 + 6 + 1 = 20$

4. RESULT AND DISCUSSION IN OCT_n

From the last tables, we developed the following sequence of cardinalities of OCT_n for small values of n. thus

n	1	2	3	4
OCT_n	1	3	8	20

Theorem 1. For all $n \geq 1$ the semigroup OCT_n contains $2^{n-2}\,(n+1)$ elements.

Proof: By Lemma 2.1 in Garba et al (2017), $\alpha \in OCTn$ if and only if each block of α is convex and also image of α is convex. Thus, if α is of height r, that is $|im\alpha| = r$, then the number of possible kernel blocks of α is the number of ways of inserting r – 1 strokes into n – 1 spaces. This equals the number of selecting r – 1 out of n – 1, thus, we have $\binom{n-1}{r-1}$ possible Kernel partitions of α. Next, there are n – r + 1 possible choices of the image of α.

Therefore, there are a total of (n – r + 1) $\binom{n-1}{r-1}$

number of order – preserving full contraction of height r. hence, the total number of elements in OCT_n is

$$|\, OCT_n \,| = \sum_{r=1}^{n} (n-r+1) \binom{n-1}{r-1}$$

It remain to prove the identity

$$\sum_{r=1}^{n} (n-r+1) \binom{n-1}{r-1} = 2^{n-2}\,(n+1)$$

But then

$$\sum_{r=1}^{n} (n-r+1) \binom{n-1}{r-1} = \sum_{r=1}^{n} (n-r) \binom{n-1}{r-1} + \sum_{r=1}^{n} \binom{n-1}{r-1}$$

$$= (n-1) \sum_{r=1}^{n} \binom{n-2}{r-1} + \sum_{r=1}^{n} \binom{n-1}{r-1}$$

$$= (n-1)2^{n-2} + 2^{n-1}$$

$$= (n-1+2)2^{n-2}$$

$$= 2^{n-2}\,(n+1)$$

Validation:
Consider any $n \geq 1$, say $n = 4$ that is T_4 and consider the tables for T_4 .Counting the cardinality of order-preserving full contraction, will see that there are exactly 20 of them. And going by the generated closed form formula, it can be seen that when $n = 4$ we have $2^{4-2}(4+1)$=20. The formula is valid for any $n \geq 1$.

5. CONCLUSION AND RECOMMENDATION

5.1 Conclusion

We have shown that the semigroup OCT_n contains $2^{n-2}\,(n+1)$ elements. These numbers have been previously found by Adeshola (2013) via different method. Our method of computation is more simple and direct and has the advantage of calculating the number of elements of a given height in OCT_n

5.2 Recommendations

We recommend that similar study to be extended to each of the following transformation semigroups:

(1) The semigroup OCI_n consisting of all partial one-to-one order-preseving contraction mappings of X_n
(2) The semigroup OCP_n consisting of all partial order-preseving contraction mappings of X_n
(3) The semigroup CT_n consisting of all full contraction mappings of X_n

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